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Entire functions of minimal growth with prescribed zeros

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Let l be a positive continuous increasing to $+\infty$ function on $\mathbb{R}$. For a positive non-decreasing on $\mathbb{R}$ function h, we found sufficient and necessary conditions under which, for an arbitrary complex sequence (ζ_n) such that $\zeta_n \to \infty$ as $n \to \infty$ and $\ln n(r) \geq l(\ln r)$ for all sufficiently large r, there exists an entire function f whose zeros are the ζ_n (with multiplicities taken into account) satisfying

$$\ln\ln M(r) = o\big(l^{-1}(\ln n(r)) \ln n_\zeta(r) h(\ln n(r))\big), \quad r \notin E,\ r \to +\infty,$$

where $E \subset [1, +\infty)$ is a set of finite logarithmic measure. Here, $n(r)$ is the counting function of the sequence (ζ_n), and $M(r)$ is the maximum modulus of the function f.

Key words and phrases: entire function, maximum modulus, Nevanlinna characteristic, zero, counting function.

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Introduction and results

Let $\mathcal{Z}$ be the class of all complex sequences $\zeta = (\zeta_n)$ such that $0 < |\zeta_1| \leq |\zeta_2| \leq \ldots$ and $\zeta_n \to \infty$ as $n \to \infty$. For any sequence $\zeta = (\zeta_n)$ belonging to $\mathcal{Z}$, by $\mathcal{E}_\zeta$ we denote the class of all entire functions whose zeros are precisely the ζ_n, where a complex number that occurs m times in the sequence ζ corresponds to a zero of multiplicity m, and, for every $r \geq 0$, let

$$n_\zeta(r) = \textstyle\sum_{|\zeta_n| \leq r} 1$$

be the counting function of this sequence.

If $x \in \mathbb{R}$, then we put $\exp_1(x) = \exp(x) = e^x$ and let $\exp_{n+1}(x) = \exp_n(e^x)$ for every integer $n \geq 1$. For all $x > 0$ we set $\ln_1 x = \ln x$, and let $\ln_{n+1} x = \ln_n \ln x$ for every integer $n \geq 1$ and any $x > \exp_n(0)$.

By H denote the class of all positive continuous non-decreasing functions on $\mathbb{R}$. If the integral $\int_{u_0}^\infty \frac{du}{\varphi(u)}$ converges (respectively, diverges) for some value of $u_0 \in \mathbb{R}$, then it converges (respectively, diverges) for every other value of $u_0 \in \mathbb{R}$, and in this case we write

$$\int^\infty \frac{du}{\varphi(u)} < +\infty, \tag{1}$$

$$\int^\infty \frac{du}{\varphi(u)} = +\infty, \tag{2}$$

respectively.

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For an arbitrary entire function f and each $r \geq 0$, we denote by $M_f(r)$ and $T_f(r)$ the maximum modulus and the Nevanlinna characteristic of the function f, respectively, i.e.

$$M_f(r) = \max\{|f(z)| : |z| = r\}, \qquad T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta.$$

W. Bergweiler [10], answering a question of A.A. Gol'dberg [19], proved the following two results.

Theorem A ([10]). *Let $\varphi \in H$ be a function for which* (1) *holds. Then for any sequence $\zeta \in \mathcal{Z}$ satisfying*

$$\varliminf_{r\to+\infty} \frac{\ln n_\zeta(r)}{\ln r} > 0, \tag{3}$$

there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1, +\infty)$ of finite logarithmic measure such that

$$\ln_2 M_f(r) = o(\ln^2 n_\zeta(r)\varphi(\ln_3 n_\zeta(r))), \quad r \notin E,\ r \to +\infty.$$

Theorem B ([10]). *Let $\varphi \in H$ be a function for which* (2) *holds and, moreover, we have*

$$\exists a > 0\ \exists u_0 > 0\ \forall u \geq u_0 : \quad \varphi(u) \leq u^a. \tag{4}$$

Then, for every $b > 0$, there exists a sequence $\zeta \in \mathcal{Z}$ satisfying

$$\varliminf_{r\to+\infty} \frac{\ln n_\zeta(r)}{\ln r} = b \tag{5}$$

such that, for any entire function $f \in \mathcal{E}_\zeta$, we have

$$\ln^2 n_\zeta(r)\varphi(\ln_3 n_\zeta(r)) = o(\ln_2 M_f(r)), \quad r \in F(f),\ r \to +\infty,$$

where $F(f) \subset [1, +\infty)$ is a set of infinite logarithmic measure.

W. Bergweiler [10] also suggested that condition (4) is probably not necessary in Theorem B. Note also that the function $\varphi \in H$ in Theorem B is considered fixed and therefore Theorem B does not indicate the dependence of the sequence ζ and the set $F(f)$ on φ. In addition to φ, the set $F(f)$ in Theorem B also depends, generally speaking, on f. This is indicated by the following result, a simple justification of which is also given in [10].

Theorem C ([10]). *Suppose that $\zeta \in \mathcal{Z}$ is a sequence satisfying* (3) *and $G \subset [1, +\infty)$ is an arbitrary unbounded set. Then there exist an entire function $f \in \mathcal{E}_\zeta$ and an unbounded sequence (r_k) of points from the set G such that*

$$\ln_2 M_f(r_k) = o(\ln^2 n_\zeta(r_k)), \quad k \to \infty.$$

This result, in some sense, cannot be improved.

