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THE FOURTH ORDER MIXED PERIODIC RECURRENCE FRACTIONS

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Abstract. Offered economical algorithm for calculation of rational shortenings of the fourth-order mixed periodic recurrence fraction.

Keywords: parafunctions of triangular matrices, recurrence fractions, algebraic equations.

1. Introduction

The author got his idea of recurrence fractions in the year 2002 in [1], where continued fractions are written in terms of parapermanents of triangular matrices, i.e. the second-order recurrence fractions. The third-order recurrence fractions were studied by the author and his postgraduate student Semenchuk A.V. in [2],[3],[4]. This study is a natural continuation of [5]; that is why, in the present paper, we use references to the theorems and formulas of the latter. Given that, we shall denote the numbers of the formulas and theorems from [5] with a stroke.

2. The Fourth-Order Mixed Periodic Recurrence Fractions

Definition 2.1. *The 4-th order recurrence fraction*

$$
\alpha=\left[\begin{array}{c|ccccccccccccccc}
q_0^* & & & & & & & & & & & & & & & \\
\frac{p_1^*}{q_1^*} & q_1^* & & & & & & & & & & & & & & \\
\frac{r_2^*}{p_2^*} & \frac{p_2^*}{q_2^*} & q_2^* & & & & & & & & & & & & & \\
\frac{s_3^*}{r_3^*} & \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* & & & & & & & & & & & & \\
0 & \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* & & & & & & & & & & & \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ddots & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & q_{l-3}^* & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* & & & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & & & & & \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & q_{k-3} & & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \frac{p_{k-2}}{q_{k-2}} & q_{k-2} & & & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \frac{r_{k-1}}{p_{k-1}} & \frac{p_{k-1}}{q_{k-1}} & q_{k-1} & & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \frac{s_k}{r_k} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots
\end{array}\right]_{\infty}, \qquad (2.1)
$$

is called the 4-th order mixed k-periodic recurrence fraction with the preperiod of l.

We shall study the fourth-order 5-periodic recurrence fractions with the preperiod of 5. In case of the 4-th order k-periodic recurrence fractions with the preperiod of l, all the considerations and relevant algorithms are similar.

For $l = 5$ and $k = 5$, the fourth-order mixed periodic recurrence fraction is written as

$$\left[\begin{array}{c|ccccccccccc}
q_0^* & & & & & & & & & & & \\
\frac{p_1^*}{q_1^*} & q_1^* & & & & & & & & & & \\
\frac{r_2^*}{p_2^*} & \frac{p_2^*}{q_2^*} & q_2^* & & & & & & & & & \\
\frac{s_3^*}{r_3^*} & \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* & & & & & & & & \\
0 & \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* & & & & & & & \\
0 & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & & & & & & \\
0 & 0 & 0 & \frac{s_2}{r_2} & \frac{r_2}{p_2} & \frac{p_2}{q_2} & q_2 & & & & & \\
0 & 0 & 0 & 0 & \frac{s_3}{r_3} & \frac{r_3}{p_3} & \frac{p_3}{q_3} & q_3 & & & & \\
0 & 0 & 0 & 0 & 0 & \frac{s_4}{r_4} & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{s_5}{r_5} & \frac{r_5}{p_5} & \frac{p_5}{q_5} & q_5 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{array}\right]_\infty,$$

where q_j^*, p_j^*, r_j^*, s_j^*, q_i, p_i, r_i and s_i are positive.

We shall decompose the numerator of the rational shortening $\frac{[q_0^*]_{n+5}}{[q_1^*]_{n+4}}$ of this fraction by the elements of the inscribed rectangular matrix $T(6)$, and the denominator — by the elements of the table $T(5)$. We shall obtain

$$\frac{[q_0^*]_{n+5}}{[q_1^*]_{n+4}} = \frac{[q_0^*]_5(q_1[q_2]_{n-1} + p_2[q_3]_{n-2} + r_3[q_4]_{n-3} + s_4[q_5]_{n-4}) +}{[q_1^*]_4(q_1[q_2]_{n-1} + p_2[q_3]_{n-2} + r_3[q_4]_{n-3} + s_4[q_5]_{n-4}) +}$$

$$\frac{+[q_0^*]_4(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) + [q_0^*]_3(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_0^*]_2 s_1[q_2]_{n-1}}{+[q_1^*]_3(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) + [q_1^*]_2(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_1^*]_1 s_1[q_2]_{n-1}}.$$

