Journal of Vasyl Stefanyk Precarpathian National University
http://jpnu.pu.if.ua
Vol. 2, No. 4 (2015), 50-57

UDC 517.988.6,517.988.8
MSC 2010: 47J05, 47J25
doi: 10.15330/jpnu.2.4.50-57

CONVERGENCE INVESTIGATION OF ITERATIVE AGGREGATION METHODS FOR LINEAR EQUATIONS IN A BANACH SPACE

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Abstract. The sufficient conditions of convergence for a class of multi-parameter iterative aggregation methods are established. These conditions do not contain the requirements of positivity for the operators and aggregating functionals. Moreover, it is not necessary that the corresponding linear continuous operators are compressing.

Keywords: decomposition, parallelization computation methods, iterative aggregation.

1. INTRODUCTION

Problems of the operator equations decomposition are still actual. It is caused by the necessity of construction parallelization computation methods. Multi-parameter iterative aggregation is an effective method for decomposition of the high dimension problems (see [1]).

Let E be a Banach space and $A : E \longmapsto E$ be a linear continuous operator. Consider the equation

$$x = Ax + b, \qquad b \in E. \tag{1.1}$$

For such equations often it is assumed that: 1) the normal cone $K \subset E$ of positive elements is given; 2) semiordering in E is introduced by such elements; 3) compression operator A and element b are positive (see, for example, [2]–[4]). These and other requirements are caused by the specificity of the corresponding problems (see, for example, [5]–[10]). More detailed results for one-parametric method are given in [2, p. 155–158] and can be described by the formula

$$x^{(n+1)} = \frac{(\varphi, b)}{(\varphi, x^{(n)} - Ax^{(n)})} Ax^{(n)} + b \qquad (n = 0, 1, \dots). \tag{1.2}$$

Here (φ, x) denotes the value of a linear functional $\varphi \in K^*$ on the elements $x \in E$, where K^* is a cone of positive elements in a dual Banach space E^*. The algorithm (1.2) is investigated in [2, p. 155–158] with the following assumptions: (i) A is a focusing operator [2, p. 77]; (ii) spectral radius $\rho(A)$ of the operator A is less than one; (iii) the functional φ is admissible. A functional φ is called an admissible if there exists a functional $g \in K^*$ such that $\varphi = A^* g$ and $(g, x) > (\varphi, x)$ for $x \in K$, $x \neq \Theta$, where A^* is conjugate to A operator and Θ is zero element in E.

In particular, if (1.1) is a system of linear algebraic equations with a matrix $A = \{a_{ij}\}$, then the focusing condition is valid when all a_{ij} are strictly positive numbers. For the linear integral operator of the following form

$$Ax = \int_a^b G(t,s)x(s)ds$$

the focusing condition is valid if the continuous function $G(t,s)$ satisfies the condition $G(t,s) \geqslant \varepsilon > 0$ for $t, s \in [a,b]$.

In [2, p. 158] it is noted that the theory of methods for iterative aggregation is not well developed and the conditions of their convergence are unknown. In particular, as it is indicated by numerous examples (see [2, p. 158]), one parametric method (1.2) often converges when the above conditions are not fulfilled.

In this work we investigate the multi-parameter algorithms of iterative aggregation using the methodology described in [11]–[15]. The established sufficient conditions of convergence do not contain the requirement of type $\rho(A) < 1$ for a spectral radius $\rho(A)$ of an operator A and condition of signs constancy for the operator A and of the aggregating functionals.

2. Construction of the Aggregative-Iterative Algorithm

We consider the equation (1.1) in a Banach space E. We do not need semiordering in E. Let the equation (1.1) is presented in the form

$$x = \sum_{j=1}^{N} A_j x + A_0 x + b, \tag{2.1}$$

where $A_0 : E \to E$, $A_j : E \to E$ $(j = 1, \ldots, N)$, $b \in E$. Set the linear continuous functionals $\varphi^{(i)}$ $(i = 0, 1, \ldots, N)$. Let us join to the equation (2.1) the auxiliary system of equations

$$y_i = \sum_{j=1}^{N} \lambda_{ij} y_j + (\varphi^{(i)}, B_i x) - (\varphi^{(i)}, b) \qquad (i = 0, 1, \ldots, N). \tag{2.2}$$

Our basic assumptions are the following.