Theorem D ([10]). *Let $b > 0$ and let ε be a function decreasing to* 0 *on* $\mathbb{R}$. *Then there exist a sequence $\zeta \in \mathcal{Z}$ satisfying* (5) *and a set $G \subset [1, +\infty)$ of upper logarithmic density* 1 *such that, for any function $f \in \mathcal{E}_\zeta$, we have*

$$\ln^2 n_\zeta(r)\varepsilon(\ln n_\zeta(r)) = o(\ln_2 M_f(r)), \quad r \in G,\ r \to +\infty.$$

By L we denote the class of all positive continuous increasing to $+\infty$ functions on $\mathbb{R}$. The following two results, which are proved in [4], generalize Theorems C and D to the case of an arbitrary growth for the function $n_\zeta(r)$.

Theorem E ([4]). *Let $l \in L$. Then for any sequence $\zeta \in \mathcal{Z}$ satisfying*

$$\ln n_\zeta(r) \ge l(\ln r), \quad r \ge r_0, \tag{6}$$

and for every unbounded set $G \subset [1, +\infty)$ there exist a function $f \in \mathcal{E}_\zeta$ and an unbounded sequence (r_k) of points from the set G such that

$$\ln_2 M_f(r_k) = o(l^{-1}(\ln n_\zeta(r_k)) \ln n_\zeta(r_k)), \quad k \to \infty.$$

Theorem F ([4]). *Let $l \in L$ and let ε be a positive function on $\mathbb{R}$ such that*

$$\varliminf_{x\to+\infty} \frac{\varepsilon(\ln[x])}{\ln x} = 0.$$

Then there exist a sequence $\zeta \in \mathcal{Z}$, for which (6) holds and $\ln n(r-0,\zeta) = l(\ln r)$ on an unbounded from above set of values r, and a set $G \subset [1, +\infty)$ of upper logarithmic density 1 such that, for any function $f \in \mathcal{E}_\zeta$, we have

$$l^{-1}(\ln n_\zeta(r))\varepsilon(\ln n_\zeta(r)) = o(\ln_2 M_f(r)), \quad r \in G,\ r \to +\infty.$$

An attempt to generalize Theorems A and B to the case of an arbitrary growth for the function $n_\zeta(r)$ was made in [2].

Theorem G ([2]). *Let $l \in L$. Then for any sequence $\zeta \in \mathcal{Z}$ satisfying (6) there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1, +\infty)$ of finite logarithmic measure such that, for every $\delta > 0$, we have*

$$\ln_2 M_f(r) = o(l^{-1}(\ln n_\zeta(r)) \ln^{1+\delta} n_\zeta(r)), \quad r \notin E,\ r \to +\infty.$$

Theorem H ([2]). *Let $l \in L$. Then there exists a sequence $\zeta \in \mathcal{Z}$ such that (6) holds, $\ln n_\zeta(r-0) = l(\ln r)$ on an unbounded from above set of values of r, and, for any function $f \in \mathcal{E}_\zeta$, we have*

$$l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r) = o(\ln_2 M_f(r)), \quad r \in F(f),\ r \to +\infty,$$

where $F(f) \subset [1, +\infty)$ is a set of infinite logarithmic measure.

For every entire function f and each $r \ge 0$ we have $T_f(r) \le \ln^+ M_f(r)$. Therefore, in Theorems A, C, E, and G, we can replace $\ln_2 M_f(r)$ by $\ln T_f(r)$. The same substitution is possible in Theorems B, D, F, and H, which is easy to see by analyzing the proofs of these results given in [2, 4, 10].

Theorems G and H are far from exact analogues of Theorems A and B in the case of arbitrary growth for the function $n_\zeta(r)$. In particular, Theorems A and B do not follow from Theorems G and H, respectively. However, Theorems G and H together with Theorems C and D allow us to make conclusions about the form of direct analogues of Theorems A and B in the case of arbitrary growth for the function $n_\zeta(r)$. In a certain sense, the following two results are such analogues.

Theorem 1. *Let $l \in L$ and let $\varphi \in H$ be a function satisfying* (1). *If for a function $\alpha \in L$ and a constant $c_1 > 0$ we have*

$$\alpha^{-1}(\alpha(x) + c_1) = O(x\alpha(x)l^{-1}(x)), \quad x \to +\infty, \tag{7}$$

then for any sequence $\zeta \in \mathcal{Z}$ satisfying (6) *there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1, +\infty)$ of finite logarithmic measure such that*

$$\ln_2 M_f(r) = o\left(l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r)\varphi(\alpha(\ln n_\zeta(r)))\right), \quad r \notin E,\ r \to +\infty. \tag{8}$$

Theorem 2. *Let $l \in L$. If for a function $\alpha \in L$ and a constant $c_2 > 0$ we have*

$$x\alpha(x)l^{-1}(x) = O(\alpha^{-1}(\alpha(x) + c_2)), \quad x \to +\infty, \tag{9}$$

and also

$$\varliminf_{y \to +\infty} \frac{l^{-1}(\alpha^{-1}(y + c_2))}{l^{-1}(\alpha^{-1}(y))} > 1, \tag{10}$$

then there exists a sequence $\zeta \in \mathcal{Z}$ such that (6) *holds, $\ln n_\zeta(r - 0) = l(\ln r)$ on an unbounded from above set of values of r, and, for every function $\varphi \in H$ satisfying* (2) *and any function $f \in \mathcal{E}_\zeta$, we have*

$$l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r)\varphi(\alpha(\ln n_\zeta(r))) = o(\ln T_f(r)), \quad r \in F(\varphi, f),\ r \to +\infty, \tag{11}$$

where $F(\varphi, f) \subset [1, +\infty)$ is a set of infinite logarithmic measure.

Let $b > 0$, $c > 0$, $l(u) = bu$ for all $u \geq u_0$, and $\alpha(x) = \ln_2 x$ for each $x \geq x_0$. Then, for all sufficiently large x, we get

$$\alpha^{-1}(\alpha(x) + c) = x^{\exp(c)}, \qquad x\alpha(x)l^{-1}(x) = x^2 \ln_2 x / b.$$

Therefore, if $c \leq \ln 2$, then condition (7) is satisfied and we obtain Theorem A from Theorem 1. If $c > \ln 2$, then conditions (9) and (10) are satisfied and we obtain Theorem B from Theorem 2; moreover, condition (4) in Theorem B turns out to be redundant and the sequence ζ can be chosen to be independent of φ.