In the numerator and denominator, the expression in the first brackets is decomposition of the parapermanent $[q_1]_n$ by the elements of the first column, so the last fraction is written as

$$\frac{[q_0^*]_5[q_1]_n \ + \ [q_0^*]_4(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) +}{[q_1^*]_4[q_1]_n \ + \ [q_1^*]_3(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) +}$$

$$\frac{+[q_0^*]_3(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_0^*]_2 s_1[q_2]_{n-1}}{+[q_1^*]_2(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_1^*]_1 s_1[q_2]_{n-1}},$$

or after grouping corresponding summands, it is written as

$$\frac{[q_0^*]_5[q_1]_n + ([q_0^*]_4 p_1 + [q_0^*]_3 r_1 + [q_0^*]_2 s_1)[q_2]_{n-1} + ([q_0^*]_4 r_2 + [q_0^*]_3 s_2)[q_3]_{n-2} + [q_0^*]_4 s_3}{[q_1^*]_4[q_1]_n + ([q_1^*]_3 p_1 + [q_1^*]_2 r_1 + [q_1^*]_1 s_1)[q_2]_{n-1} + ([q_1^*]_3 r_2 + [q_1^*]_2 s_2)[q_3]_{n-2} + [q_1^*]_3 s_3}.$$

We shall divide the numerator and denominator of the last fraction by $[q_4]_{n-3}$ and obtain the fraction

$$\frac{[q_0^*]_5 \frac{[q_1]_n}{[q_2]_{n-1}} \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} + ([q_0^*]_4 p_1 + [q_0^*]_3 r_1 + [q_0^*]_2 s_1) \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} +}{[q_1^*]_4 \frac{[q_1]_n}{[q_2]_{n-1}} \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} + ([q_1^*]_3 p_1 + [q_1^*]_2 r_1 + [q_1^*]_1 s_1) \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} +}$$

$$\frac{+([q_0^*]_4 r_2 + [q_0^*]_3 s_2)\frac{[q_3]_{n-2}}{[q_4]_{n-3}} + [q_0^*]_4 s_3}{+([q_1^*]_3 r_2 + [q_1^*]_2 s_2)\frac{[q_3]_{n-2}}{[q_4]_{n-3}} + [q_1^*]_3 s_3}$$

Let us have the following limits

$$\lim_{n\to\infty}\frac{[q_0^*]_{n+4}}{[q_1^*]_{n+3}} = x^*,\ \lim_{n\to\infty}\frac{[q_1]_n}{[q_2]_{n-1}} = x,\ \lim_{n\to\infty}\frac{[q_2]_{n-1}}{[q_3]_{n-2}} = y,\ \lim_{n\to\infty}\frac{[q_3]_{n-2}}{[q_4]_{n-3}} = z.$$

Then the last expression is written as

$$x^* = \frac{[q_0^*]_5 xyz + ([q_0^*]_4 p_1 + [q_0^*]_3 r_1 + [q_0^*]_2 s_1)yz + ([q_0^*]_4 r_2 + [q_0^*]_3 s_2)z + [q_0^*]_4 s_3}{[q_1^*]_4 xyz + ([q_1^*]_3 p_1 + [q_1^*]_2 r_1 + [q_1^*]_1 s_1)yz + ([q_1^*]_3 r_2 + [q_1^*]_2 s_2)z + [q_1^*]_3 s_3},$$

where x, y, z are solutions of simultaneous equations

$$\begin{cases} x = q_1 + \frac{p_2}{y} + \frac{r_3}{yz} + \frac{s_4}{yzu}, \\ y = q_2 + \frac{p_3}{z} + \frac{r_4}{zu} + \frac{s_5}{zuv}, \\ z = q_3 + \frac{p_4}{u} + \frac{r_5}{uv} + \frac{s_1}{uvx}, \\ u = q_4 + \frac{p_5}{v} + \frac{r_1}{vx} + \frac{s_2}{vxy}, \\ v = q_5 + \frac{p_1}{x} + \frac{r_2}{xy} + \frac{s_3}{xyz}, \end{cases} \tag{2.2}$$