A) The equalities

$$(\varphi^{(i)}, A_j x) = \lambda_{ij} (\varphi^{(j)}, x) \qquad (j = 1, \ldots, N;\ i = 0, 1, \ldots, N) \tag{2.3}$$

hold.

B) There exists the operators $B_i : E \to E$ $(i = 0, 1, \ldots, N)$ such that

$$(\varphi^{(i)}, A_0 x + B_i x) = \lambda_{i0} (\varphi^{(0)}, x) \qquad (i = 0, 1, \ldots, N). \tag{2.4}$$

Let us construct the iterative process by the formulas

$$x^{(n+1)} = \sum_{j=1}^{N} A_j x^{(n)} + A_0 x^{(n)} + \sum_{j=1}^{N} a_j^{(n)} (y_j^{(n)} - y_j^{(n+1)}) + a_0^{(n)} (y_0^{(n)} - y_0^{(n+1)}) + b, \tag{2.5}$$

$$\begin{aligned} y_i^{(n+1)} &= \sum_{j=1}^{N} \lambda_{ij} y_j^{(n+1)} + \lambda_{i0} y_0^{(n+1)} + (\varphi^{(i)}, B_i x^{(n)}) + \sum_{j=1}^{N} \alpha_{ij}^{(n)} (y_j^{(n)} - y_j^{(n+1)}) \\ &+ \alpha_{i0}^{(n)} (y_0^{(n)} - y_0^{(n+1)}) - (\varphi^{(i)}, b) \qquad (i = 0, 1, \ldots, N), \end{aligned} \tag{2.6}$$

where the elements $a_j^{(n)} = a_j(x^{(n)}) \in E$ and real numbers $\alpha_{ij}^{(n)} = \alpha_{ij}(x^{(n)})$ satisfy the condition:

$$(\varphi^{(i)}, a_j(x)) + \lambda_{ij}(x) = \lambda_{ij} \qquad (x \in E,\ i, j = 0, 1, \ldots, N) \tag{2.7}$$

where λ_{ij} are real numbers and $a_j(x)$, $\alpha_{ij}(x)$ are continuous functions for $x \in E$.

3. Main Lemma

Let E' be an Euclid space of dimension $N+1$. Consider the set of elements $x \in E$ and vectors $y = \{y_0, y_1, \ldots, y_N\}^T \in E'$, such that the equalities

$$(\varphi^{(i)}, x) + y_i = 0 \qquad (i = 0, 1, \ldots, N) \tag{3.1}$$

hold. Denote this set by ε_0. It is clear that ε_0 is a subspace of the space $\widetilde{E} = E \times E'$ equipped by the norm

$$\|(x, y)\| = \sqrt{\|x\|^2 + |y|^2},$$

where $\|x\|$ is the norm of an element $x \in E$ and $|y|$ is the Euclidean norm of a vector $y \in E'$.

Lemma 3.1. *Let the conditions A) and B) be satisfied. Let the matrix $I - \Lambda$ be nondegenerate, where I is the unit matrix in E' and*

$$\Lambda = \{\lambda_{ij}\} \qquad (i, j = 0, 1, \ldots, N). \tag{3.2}$$

Then solution $\{x^*, y^*\} \in \widetilde{E}$ of the system (2.1), (2.2) *belongs to ε_0, i.e. $\{x^*, y^*\} \in \varepsilon_0$.*

Proof. From the formulas (2.1)–(2.4) for $x = x^*$, $y_i = y_i^*$ we have

$$\begin{aligned} (\varphi^{(i)}, x^*) + y_i^* &= \sum_{j=1}^{N} (\varphi^{(i)}, A_j x^*) + (\varphi^{(i)}, A_0 x^*) + (\varphi^{(i)}, b) \\ &+ \sum_{j=1}^{N} \lambda_{ij} y_j^* + (\varphi^{(i)}, B_i x^*) - (\varphi^{(i)}, b) \\ &= \sum_{j=1}^{N} \lambda_{ij} \left[(\varphi^{(j)}, x^*) + y_j^* \right] + \left[(\varphi^{(i)}, A_0 x^* + B_i x^*) + \lambda_{i0} y_0^* \right] \\ &= \sum_{j=1}^{N} \lambda_{ij} \left[(\varphi^{(j)}, x^*) + y_j^* \right] + \lambda_{i0} \left[(\varphi^{(0)}, x^*) + y_0^* \right] \qquad (i = 0, 1, \ldots, N). \end{aligned}$$

Note, that the matrix $I - \Lambda$ is nondegenerate, so the lemma is proved.