Below we give some other consequences from Theorems 1 and 2.

We note that some related problems regarding the description of the minimal growth of entire functions with a given sequence of zeros $\zeta \in \mathcal{Z}$ have been considered in many works. In particular, by conditions from above on the growth of $n_\zeta(r)$, relations between $\ln M_f(r)$ and $n_\zeta(r)$, which describes the minimal growth of functions f from the class $\mathcal{E}_\zeta$ and hold along some increasing to $+\infty$ sequence of values of r, were established in [3, 9, 22]. Without any conditions on the growth of $n_\zeta(r)$, similar relations for $N_\zeta(r) = \int_0^r n_\zeta(t)d(\ln t)$ instead of $n_\zeta(r)$ were obtained in [1]. Analogues of Theorems E and F for $N_\zeta(r)$ instead of $n_\zeta(r)$ were established in [6].

For many relations between the characteristics of analytic functions, as well as for relation (8), it is typical that they are guaranteed to hold only outside certain exceptional sets. In this connection, M.M. Sheremeta [24] (see also [18]) established a relation between $\ln M_f(r)$ and $n_\zeta(r)$, which describes the minimal growth of functions f from the class $\mathcal{E}_\zeta$ and holds for all sufficiently large values of r (that is, without an exceptional set). A.A. Gol'dberg [19] proved that for an arbitrary sequence $\zeta \in \mathcal{Z}$ there exists an entire function $f \in \mathcal{E}_\zeta$ such that the relation $\ln_2 M_f(r) = o(N_\zeta(r))$ holds as $r \to +\infty$ outside an exceptional set of finite logarithmic measure. The existence of an exceptional set in this relation was justified in [1]; in addition, in [1], the estimate of this set obtained in [19] has been significantly improved. An exact estimate of an exceptional set in another relation of such kind was found in [5].

Finally, we mention other investigations related to establishing the existence of exceptional sets and obtaining exact estimates for these sets in relations between characteristics of analytic functions. In particular, such investigations were carried out for Wiman's asymptotic equality between the maximum modulus and the maximum of the real part of entire functions (see [8, 17, 25]), for the Borel relation between the maximum modulus and the maximal term of entire functions presented by power series (see [13, 15, 16]), for the lemma on the logarithmic derivative (see [14]), for inequalities of the Wiman-Valiron type or Kövari type between the maximum modulus and the maximal term of analytic functions presented by power series (see [25, 26]), for inequalities of the Wiman-Valiron type between the maximum modulus and the maximal term of multiple power series (see [12, 21]), for asymptotic equalities between the maximal term and the sum of entire Dirichlet series, multiple Dirichlet series or Taylor-Dirichlet series (see [7, 11, 27, 28]).

1 Auxiliary results

Let $z \in \mathbb{C}$, and let $p \geq 0$ be an integer. By $E(z,p)$ we denote the usual Weierstrass primary factor, i.e.

$$E(z,p) = \begin{cases} 1-z, & \text{if } p = 0, \\ (1-z)\exp\left(\sum\limits_{n=1}^{p} \dfrac{z^n}{n}\right), & \text{if } p \geq 1. \end{cases}$$

To prove Theorem 1, we need the following two lemmas.

Lemma 1 ([2]). *Let $\zeta \in \mathcal{Z}$. Then there exists a nonnegative sequence (λ_n) with the following properties:*

(i) $\lambda_n \sim \ln n / \ln |\zeta_n|$ as $n \to \infty$;

(ii) for any sequence (p_n) of non-negative integers such that $p_n \geq [\lambda_n]$ for all sufficiently large n, the series

$$\sum_{n=1}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{p_n+1} \tag{12}$$

converges for every $r \geq 0$, and the product

$$\prod_{n=1}^{\infty} E\left(\frac{z}{\zeta_n}, p_n\right) \tag{13}$$

converges uniformly and absolutely to an entire function $f \in \mathcal{E}_\zeta$ on any compact subset of $\mathbb{C}$, and for all $r \geq 0$ we have $\ln M_f(r) \leq G(r)$, where $G(r)$ is the sum of series (12).

Lemma 2. *Let $r_0 > 1$, u be a non-decreasing unbounded on $[r_0, +\infty)$ function, and $\psi \in H$ be a function such that*

$$\int^{\infty} \frac{du}{\psi(u)} < +\infty. \tag{14}$$

Then, for each fixed number $c > 0$, the set

$$E = \Big\{ r \geq r_0 : \ u\Big(r \exp\Big(\frac{1}{\psi(u(r))} \Big) \Big) > u(r) + c \Big\}$$

has finite logarithmic measure.

Note that Lemma 2 is a version of the classical Borel-Nevanlinna theorem (see, for example, [20, p. 90]) and can be easily deduced from this theorem.