for the 5-periodic recurrence fraction

$$\left[\begin{array}{c|cccccc} q_1 & & & & & & \\ \frac{p_2}{q_2} & q_2 & & & & & \\ \frac{r_3}{p_3} & \frac{p_3}{q_3} & q_3 & & & & \\ \frac{s_4}{r_4} & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 & & & \\ 0 & \frac{s_5}{r_5} & \frac{r_5}{p_5} & \frac{p_5}{q_5} & q_5 & & \\ 0 & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & \\ \vdots & \dots & \dots & \dots & \dots & \dots & \ddots \end{array}\right], \tag{2.3}$$

where

$$u = \lim_{n\to\infty}\frac{[q_4]_{n-3}}{[q_5]_{n-4}},\ v = \lim_{n\to\infty}\frac{[q_5]_{n-4}}{[q_1]_{n-5}}$$

(see [5], p.2 on page 13).

Thus, the theorem is true.

Theorem 2.2. *Let $l = 5,\ k = 5$, and $q_j^*,\ p_j^*,\ r_j^*,\ s_j^*,\ q_i,\ p_i,\ r_i,\ s_i > 0$, and the limits*

$$lim_{n\to\infty}\frac{[q_0^*]_{n+4}}{[q_1^*]_{n+3}} = x^*,\ lim_{n\to\infty}\frac{[q_1]_n}{[q_2]_{n-1}} = x,\ lim_{n\to\infty}\frac{[q_2]_{n-1}}{[q_3]_{n-2}} = y,\ lim_{n\to\infty}\frac{[q_3]_{n-2}}{[q_4]_{n-3}} = z.$$

Then

$$x^* = \frac{[q_0^*]_5 xyz + ([q_0^*]_4 p_1 + [q_0^*]_3 r_1 + [q_0^*]_2 s_1)yz + ([q_0^*]_4 r_2 + [q_0^*]_3 s_2)z + [q_0^*]_4 s_3}{[q_1^*]_4 xyz + ([q_1^*]_3 p_1 + [q_1^*]_2 r_1 + [q_1^*]_1 s_1)yz + ([q_1^*]_3 r_2 + [q_1^*]_2 s_2)z + [q_1^*]_3 s_3},$$

where x, y, z are solutions of the simultaneous equations (2.2) for the 5-periodic recurrence fraction (2.3).

Example 2.3. Let

$$q_0^* = 3, q_1^* = 2, q_2^* = 3, q_3^* = 2, q_4^* = 3, p_1^* = 4, p_2^* = 3,$$
$$p_3^* = 4, p_4^* = 3, r_2^* = 2, r_3^* = 5, r_4^* = 2, s_3^* = 1, s_4^* = 1,$$
$$q_1 = 1, p_2 = 1, r_3 = 1, s_4 = 1, q_2 = 1, p_3 = 1, r_4 = 1, s_5 = 1, q_3 = 2, p_4 = 2,$$
$$r_5 = 2, s_1 = 2, q_4 = 1, p_5 = 1, r_1 = 1, s_2 = 1, q_5 = 2, p_1 = 2, r_2 = 2, s_3 = 2,$$

then the mixed recurrence fraction is written as

$$\left[\begin{array}{c|cccccccccc}
3 & & & & & & & & & & \\
\frac{4}{2} & 2 & & & & & & & & & \\
\frac{2}{3} & \frac{3}{3} & 3 & & & & & & & & \\
\frac{1}{5} & \frac{5}{4} & \frac{4}{2} & 2 & & & & & & & \\
0 & \frac{1}{2} & \frac{2}{3} & \frac{3}{3} & 3 & & & & & & \\
0 & 0 & \frac{2}{1} & \frac{1}{2} & \frac{2}{1} & 1 & & & & & \\
0 & 0 & 0 & \frac{1}{2} & \frac{2}{1} & \frac{1}{1} & 1 & & & & \\
0 & 0 & 0 & 0 & \frac{2}{1} & \frac{1}{1} & \frac{1}{2} & 2 & & & \\
0 & 0 & 0 & 0 & 0 & \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{2}{1} & \frac{1}{2} & 2 & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{array}\right]_\infty$$