Lemma 3.2. *Let the conditions A), B) and* (2.7) *be satisfied. If the matrix $I-\Lambda$ is nondegenerate and $\{x^{(0)}, y^{(0)}\} \in \varepsilon_0$, then $\{x^{(n)}, y^{(n)}\} \in \varepsilon_0$ for $n = 0, 1, \ldots$*

Proof. From the equalities (2.5)–(2.7) we have

$$\begin{aligned}
(\varphi^{(i)}, x^{(n+1)}) + y_i^{(n+1)} &= \sum_{j=1}^{N}(\varphi^{(i)}, A_j x^{(n)}) + (\varphi^{(i)}, A_0 x^{(n)}) + \sum_{j=1}^{N}(\varphi^{(i)}, a_j^{(n)})(y_j^{(n)} - y_j^{(n+1)}) \\
&+ (\varphi^{(i)}, a_0^{(n)})(y_0^{(n)} - y_0^{(n+1)}) + (\varphi^{(i)}, b) + \sum_{j=1}^{N}\lambda_{ij} y_j^{(n+1)} + \lambda_{i0} y_0^{(n+1)} + (\varphi^{(1)}, B_0 x^{(n)}) \\
&+ \sum_{j=1}^{N}\alpha_{ij}^{(n)}(y_j^{(n)} - y_j^{(n+1)}) + \alpha_{i0}^{(n)}(y_0^{(n)} - y_j^{(n+1)}) - (\varphi^{(i)}, b) = \sum_{j=1}^{N}\lambda_{ij}(\varphi^{(j)}, x^{(n)}) \\
&+ (\varphi^{(i)}, A_0 x^{(n)} + B_i x^{(n)}) + \left[(\varphi^{(i)}, a_0^{(n)}) + \alpha_{i0}^{(n)}\right] y_0^{(n)} + \sum_{j=1}^{N}\left[(\varphi^{(i)}, a_j^{(n)}) + \alpha_{ij}^{(n)}\right] y_j^{(n)} \\
&+ \left[\lambda_{i0} - (\varphi^{(i)}, a_0^{(n)}) + \alpha_{i0}^{(n)}\right] y_0^{(n+1)} + \left[\lambda_{ij} - (\varphi^{(i)}, a_j^{(n)}) - \alpha_{ij}^{(n)}\right] y_j^{(n+1)} \\
&= \sum_{j=1}^{N}\lambda_{ij}\left[(\varphi^{(j)}, x^{(n)}) + y_j^{(n)}\right] + \lambda_{i0}\left[(\varphi^{(0)}, x^{(n)}) + y_0^{(n)}\right] \qquad (i = 0, 1, \ldots, N).
\end{aligned} \tag{3.3}$$

Since $\{x^{(0)}, y^{(0)}\} \in \varepsilon_0$, equalities (3.3) are the reason for using of the induction principle. The proof is complete.

From these two lemmas we obtain the following assertion.

Lemma 3.3. *Let the conditions A), B) and* (2.7) *be satisfied. If there exists the matrix $(I-\Lambda)^{-1}$, $\{x^{(0)}, y^{(0)}\} \in \varepsilon_0$, and $\{x^*, y^*\}$ is the solution of the system* (2.1), (2.2) *in $\widetilde{E}$, then*

$$(\varphi^{(i)}, x^{(n)} - x^*) + y_i^{(n)} - y_i^* = 0 \qquad (i = 0, 1, \ldots, N;\ n = 0, 1, \ldots). \tag{3.4}$$

Proof. It is enough to note that (3.4) is a consequence of the equalities (3.1) for $\{x^{(0)}, y^{(0)}\}$ and $\{x^*, y^*\}$.