2 Proof of Theorems

Proof of Theorem 1. Suppose that $l, \alpha \in L$ and the functions α and l satisfy condition (7) with some constant $c_1 > 0$. Let φ be a function for which (1) holds. Then it is easy to prove that there exists a function $\psi \in H$ satisfying (14) such that

$$\psi(u) = o(\varphi(u)), \quad u \to +\infty. \tag{15}$$

Note also that condition (1) implies the relation

$$u = o(\varphi(u)), \quad u \to +\infty. \tag{16}$$

Let $\zeta \in \mathcal{Z}$ be an arbitrary sequence such that (6) holds. Fix some $r_0 \geq |\zeta_0|$ and, for all $r \geq r_0$, we put

$$s = s(r) = r \exp\Big(\frac{1}{\psi(\alpha(\ln n_\zeta(r)))} \Big).$$

We also put $p_n = [2\psi(\alpha(\ln n)) \ln n]$ for every integer $n \geq 1$ and consider product (13). By Lemma 1 this product defines an entire function $f \in \mathcal{E}_\zeta$ such that

$$\ln M_f(r) \leq \sum_{n=1}^{\infty} \Big(\frac{r}{|\zeta_n|} \Big)^{p_n+1} = S_1(r) + S_2(r) + S_3(r), \tag{17}$$

where, for all $r \geq r_0$, we used notation

$$S_1(r) = \sum_{|\zeta_n| \leq r} \Big(\frac{r}{|\zeta_n|} \Big)^{p_n+1}, \quad S_2(r) = \sum_{r < |\zeta_n| \leq s} \Big(\frac{r}{|\zeta_n|} \Big)^{p_n+1}, \quad S_3(r) = \sum_{|\zeta_n| > s} \Big(\frac{r}{|\zeta_n|} \Big)^{p_n+1}.$$

Let us prove that (8) holds for the function f with a set $E \subset [1, +\infty)$ of finite logarithmic measure. Let us evaluate each of the sums $S_j(r)$, $j = 1, 2, 3$, assuming that $r \geq r_0$.

For the first sum we have

$$S_1(r) \leq n_\zeta(r) \Big(\frac{r}{|\zeta_1|} \Big)^{p_{n_\zeta(r)}+1},$$

and therefore, taking into account (15), we get

$$\begin{aligned} \ln S_1(r) &\leq \ln n_\zeta(r) + (p_{n_\zeta(r)} + 1) \ln \frac{r}{|\zeta_1|} \leq 3\psi(\alpha(\ln n_\zeta(r))) \ln n_\zeta(r) \ln r \\ &= o(l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r) \varphi(\alpha(\ln n_\zeta(r)))), \quad r \to +\infty. \end{aligned} \tag{18}$$

Further, noting that $|\zeta_n| > r$ for all integers $n > n_\zeta(r)$, we have $S_2(r) \le \sum\limits_{r<|\zeta_n|\le s} 1 < n_\zeta(s)$. By Lemma 2 with $u(r) = \alpha(\ln n_\zeta(r))$ for all $r \ge r_0$, the set

$$E = \{r \ge r_0 : \alpha(\ln n_\zeta(s)) > \alpha(\ln n_\zeta(r)) + c_1\}$$

has finite logarithmic measure. According to (7) and (16), outside the set E we get

$$\begin{aligned} \ln S_2(r) < \ln n_\zeta(s) &\le \alpha^{-1}(\alpha(\ln n_\zeta(r)) + c_1) \\ &= O(\ln n_\zeta(r)\alpha(\ln n_\zeta(r))l^{-1}(\ln n_\zeta(r))) \\ &= o(l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r)\varphi(\alpha(\ln n_\zeta(r)))), \quad r \to +\infty. \end{aligned} \tag{19}$$

In addition, for all integers $n > n_\zeta(s)$ we have $|\zeta_n| > s$, and hence

$$\frac{r}{|\zeta_n|} < \exp\Big(-\frac{1}{\psi(\alpha(\ln n_\zeta(r)))}\Big) \le \exp\Big(-\frac{1}{\psi(\alpha(\ln n))}\Big).$$

This implies that

$$S_3(r) \le \sum_{n>n_\zeta(s)} \exp\Big\{-\frac{p_n+1}{\psi(\alpha(\ln n))}\Big\} \le \sum_{n>n_\zeta(s)} \frac{1}{e^{2\ln n}} \le \sum_{n\ge 2}\frac{1}{n^2} < 1. \tag{20}$$

Therefore, from (17)–(20) we get (8). Theorem 1 is proved. □

Proof of Theorem 2. Suppose that functions $l, \alpha \in L$ and a constant $c_2 > 0$ satisfy conditions (9) and (10). We can assume without loss of generality that $c_2 = 1$, since otherwise we can put $\alpha_0(x) = \alpha(x)/c_2$ for all $x \in \mathbb{R}$ and write α_0 instead of α everywhere below.

It follows from conditions (9) and (10) with $c_2 = 1$ that there exist constants $\varepsilon > 0$ and $q > 1$ such that, for all sufficiently large y, we have

$$\alpha^{-1}(y+1) \ge \varepsilon\alpha^{-1}(y)yl^{-1}(\alpha^{-1}(y)), \tag{21}$$

$$l^{-1}(\alpha^{-1}(y+1)) \ge ql^{-1}(\alpha^{-1}(y)). \tag{22}$$

From (21) we see that the length of the interval $[\exp(\alpha^{-1}(y)), \exp(\alpha^{-1}(y+1))]$ goes to $+\infty$ as $y \to +\infty$. Therefore, for all sufficiently large y, say for $y \ge y_0 > 0$, each such interval contains an integer. For an arbitrary integer $k \ge 1$, we choose some integer in the interval $[\exp(\alpha^{-1}(y_0+3k-1)), \exp(\alpha^{-1}(y_0+3k))]$, which we denote by n_k, and set $t_k = \alpha(\ln n_k)$. Note that

$$y_0 + 3k - 1 \le t_k \le y_0 + 3k. \tag{23}$$

We put $m_1 = n_1$, $r_1 = 1$, and let $m_{k+1} = n_{k+1} - n_k$ and $r_{k+1} = \exp(l^{-1}(\ln n_k))$ for all integers $k \ge 1$. Since, by (21),

$$\begin{aligned} \ln n_{k+1} \ge \alpha^{-1}(y_0+3k+2) &\ge \varepsilon\alpha^{-1}(y_0+3k+1)(y_0+3k+1)l^{-1}(\alpha^{-1}(y_0+3k+1)) > \\ &> \varepsilon^2\alpha^{-1}(y_0+3k)(y_0+3k)^2(l^{-1}(\alpha^{-1}(y_0+3k)))^2 \ge \varepsilon^2\ln n_k\cdot(y_0+3k)^2(l^{-1}(\ln n_k))^2, \end{aligned}$$

we have $\ln m_k \sim \ln n_k$ as $k \to \infty$ and

$$\ln m_{k+1} \ge 4\ln n_k\cdot(y_0+3k)^2 l^{-1}(\ln n_k), \quad k \ge k_1. \tag{24}$$