The value of the mixed periodic recurrence fraction converges to the number

$$x^* = \frac{560xyz + 337yz + 317z + 276}{125xyz + 75yz + 71z + 62},$$

where x is the root of the equation modulo maximum

$$9x^4 - 9x^3 - 9x^2 - 8x - 16 = 0,$$

such that

$$x = \frac{1}{4} + \frac{1}{2}\left(\sqrt{-\frac{11}{8} + 2v} + \sqrt{\frac{33}{8} - 2v + \frac{\frac{109}{36}}{\sqrt{-\frac{11}{8} + 2v}}}\right) \approx 1.969558741906025,$$

where

$$v = \frac{1}{2}\left(\frac{55}{24} + \frac{1}{9}\sqrt[3]{-2007 + 144\sqrt{622}} - \frac{23}{\sqrt[3]{-2007 + 144\sqrt{622}}}\right),$$

and

$$y = \frac{4 - x}{3x^3 - 3x^2 - 3x - 4},$$
$$z = \frac{4 - x}{3x^2y - 3xy - 4y + x - 4}.$$

Thus,

$$x^* \approx 4.47989948650800763333.$$

Let us find rational shortenings of this fraction; we shall get

$$\delta_1 = \frac{10}{2} = 5, \delta_2 = \frac{41}{9} = 4,56, \delta_3 = \frac{138}{31} = 4,452, \delta_4 = \frac{560}{125} \approx 4,48, \delta_5 = \frac{897}{200} \approx 4,485,$$

$$\delta_6 = \frac{1774}{396} \approx 4,479798, \delta_7 = \frac{5281}{1179} \approx 4,47922, \delta_8 = \frac{10286}{2296} \approx 4,47997, \delta_9 = \frac{30298}{6763} \approx 4,47996,$$

$$\delta_{10} = \frac{59699}{13326} \approx 4,479889, \delta_{11} = \frac{115850}{25860} \approx 4,4798917, \delta_{12} = \frac{342269}{76401} \approx 4,479902,$$

$$\delta_{13} = \frac{663966}{148210} \approx 4,4799001, \delta_{14} = \frac{1961600}{437867} \approx 4,47989915, \delta_{15} = \frac{3863501}{862408} \approx 4,47989931,$$

$$\delta_{16} = \frac{7495302}{1673096} \approx 4,47989954, \delta_{17} = \frac{22143637}{4942887} \approx 4,47989950003,$$

$$\delta_{18} = \frac{42959342}{9589354} \approx 4,4798994802.$$

3. Algorithm for Calculation of Rational Shortenings of the Fourth-Order Mixed Periodic Recurrence Fraction

Let us construct the algorithm for calculation of rational shortenings of the fourth-order mixed periodic recurrence fractions, which is much more practical than the algorithm described in the previous section.

Let n be the order of the parameter of its rational shortening, and $n = sk + l,\ s = 1, 2, 3, \ldots$. Then the following theorem is true.

Theorem 3.1. *The rational shortening*

$$\delta_n^* = \frac{P_n^*}{Q_n^*}$$

of the fourth-order mixed periodic recurrence fraction (2.1), with the period of $k \geqslant 2$, equals the value of the expression

$$q_0^* + p_1^* \cdot \frac{B}{A} + r_2^* \cdot \frac{C}{A} + s_3^* \cdot \frac{D}{A},$$

where A , B , C and D are defined by the recurrence equalities

$$A = s_3\alpha_{l-1}D_{ks-3}^{s-1} + (s_2\alpha_{l-2} + r_2\alpha_{l-1})C_{ks-2}^{s-1} + (s_1\alpha_{l-3} + r_1\alpha_{l-2} + p_1\alpha_{l-1})B_{ks-1}^{s-1} + \alpha_l A_{ks}^s, \quad (3.1)$$