4. Sufficient Conditions for the Convergence of the Algorithm (2.5), (2.6)

Denote $a^{(n)} = \{a_0^{(n)}, a_1^{(n)}, \ldots, a_N^{(n)}\}$, $[\varphi, b] = \{(\varphi^{(0)}, b), (\varphi^{(1)}, b), \ldots, (\varphi^{(N)}, b)\}^T$, $[\varphi, Bx] = \{(\varphi^{(0)}, B_0 x), (\varphi^{(1)}, B_1 x), \ldots, (\varphi^{(N)}, B_N x)\}^T$. Let us rewrite the formulas (2.2), (2.5), (2.6) in the form

$$y = \Lambda y + [\varphi, Bx] - [\varphi, b], \tag{4.1}$$

$$x^{(n+1)} = A x^{(n)} + a^{(n)}(y^{(n)} - y^{(n+1)}) + b, \tag{4.2}$$

$$y^{(n+1)} = \Lambda y^{(n+1)} + [\varphi, B x^{(n)}] + \alpha^{(n)}(y^{(n)} - y^{(n+1)}) - [\varphi, b], \tag{4.3}$$

respectively, where the matrix Λ is defined by (3.2).

From the formulas (1.1), (4.1) and (4.2), (4.3) we obtain

$$x^{(n+1)} - x^* = A(x^{(n)} - x^*) + a^{(n)}(y^{(n)} - y^*) - a^{(n)}(y^{(n+1)} - y^*), \tag{4.4}$$

$$y^{(n+1)} - y^* = (\Lambda - \alpha^{(n)})(y^{(n+1)} - y^*) + \alpha^{(n)}(y^{(n)} - y^*) + [\varphi, B(x^{(n)} - x^*)]. \tag{4.5}$$

Hence

$$y^{(n+1)} - y^* = (I - \Lambda + \alpha^{(n)})^{-1}\alpha^{(n)}(y^{(n)} - y^*) + (I - \Lambda + \alpha^{(n)})^{-1}[\varphi, B(x^{(n)} - x^*)]. \tag{4.6}$$

This equality together with (4.4) imply the equality

$$\begin{aligned} x^{(n+1)} - x^* =& A(x^{(n)} - x^*) + a^{(n)}(y^{(n)} - y^*) - a^{(n)}(I - \Lambda + \alpha^{(n)})^{-1}\alpha^{(n)}(y^{(n)} - y^*) - \\ & - a^{(n)}(I - \Lambda + \alpha^{(n)})^{-1}[\varphi, B(x^{(n)} - x^*)]. \end{aligned} \tag{4.7}$$

Therefore, using (4.7) and (3.4), we obtain

$$x^{(n+1)} - x^* = A(x^{(n)} - x^*) - a^{(n)}(I - \Lambda + \alpha^{(n)})^{-1}((I - \Lambda)[\varphi, x^{(n)} - x^*] + [\varphi, B(x^{(n)} - x^*)]). \tag{4.8}$$

From (4.6), (4.8) it follows the next assertion.

Theorem 4.1. *Let the conditions of Lemma 3.3 be satisfied. Let the operator, generated by the right part of the equalities* (4.6), (4.7) *with respect to the pair* $\{x - x^*, y - y^*\}$ *with* $(x, y) \in \varepsilon_0$*, be compression. This means that the operator* $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ *is compression with respect to the pair* $\{w, z\}$*, where* $w \in E$*,* $z \in E'$*,* $\{w, z\} \in \varepsilon_0$*,*

$$h_{11}w = Aw + a(x)(I - \Lambda + \alpha(x))^{-1}[\varphi, Bw],$$

$$h_{12}z = a(x)(I - \Lambda + \alpha(x))^{-1}(I - \Lambda)z,$$

$$h_{21}w = (I - \Lambda + \alpha(x))^{-1}[\varphi, Bw],$$

$$h_{22}z = (I - \Lambda + \alpha(x))^{-1}\alpha(x)z.$$

Then a sequence $\{x^{(n)}\}$*, obtained by the algorithm* (4.2), (4.3) *converges to the solution* $x^* \in E$ *of the equation* (1.1) *not slower than a geometric progression with common ratio* $q < 1$*, where* q *is a norm of the operator* H *in the space* $\widetilde{E}$*.*

From the Theorem 4.1 we can get next proposition.

Theorem 4.2. *Let the conditions of Lemma 3.3 be satisfied. Define the operator* H_0 *by the formula*

$$H_0 w = Aw - a(x)(I - \Lambda + \alpha(x))^{-1}((I - \Lambda)[\varphi, w] + [\varphi, Bw]). \tag{4.9}$$

If for $(x, y) \in \varepsilon_0$ *the operator* H_0 *is compression with a compression constant* $q_0 < 1$*, then a sequence* $\{x^{(n)}\}$*, obtained by* (4.2), (4.3)*, converges to the solution* x^* *of the equation* (1.1) *not slower than a geometric progression with common ratio* q_0*.*

Proof. Rewrite the equalities (3.4) in the form

$$[\varphi, x^{(n)} - x^*] + (y^{(n)} - y^*)^T = \Theta, \tag{4.10}$$

where Θ is zero column vector. From (4.9), (4.10) we obtain that the theorem is proved.