Also it is easy to see that

$$k = o(\ln \ln m_k), \quad k \to +\infty. \tag{25}$$

In addition, according to (22) and (23) for all integers $k \geq 2$, we have

$$\ln r_{k+1} = l^{-1}(\alpha^{-1}(t_k)) \geq l^{-1}(\alpha^{-1}(t_{k-1}+2)) \geq q^2 l^{-1}(\alpha^{-1}(t_{k-1})) = q^2 \ln r_k. \tag{26}$$

Let us form the sequence $\zeta = (\zeta_n)$ as follows

$$\underbrace{r_1, \ldots, r_1}_{m_1 \text{ times}}, \underbrace{r_2, \ldots, r_2}_{m_2 \text{ times}}, \ldots, \underbrace{r_k, \ldots, r_k}_{m_k \text{ times}}, \ldots,$$

that is, we set $\zeta_n = r_k$ for all integers $n \in (n_k - m_k, n_k]$ and $k \geq 1$. Then $n(r, \zeta) = 0$ for all $r \in [0, 1)$. If $r \in [r_k, r_{k+1})$ for some integer $k \geq 1$, then

$$n_\zeta(r) = \sum_{j=1}^{k} m_j = n_k = \exp(l(\ln r_{k+1})) > \exp(l(\ln r)).$$

Therefore, the sequence ζ satisfies (6). In addition, $\ln n_\zeta(r_{k+1} - 0) = \ln n_k = l(\ln r_{k+1})$ for all integers $k \geq 1$.

Now let $\varphi \in H$ be a function satisfying (2) and let $f \in \mathcal{E}_\zeta$. Let us prove that (11) holds with a set $F(\varphi, f) \subset [1, +\infty)$ of infinite logarithmic measure.

It is easy to prove that there exists a function $\psi \in H$ such that

$$\int^{+\infty} \frac{du}{\psi(u)} = +\infty, \tag{27}$$

and also

$$\varphi(u) = o(\psi(u)), \quad u \to +\infty. \tag{28}$$

For all integers $k \geq 1$, we put $s_k = r_k \exp(-1/(2\psi(t_k)))$. Using (26), we get

$$\ln s_{k+1} = \ln r_{k+1} - \frac{1}{2\psi(t_{k+1})} \geq q \ln r_k, \quad k \geq k_2. \tag{29}$$

Let $k_0 = \max\{k_1, k_2, k_3\}$. Fix an arbitrary integer $j_0 > k_0/2$. Since, by (23), we have $t_{2j} \leq y_0 + 6j$ for each integer $j \geq 1$, then from (27) we obtain $\sum\limits_{j=j_0}^{\infty} \frac{1}{\psi(t_{2j})} = +\infty$. Hence, if J is the set of all integers $j \geq j_0$ satisfying the inequality $\psi(t_{2j}) \leq j^2$, then

$$\sum_{j \in J} \frac{1}{\psi(t_{2j})} = +\infty. \tag{30}$$

The function f has no zeros in the open unit disk, and therefore there exists an analytic in this disk function $g(z) = \sum\limits_{n=0}^{\infty} \gamma_n z^n$ such that $f(z) = \exp(g(z))$, if $|z| < 1$. For every fixed $r > 0$ let $c_p(r)$ be the p-th Fourier coefficient of the function $\ln |f(re^{i\theta})|$, i.e.

$$c_p(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ip\theta} \ln |f(re^{i\theta})| d\theta, \quad p \in \mathbb{Z}.$$

Since all the ζ_n are positive, we have (see, for example, [20, p. 7])

$$c_p(r) = \frac{1}{2}\gamma_p r^p + \frac{1}{2p}\sum_{\zeta_n<r}\Big(\Big(\frac{r}{\zeta_n}\Big)^p - \Big(\frac{\zeta_n}{r}\Big)^p\Big) \tag{31}$$

for each integer $p \geq 1$. Using (31), for $R > r > 0$ we obtain the following equality

$$c_p(R) - \Big(\frac{R}{r}\Big)^p c_p(r) = \frac{1}{2p}\sum_{r\leq|\zeta_n|<R}\Big(\Big(\frac{R}{\zeta_n}\Big)^p - \Big(\frac{\zeta_n}{R}\Big)^p\Big) + \frac{1}{2p}\sum_{|\zeta_n|<r}\Big(\Big(\frac{\zeta_n R}{r^2}\Big)^p - \Big(\frac{\zeta_n}{R}\Big)^p\Big).$$

Both terms in the right-hand side of the above equality are non-negative, and thus we obtain

$$|c_p(R)| + \Big(\frac{R}{r}\Big)^p |c_p(r)| \geq \frac{1}{2p}\sum_{r\leq|\zeta_n|<R}\Big(\Big(\frac{R}{\zeta_n}\Big)^p - \Big(\frac{\zeta_n}{R}\Big)^p\Big). \tag{32}$$

It is also well known (see, for example, [20, p. 62]) that, for all $r > 0$ and all integers p, we have

$$|c_p(r)| \leq 2T_f(r). \tag{33}$$

Let us put $\delta = (q-1)/(5q)$, and let $j \in J$ and $k = 2j-1$. Consider the following two possible cases.