$$B = s_3\beta_{l-2}D_{ks-3}^{s-1} + (s_2\beta_{l-3} + r_2\beta_{l-2})C_{ks-2}^{s-1} + (s_1\beta_{l-4} + r_1\beta_{l-3} + p_1\beta_{l-2})B_{ks-1}^{s-1} + \beta_{l-1} A_{ks}^s, \quad (3.2)$$

$$C = s_3\gamma_{l-3}D_{ks-3}^{s-1} + (s_2\gamma_{l-4} + r_2\gamma_{l-3})C_{ks-2}^{s-1} + (s_1\gamma_{l-5} + r_1\gamma_{l-4} + p_1\gamma_{l-3})B_{ks-1}^{s-1} + \gamma_{l-2} A_{ks}^s, \quad (3.3)$$

$$D = s_3\eta_{l-4}D_{ks-3}^{s-1} + (s_2\eta_{l-5} + r_2\eta_{l-4})C_{ks-2}^{s-1} + (s_1\eta_{l-6} + r_1\eta_{l-5} + p_1\eta_{l-4})B_{ks-1}^{s-1} + \eta_{l-3} A_{ks}^s, \quad (3.4)$$

where

$$\alpha_l = \begin{bmatrix} q_1^* &&&&&&&&& \\ \frac{p_2^*}{q_2^*} & q_2^* &&&&&&&& \\ \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* &&&&&&& \\ \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_{4^*} &&&&&& \\ 0 & \frac{s_5^*}{r_5^*} & \frac{r_5^*}{p_5^*} & \frac{p_5^*}{q_5^*} & q_{5^*} &&&&& \\ \vdots & \dots & \dots & \dots & \dots & \ddots &&&& \\ 0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* &&& \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* && \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \end{bmatrix}, \tag{3.5}$$

$$\beta_{l-1} = \begin{bmatrix} q_2^* &&&&&&&&& \\ \frac{p_3^*}{q_3^*} & q_3^* &&&&&&&& \\ \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* &&&&&&& \\ \frac{s_5^*}{r_5^*} & \frac{r_5^*}{p_5^*} & \frac{p_5^*}{q_5^*} & q_{5^*} &&&&&& \\ 0 & \frac{s_6^*}{r_6^*} & \frac{r_6^*}{p_6^*} & \frac{p_6^*}{q_6^*} & q_{6^*} &&&&& \\ \vdots & \dots & \dots & \dots & \dots & \ddots &&&& \\ 0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* &&& \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* && \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \end{bmatrix}, \tag{3.6}$$

$$\gamma_{l-2} = \begin{bmatrix} q_3^* &&&&&&&&& \\ \frac{p_4^*}{q_4^*} & q_4^* &&&&&&&& \\ \frac{r_5^*}{p_5^*} & \frac{p_5^*}{q_5^*} & q_5^* &&&&&&& \\ \frac{s_6^*}{r_6^*} & \frac{r_6^*}{p_6^*} & \frac{p_6^*}{q_6^*} & q_{6^*} &&&&&& \\ 0 & \frac{s_7^*}{r_7^*} & \frac{r_7^*}{p_7^*} & \frac{p_7^*}{q_7^*} & q_{7^*} &&&&& \\ \vdots & \dots & \dots & \dots & \dots & \ddots &&&& \\ 0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* &&& \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* && \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \end{bmatrix}, \tag{3.7}$$

$$\eta_{l-3}=\begin{bmatrix}
q_4^* &&&&&&&&& \\
\frac{p_5^*}{q_5^*} & q_5^* &&&&&&&& \\
\frac{r_6^*}{p_6^*} & \frac{p_6^*}{q_6^*} & q_6^* &&&&&&& \\
\frac{s_7^*}{r_7^*} & \frac{r_7^*}{p_7^*} & \frac{p_7^*}{q_7^*} & q_{7^*} &&&&&& \\
0 & \frac{s_8^*}{r_8^*} & \frac{r_8^*}{p_8^*} & \frac{p_8^*}{q_8^*} & q_{8^*} &&&&& \\
\vdots & \dots & \dots & \dots & \dots & \ddots &&&& \\
0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* &&& \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* && \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^*
\end{bmatrix}, \tag{3.8}$$