5. Multi-Parameter Iterative Aggregation

Define elements $a_j(x)$ by the formula

$$a_j(x) = \frac{A_j x}{(\varphi^{(j)}, x)} \qquad (j = 0, 1, \ldots, N,\ x \in E). \tag{5.1}$$

In this case the algorithm (2.5), (2.6) can be defined by the interpolation formula

$$x^{(n+1)} = \sum_{j=1}^{N} \frac{(\varphi^{(j)}, x^{(n+1)})}{(\varphi^{(j)}, x^{(n)})} A_j x^{(n)} + b + \frac{(\varphi^{(0)}, x^{(n+1)})}{(\varphi^{(0)}, x^{(n)})}. \tag{5.2}$$

This algorithm is an analogue of the method (19.12), (19.13) from [2, p.156]. From the nondegeneracy of the matrices $I - \Lambda$, $I - \Lambda + \alpha(x)$ for $\{x, y\} \in \varepsilon_0$ $(x \in E,\ y \in E')$ it follows that we can choose the aggregation functionals $\varphi^{(i)}$, matrices $\Lambda = \{\lambda_{ij}\}$ and $\alpha(x) = \{\alpha_{ij}(x)\}$, which are used in described above methodology.

If $\alpha(x)$ is a zero matrix, then the algorithm (2.5), (2.6) does not converted to one of the projection-iterative methods, that are investigated in [16, 17].

It is also possible to construct other multi-parameter algorithms of iterative aggregation. For example,

$$x^{(n+1)} = A_0 x^{(n)} + \sum_{j=1}^{N} \frac{(\varphi^{(j)}, x^{(n+1)})}{(\varphi^{(j)}, x^{(n)})} A_j x^{(n)} + b. \tag{5.3}$$

Let us consider the case, when we use the formulas

$$x^{(n+1)} = \sum_{j=1}^{N} A_j x^{(n)} + \sum_{j=1}^{N} a_j^{(n)} (y_j^{(n)} - y_j^{(n+1)}) + A_0 x^{(n)} + b, \tag{5.4}$$

$$y^{(n+1)} = \Lambda y^{(n+1)} + [\varphi, B x^{(n)}] + \alpha^{(n)} (y^{(n)} - y^{(n+1)}) - [\varphi, A_0 x^{(n)}] - [\varphi, b] \tag{5.5}$$

instead of the formulas (2.5), (2.6) respectively.

Everywhere in the formulas (5.3)–(5.5) all indices i, j take values from 1 to N, i.e. $i \neq 0$ and $j \neq 0$.

For the algorithm (5.4), (5.5) we remain the structure of the matrix H and of the set ε_0. In this case we have

$$h_{11} w = Aw + a(x)(I - \Lambda + \alpha(x))^{-1} [\varphi, (B - A_0) w],$$

$$h_{12} z = a(x)(I - \Lambda + \alpha(x))^{-1} (I - \Lambda) z,$$

$$h_{21} w = (I - \Lambda + \alpha(x))^{-1} [\varphi, (B - A_0) w],$$

$$h_{22} z = (I - \Lambda + \alpha(x))^{-1} \alpha(x) z.$$

In these circumstances, the Theorem 4.2 is still valid for the algorithm (5.4), (5.5).

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Received: 12.05.2015; **revised:** 17.11.2015.

Копач М.І., Обшта А.Ф., Шувар Б.А. Дослідження збіжності методів ітеративного агрегування для лінійних рівнянь в банаховому просторі. *Журнал Прикарпатського університету імені Василя Стефаника,* **2** (4) (2015), 50-57.

У роботі встановленні достатні умови збіжності одного класу багатопараметричних агрегаційно-ітеративних методів. Отримані результати не містять вимог про додатність операторів і агрегуючих функціоналів, а також не потребують, щоб відповідні лінійні оператори були стискуючими.

Ключові слова: декомпозиція, розпаралелення обчислень, ітеративне агрегування.