Case 1: there exists an integer p such that

$$\ln|c_p(s_{k+1})| \geq \delta l^{-1}(\ln n_k)\ln n_k \cdot \psi(\alpha(\ln n_k)). \tag{34}$$

Case 2: inequality (34) does not hold for any integer p.

Let $F_j = [s_{k+1}, r_{k+1}]$ in Case 1 and let $F_j = [s_k, r_k]$ in Case 2. Note that, if $m = k$ or $m = k+1$, then

$$\int_{s_m}^{r_m}\frac{dr}{r} = \ln\frac{r_m}{s_m} = \frac{1}{2\psi(t_m)} \geq \frac{1}{2\psi(t_{2j})}.$$

Let us put $F = F(\varphi, f) = \cup_{j\in J}F_j$. Using (29), we see that for arbitrary $j', j'' \in J$ such that $j' < j''$ the inequality $\max F_{j'} < \min F_{j''}$ holds, and therefore

$$\int_F \frac{dr}{r} = \sum_{j\in J}\int_{F_j}\frac{dr}{r} \geq \sum_{j\in J}\frac{1}{2\psi(t_{2j})}.$$

It follows from this and from (30) that the set F has infinite logarithmic measure.

Using (33) and (34), in Case 1, for all $r \in F_j = [s_{k+1}, r_{k+1}]$, we have

$$\ln T_f(r) \geq \ln T_f(s_{k+1}) \geq \delta l^{-1}(\ln n_\zeta(r))\ln n_\zeta(r)\psi(\alpha(\ln n_\zeta(r))) - \ln 2. \tag{35}$$

Now let us consider Case 2. Let $p = [\psi(\alpha(\ln n_k))\ln m_k]$. Then

$$\ln|c_p(s_{k+1})| < \delta l^{-1}(\ln n_k)\ln n_k \cdot \psi(\alpha(\ln n_k)). \tag{36}$$

Using (32) with $R = s_{k+1}$ and $r = s_k$, as well as taking into account (29), we get

$$|c_p(s_{k+1})| + \Big(\frac{s_{k+1}}{s_k}\Big)^p |c_p(s_k)| \geq \frac{m_k}{2p}\Big(\Big(\frac{s_{k+1}}{r_k}\Big)^p - \Big(\frac{r_k}{s_{k+1}}\Big)^p\Big) \geq \frac{m_k}{4p}\Big(\frac{s_{k+1}}{r_k}\Big)^p \tag{37}$$

for all j large enough, say $j \geq j_1$. Since, according to (29),

$$\ln \frac{s_{k+1}}{r_k} \geq \frac{q-1}{q} \ln s_{k+1} = 5\delta \ln s_{k+1} = 5\delta\Big(\ln r_{k+1} - \frac{1}{2\psi(t_{k+1})}\Big),$$

then, using (36), for $j \geq j_2$ we have

$$\begin{aligned}\ln \Big(\frac{m_k}{4p}\Big(\frac{s_{k+1}}{r_k}\Big)^p\Big) &\geq \ln m_k - \ln(4p) + 4p\delta \ln r_{k+1} \geq 3p\delta \ln r_{k+1} = 3p\delta l^{-1}(\ln n_k)\\ &\geq 2\delta l^{-1}(\ln n_k) \ln n_k \cdot \psi(\alpha(\ln n_k)) \geq \ln |c_p(s_{k+1})| + \ln 2.\end{aligned}$$

From this and (37) we see that

$$\Big(\frac{s_{k+1}}{s_k}\Big)^p |c_p(s_k)| \geq \frac{m_k}{4p}\Big(\frac{s_{k+1}}{r_k}\Big)^p - |c_p(s_{k+1})| \geq \frac{m_k}{8p}\Big(\frac{s_{k+1}}{r_k}\Big)^p,$$

and therefore, taking into account that

$$\psi(t_{k-1}) \leq \psi(t_k) \leq \psi(t_{k+1}) = \psi(t_{2j}) \leq j^2 = (k+1)^2/4,$$

and using (25) and (24), for $j \geq j_3$ we obtain

$$\begin{aligned}\ln |c_p(s_k)| &\geq \ln \Big(\frac{m_k}{8p}\Big(\frac{s_k}{r_k}\Big)^p\Big) \geq \ln m_k - \ln(8\psi(\alpha(\ln n_k)) \ln m_k) + \psi(\alpha(\ln n_k)) \ln m_k \ln \frac{s_k}{r_k}\\ &= \ln m_k - \ln(8 \ln m_k) - \ln \psi(t_k) - \frac{1}{2} \ln m_k \geq \frac{1}{3} \ln m_k - \ln \psi(t_k)\\ &\geq \frac{1}{3} \ln m_k - 2 \ln k \geq \frac{1}{4} \ln m_k \geq \ln n_{k-1} \cdot (y_0 + 3k - 3)^2 l^{-1}(\ln n_{k-1})\\ &\geq l^{-1}(\ln n_{k-1}) \ln n_{k-1} \psi(t_{k-1}) = l^{-1}(\ln n_{k-1}) \ln n_{k-1} \psi(\alpha(\ln n_{k-1})).\end{aligned}$$

This implies that in Case 2, for all $r \in F_j = [s_k, r_k]$ and $j \geq j_3$, we have

$$\ln T_f(r) \geq \ln T_f(s_k) \geq \ln |c_p(s_k)| - \ln 2 \geq l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r) \psi(\alpha(\ln n_\zeta(r))) - \ln 2. \tag{38}$$

Therefore, if j is sufficiently large, then for all $r \in F_j$ we have (35) or (38). Recalling (28), we see that (11) holds. Theorem 2 is proved. □

3 Consequences of Theorems

In this section, we give examples of functions $l, \alpha \in L$ for which all the conditions of Theorems 1 and 2 are simultaneously satisfied, and we formulate the corresponding consequences of these theorems.