and A_{sk}^s, B_{sk-1}^{s-1}, C_{sk-2}^{s-1}, D_{sk-3}^{s-1} *are respectively defined by the recurrences (9'), (10'), (11'), (12'). If* $k=2,3,4$, *we believe* $\xi_{<0}=\tau_{<0}=\psi_{<0}=\varphi_{<0}=0$, $\varphi_0=\psi_0=\tau_0=\xi_0=1$. *Likewise, if* $l=2,3,4$, *we consider* $\alpha_{<0}=\beta_{<0}=\gamma_{<0}=\eta_{<0}=0$, $\alpha_0=\beta_0=\gamma_0=\eta_0=1$.

Proof. For the fraction (2.1), the parapermanents P_{sk}^* and Q_{sk}^* are written as

$$P_{sk}^*=\begin{bmatrix}
q_0^* \\
\frac{p_1^*}{q_1^*} & q_1^* \\
\frac{r_2^*}{p_2^*} & \frac{p_2^*}{q_2^*} & q_2^* \\
\frac{s_3^*}{r_3^*} & \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* \\
0 & \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* \\
\vdots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & q_{k-3} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{k-2}}{q_{k-2}} & q_{k-2} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{k-1}}{p_{k-1}} & \frac{p_{k-1}}{q_{k-1}} & q_{k-1} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \frac{s_k}{r_k} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & q_k
\end{bmatrix}, \tag{3.9}$$

$$Q_{sk}^* = \begin{bmatrix}
q_1^* \\
\frac{p_2^*}{q_2^*} & q_2^* \\
\frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* \\
\frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* \\
0 & \frac{s_5^*}{r_5^*} & \frac{r_5^*}{p_5^*} & \frac{p_5^*}{q_5^*} & q_5^* \\
\vdots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots q_{l-3}^* \\
0 & 0 & 0 & 0 & 0 & \dots \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* \\
0 & 0 & 0 & 0 & 0 & \dots \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* \\
0 & 0 & 0 & 0 & 0 & \dots \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \\
0 & 0 & 0 & 0 & 0 & \dots 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & \dots q_{k-3} \\
0 & 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & \dots \frac{p_{k-2}}{q_{k-2}} & q_{k-2} \\
0 & 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & \dots \frac{r_{k-1}}{p_{k-1}} & \frac{p_{k-1}}{q_{k-1}} & q_{k-1} \\
0 & 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & \dots \frac{s_k}{r_k} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k \\
0 & 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & \dots 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & \dots q_k
\end{bmatrix}.$$

We shall denote the parapermanent, derived from the parapermanent (3.9) as a result of deleting the first column, by A, the parapermanent, derived as a result of deleting the first two columns, — by B, the parapermanent, derived as a result of deleting the first three columns, — by C, and the parapermanent, derived as a result of deleting the first four columns, — by D. Then, decomposing the first parapermanent by the elements of the first column, we shall get the equality $P_{sk}^* = q_0^* A + p_1^* B + r_2^* C + s_3^* D$, which, by taking the equality $Q_{sk}^* = A$ into account, will lead to the equality $\delta_n^* = q_0^* + p_1^* \frac{B}{A} + r_2^* \frac{C}{A} + s_3^* \frac{D}{A}$.

We shall denote the parapermanent, derived from the parapermanent (3.9) as a result of deleting the first $(l+1)$ columns, by A_{sk}^s (it contains s periods of the fraction). The parapermanent, derived from the parapermanent A_{sk}^s as a result of deleting the first column, is denoted by B_{sk-1}^{s-1}. The parapermanent, derived from the parapermanent A_{sk}^s as a result of deleting the first two columns, is denoted by C_{sk-2}^{s-1}, and the parapermanent, derived from the parapermanent A_{sk}^s as a result of deleting the first three columns, — by D_{sk-3}^{s-1}. We shall decompose the parapermanent A by the elements of the inscribed rectangular matrix $T(l+1)$. Taking (3.5) into account and applying the theorem on incomplete decomposition of a parapermanent by the elements of the $(l+1)$-th column, we obtain the equality (3.1). We shall make similar transformations with the parapermanent B. We decompose it by the elements of the table $T(l)$, taking (3.6) into account, at that we get the equality (3.2). We shall also make similar transformations with the parapermanents C, D. We decompose them by the elements of the tables $T(l-1), T(l-2)$ respectively and take (3.7), (3.8) into account. At that, we get the equalities (3.3), (3.4).