We start with the following simple statement.

Lemma 3. *Let $c > 0$ and let β be a positive continuous function on $[a, +\infty)$. If*

$$\varliminf_{x \to +\infty} \frac{\beta(x+c)}{\beta(x)} = \Delta > 1, \tag{39}$$

then

$$\varliminf_{x \to +\infty} \frac{\ln \beta(x)}{x} \geq \frac{\ln \Delta}{c}. \tag{40}$$

Proof. We fix an arbitrary $q \in (1, \Delta)$. From (39) for some $x_0 \in \mathbb{R}$ and all $x \geq x_0$ we have $\beta(x+c) \geq q\beta(x)$. Put $m = \min\{\beta(x) : x \in [x_0, x_0+c]\}$ and let $x \geq x_0$. Then $x_0 + nc \leq x \leq x_0 + (n+1)c$ for some integer $n \geq 0$, and therefore

$$\beta(x) \geq q^n \beta(x - nc) \geq q^n m \geq q^{(x-x_0)/c} q^{-1} m.$$

So,

$$\varliminf_{x\to+\infty} \frac{\ln \beta(x)}{x} \geq \frac{\ln q}{c}.$$

Since $q \in (1, \Delta)$ is arbitrary, we have (40). $\square$

Let $l, \alpha \in L$ and $c_2 > 0$. If condition (9) is satisfied, then $\alpha^{-1}(y+c_2)/(\alpha^{-1}(y)y) \to +\infty$ as $y \to +\infty$. We put $\beta(y) = \alpha^{-1}(y) e^{-y \ln y / c_2}$ for all $y \geq y_0$. Then

$$\varliminf_{y\to+\infty} \frac{\beta(y+c_2)}{\beta(y)} = \varliminf_{y\to+\infty} \frac{\alpha^{-1}(y+c_2)}{\alpha^{-1}(y) e^{((y+c_2)\ln(y+c_2) - y\ln y)/c_2}} = \varliminf_{y\to+\infty} \frac{\alpha^{-1}(y+c_2)}{\alpha^{-1}(y) e y} = +\infty.$$

By Lemma 3 we have $\ln \beta(y)/y \to +\infty$ as $y \to +\infty$. It is easy to see that this implies the relation $y \ln y = O(\ln \alpha^{-1}(y))$ as $y \to +\infty$, which is equivalent to the relation

$$\alpha(x) = O(\ln x / \ln_2 x), \quad x \to +\infty. \tag{41}$$

In addition, if (10) holds, then by Lemma 3 we have

$$\varliminf_{x\to+\infty} \frac{\ln l^{-1}(\alpha^{-1}(y))}{y} > 0. \tag{42}$$

Therefore, we will not be able to use Theorem 2 if at least one of conditions (41) or (42) is not satisfied.

Let us now consider special cases of functions $\alpha \in L$ for which (41) is satisfied.

Case 1: $\alpha(x) = \ln x / \ln_2 x$ for all $x \geq x_0$. Then $\ln \alpha^{-1}(y) \sim y \ln y$ as $y \to +\infty$, and therefore $\ln_2 \alpha^{-1}(y) = \ln y + \ln_2 y + o(1)$ as $y \to +\infty$. Let $c > 0$. Then

$$\begin{gathered}(y+c)\ln_2 \alpha^{-1}(y+c) - y \ln_2 \alpha^{-1}(y) \leq c \ln_2 \alpha^{-1}(y+c) = c(\ln y + \ln_2 y) + o(1), \\ (y+c)\ln_2 \alpha^{-1}(y+c) - y \ln_2 \alpha^{-1}(y) \geq c \ln_2 \alpha^{-1}(y) = c(\ln y + \ln_2 y) + o(1)\end{gathered}$$

as $y \to +\infty$. So we obtain

$$\alpha^{-1}(y+c)/(\alpha^{-1}(y)y) = \exp((y+c)\ln_2 \alpha^{-1}(y+c) - y \ln_2 \alpha^{-1}(y))/y \sim y^{c-1} \ln^c y, \quad y \to +\infty. \tag{43}$$

From (43) we see that if $c_1 < 1$, then condition (7) is satisfied for an arbitrary function $l \in L$, and therefore we have the following consequence of Theorem 1.

Corollary 1. *Let $l \in L$ and let $\varphi \in H$ be a function satisfying* (1)*. Then for any sequence $\zeta \in \mathcal{Z}$ satisfying* (6) *there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1, +\infty)$ of finite logarithmic measure such that*

$$\ln_2 M_f(r) = o\big(l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r) \varphi(\ln_2 n_\zeta(r)/\ln_3 n_\zeta(r))\big), \quad r \notin E,\ r \to +\infty.$$

From (43) we also see that if $c_2 > 1$, then, for some function $l \in L$, condition (9) can be satisfied, but Theorem 2 cannot be applied, since it follows from (9) that (42) cannot be satisfied. In fact, if (9) holds for some $c_2 > 1$, then, using (43), we get $l^{-1}(\alpha^{-1}(y)) = O(y^{c_2-1}\ln^{c_2} y)$ as $y \to +\infty$, and this contradicts (42).