We shall decompose the parapermanent A_{sk}^{s} by the elements of the inscribed rectangular table $T(k+1)$, then we get the recurrence (9′). Likewise, we shall deal with the parapermanents B_{sk-1}^{s-1}, C_{sk-2}^{s-1} and D_{sk-3}^{s-1}, decomposing them by the elements of the tables $T(k)$, $T(k-1)$ and $T(k-2)$ respectively. At that, we get the recurrences (10′), (11′), (12′). □

Example 3.2. Let us have the fourth-order mixed 5-periodic recurrence fraction from Example 2.3.

Let us find its rational shortenings applying Theorem *3.1.*

We shall find

$$\eta_{-2},\eta_{-1},\eta_0,\eta_1,\gamma_{-1},\gamma_0,\gamma_1,\gamma_2,\beta_0,\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2,\alpha_3,\alpha_4,$$
$$\xi_{-1},\xi_0,\xi_1,\xi_2,\tau_0,\tau_1,\tau_2,\tau_3,\psi_1,\psi_2,\psi_3,\psi_4,\varphi_2,\varphi_3,\varphi_4,\varphi_5$$

from the equalities *(3.5), (3.6), (3.7), (3.8), (13′), (14′), (15′), (16′):*

$$\eta_{-2}=0,\eta_{-1}=0,\eta_0=1,\eta_1=3,\gamma_{-1}=0,\gamma_0=1,\gamma_1=2,\gamma_2=\left[\begin{array}{cc} 2 & \\ \frac{3}{3} & 3 \end{array}\right]=9,$$

$$\beta_0=1,\quad \beta_1=3,\quad \beta_2=\left[\begin{array}{cc} 3 & \\ \frac{4}{2} & 2 \end{array}\right]=10,\quad \beta_3=\left[\begin{array}{ccc} 3 & & \\ \frac{4}{2} & 2 & \\ \frac{2}{3} & \frac{3}{3} & 3 \end{array}\right]=41,$$

$$\alpha_1=2,\ \alpha_2=\left[\begin{array}{cc} 2 & \\ \frac{3}{3} & 3 \end{array}\right]=9,\ \alpha_3=\left[\begin{array}{ccc} 2 & & \\ \frac{3}{3} & 3 & \\ \frac{5}{4} & \frac{4}{2} & 2 \end{array}\right]=31,\ \alpha_4=\left[\begin{array}{cccc} 2 & & & \\ \frac{3}{3} & 3 & & \\ \frac{5}{4} & \frac{4}{2} & 2 & \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{3} & 3 \end{array}\right]=125,$$

$$\xi_{-1}=0,\quad \xi_0=1,\quad \xi_1=1,\quad \xi_2=\left[\begin{array}{cc} 1 & \\ \frac{1}{2} & 2 \end{array}\right]=4,$$

$$\tau_0=1,\quad \tau_1=2,\quad \tau_2=\left[\begin{array}{cc} 2 & \\ \frac{2}{1} & 1 \end{array}\right]=4,\quad \tau_3=\left[\begin{array}{ccc} 2 & & \\ \frac{2}{1} & 1 & \\ \frac{2}{1} & \frac{1}{2} & 2 \end{array}\right]=12,$$

$$\psi_1=1,\quad \psi_2=\left[\begin{array}{cc} 1 & \\ \frac{1}{2} & 2 \end{array}\right]=3,\quad \psi_3=\left[\begin{array}{ccc} 1 & & \\ \frac{1}{2} & 2 & \\ \frac{1}{2} & \frac{2}{1} & 1 \end{array}\right]=6,$$

$$\psi_4=\left[\begin{array}{cccc} 1 & & & \\ \frac{1}{2} & 2 & & \\ \frac{1}{2} & \frac{2}{1} & 1 & \\ \frac{1}{2} & \frac{2}{1} & \frac{1}{2} & 2 \end{array}\right]=11,\quad \varphi_2=\left[\begin{array}{cc} 1 & \\ \frac{1}{1} & 1 \end{array}\right]=2,\quad \varphi_3=\left[\begin{array}{ccc} 1 & & \\ \frac{1}{1} & 1 & \\ \frac{1}{1} & \frac{1}{2} & 2 \end{array}\right]=6,$$

$$\varphi_4=\left[\begin{array}{cccc} 1 & & & \\ \frac{1}{1} & 1 & & \\ \frac{1}{1} & \frac{1}{2} & 2 & \\ \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & 1 \end{array}\right]=12,\quad \varphi_5=\left[\begin{array}{ccccc} 1 & & & & \\ \frac{1}{1} & 1 & & & \\ \frac{1}{1} & \frac{1}{2} & 2 & & \\ \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & 1 & \\ 0 & \frac{1}{2} & \frac{2}{1} & \frac{1}{2} & 2 \end{array}\right]=35.$$

Thus, the recurrences (3.1), (3.2), (3.3), (3.4), (9′), (10′), (11′), (12′) will be written respectively as

$$A=62D_{5s-3}^{s-1}+71C_{5s-2}^{s-1}+75B_{5s-1}^{s-1}+125A_{5s}^{s},$$

$$B = 20D^{s-1}_{5s-3} + 23C^{s-1}_{5s-2} + 25B^{s-1}_{5s-1} + 41A^{s}_{5s},$$

$$C = 4D^{s-1}_{5s-3} + 5C^{s-1}_{5s-2} + 5B^{s-1}_{5s-1} + 9A^{s}_{5s},$$

$$D = 2D^{s-1}_{5s-3} + 2C^{s-1}_{5s-2} + 2B^{s-1}_{5s-1} + 3A^{s}_{5s},$$

$$A^{s}_{5s} = 24D^{s-2}_{5s-8} + 30C^{s-2}_{5s-7} + 34B^{s-2}_{5s-6} + 35A^{s-1}_{5s-5},$$

$$B^{s-1}_{5s-1} = 12D^{s-2}_{5s-8} + 15C^{s-2}_{5s-7} + 17B^{s-2}_{5s-6} + 18A^{s-1}_{5s-5},$$

$$C^{s-1}_{5s-2} = 8D^{s-2}_{5s-8} + 10C^{s-2}_{5s-7} + 12B^{s-2}_{5s-6} + 12A^{s-1}_{5s-5},$$

$$D^{s-1}_{5s-3} = 2D^{s-2}_{5s-8} + 3C^{s-2}_{5s-7} + 3B^{s-2}_{5s-6} + 3A^{s-1}_{5s-5}.$$

s-th convergence to the value of the given recurrence fraction by the algorithm of Theorem 3.1 is written as

$$\gamma_s = 3 + 4\frac{B}{A} + 2\frac{C}{A} + \frac{D}{A}.$$

Since

$$D^0_2 = \xi_2 = 3, \quad C^0_3 = \tau_3 = 12, \quad B^0_4 = \psi_4 = 18, \quad A^1_5 = \varphi_5 = 35,$$

then

$$\gamma_1 = \frac{59699}{13326} \approx 4,479889, \ \gamma_2 = \frac{3863501}{862408} \approx 4,47989931, \ \gamma_3 = \frac{1780047}{658189} \approx 2.70446179,$$

$$\gamma_4 = \frac{98034217}{36249057} \approx 2.70446180035, \ \gamma_5 = \frac{5399131407}{1996379245} \approx 2.70446180029286,$$

$$\gamma_6 = \frac{297351484441}{109948487499} \approx 2.704461800292654.$$

Thus, s-th convergence γ_s, found with the help of the algorithm of Theorem 3.1 coincides with the $(5s+l)$–th convergence δ_{5s+l}, found with the help of established recurrences.

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У роботі вивчаються змішані періодичні рекурентні дроби четвертого порядку. Запропоновано алгоритми їх обчислення і встановлено звязки з відповідними алгебраїчними рівняннями четвертого порядку.

Ключові слова: парафункції трикутних матриць, рекурентні дроби, алгебраїчні рівняння.