Case 2: $\alpha(x) = \ln^{\delta} x$ for some fixed $\delta \in (0,1)$ and all $x \geq x_0$. Then $\alpha^{-1}(y) = \exp(y^{1/\delta})$ for all $y \geq y_0$, and for each $c > 0$ we have

$$\alpha^{-1}(y+c)/(\alpha^{-1}(y)y) = \exp((y+c)^{1/\delta} - y^{1/\delta})/y = \exp((1+o(1))cy^{(1-\delta)/\delta}), \quad y \to +\infty. \tag{44}$$

Let $l \in L$. From (44) we see that (7) and (9) are satisfied simultaneously with some positive constants c_1 and c_2, respectively, if and only if there exist positive constants d_1 and d_2 such that

$$\exp(d_1 y^{(1-\delta)/\delta}) \leq l^{-1}(\alpha^{-1}(y)) \leq \exp(d_2 y^{(1-\delta)/\delta}), \quad y \geq y_1. \tag{45}$$

It follows from (45) that (42) holds if and only if $(1-\delta)/\delta \geq 1$, i.e. $\delta \in (0,1/2]$. Rewriting (45) in the form $\exp(d_1 \ln^{1-\delta} x) \leq l^{-1}(x) \leq \exp(d_2 \ln^{1-\delta} x)$, $x \geq x_1$, for each $\delta \in (0,1/2]$ we can easily choose a function $l \in L$ satisfying (10) with some constant $c_2 > 0$. For example, if $l(y) = \exp((a\ln y + b)^{1/(1-\delta)})$ for some constants $a > 0$ and $b \in \mathbb{R}$ and all $y \geq y_2$, then $l^{-1}(x) = \exp(A\ln^{1-\delta} x + B)$ for all $x \geq x_2$, where $A = 1/a$ and $B = -b/a$, and therefore for each $c > 0$ we have

$$\begin{aligned} l^{-1}(\alpha^{-1}(y+c))/l^{-1}(\alpha^{-1}(y)) &= \exp(A(y+c)^{(1-\delta)/\delta} - Ay^{(1-\delta)/\delta}) \\ &= \exp\big((1+o(1))Ac(1-\delta)y^{(1-2\delta)/\delta}/\delta\big), \quad y \to +\infty. \end{aligned}$$

This implies (10) with an arbitrary constant $c_2 > 0$. Hence, in the case when $\delta \in (0,1/2]$ all the conditions of Theorems 1 and 2 hold for the given function l, and therefore we have the following consequence of these theorems.

Corollary 2. *Let $\delta \in (0,1/2]$, $a > 0$, and $b \in \mathbb{R}$. Then:*

(i) for every function $\varphi \in H$ satisfying (1) *and any sequence $\zeta \in \mathcal{Z}$ satisfying*

$$\ln_2 n_\zeta(r) \geq (a\ln_2 r + b)^{1/(1-\delta)}, \quad r \geq r_0, \tag{46}$$

there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1,+\infty)$ of finite logarithmic measure such that

$$\ln_2 M_f(r) = o\big(\exp(\ln_2^{1-\delta} n_\zeta(r)/a)\ln n_\zeta(r)\varphi(\ln_2^{\delta} n_\zeta(r))\big), \quad r \notin E,\ r \to +\infty;$$

(ii) there exists a sequence $\zeta \in \mathcal{Z}$ such that (46) *holds,* $\ln_2 n_\zeta(r-0) = (a\ln_2 r + b)^{1/(1-\delta)}$ *on an unbounded from above set of values of r, and, for every function $\varphi \in H$ satisfying* (2) *and any function $f \in \mathcal{E}_\zeta$, we have*

$$\exp(\ln_2^{1-\delta} n_\zeta(r)/a)\ln n_\zeta(r)\varphi(\ln_2^{\delta} n_\zeta(r)) = o(\ln T_f(r)), \quad r \in F(\varphi,f),\ r \to +\infty,$$

where $F(\varphi,f) \subset [1,+\infty)$ is a set of infinite logarithmic measure.

Case 3: $\alpha(x) = \ln_2^\delta x$ for some fixed $\delta > 1$ and all $x \geq a$. Then $\alpha^{-1}(y) = \exp_2(y^{1/\delta})$ for all $y \geq b$, and for each $c > 0$ we have

$$\begin{aligned}\alpha^{-1}(\alpha(x)+c)/(x\alpha(x)) &= \exp\left(e^{(\ln_2^\delta x+c)^{1/\ delta}} - \ln x - \delta\ln_3 x\right)\\ &= \exp\left((1+o(1))c\ln x/(\delta\ln_2^{\delta-1}x)\right), \quad x\to+\infty.\end{aligned} \tag{47}$$

Let us consider a function $l \in L$ such that $l^{-1}(x) = \exp(A\ln x/\ln_2^{\delta-1}x)$ for all $x \geq x_0$, where A is some positive number, and we choose the constants c_1 and c_2 so that $0 < c_1 < \delta/A$ and $c_2 > \delta/A$. Then from (47) we see that (7) and (9) are satisfied. In addition, if $c > 0$, then we have

$$\begin{aligned}\ln l^{-1}(\alpha^{-1}(y+c)) - \ln l^{-1}(\alpha^{-1}(y)) &= A\left(e^{(y+c)^{1/\delta}}/(y+c)^{(\delta-1)/\delta} - e^{y^{1/\delta}}/y^{(\delta-1)/\delta}\right)\\ &\sim Ace^{y^{1/\delta}}/(\delta y^{2(\delta-1)/\delta}), \quad y\to+\infty,\end{aligned}$$

and therefore (10) holds. Noting that the inequality $\ln n_\zeta(r) \geq l(\ln r)$ from condition (6) is equivalent, for all sufficiently large r, to the inequality $l^{-1}(\ln n_\zeta(r)) \geq \ln r$, which in our case after double logarithmization takes the form

$$\ln A + \ln^{1/\delta} n_\zeta(r) - (\delta-1)\ln_2 n_\zeta(r)/\delta \geq \ln_3 r, \tag{48}$$

we can formulate the following consequence of Theorems 1 and 2.

Corollary 3. *Let $\delta > 1$, and $A > 0$. Then:*

(i) for every function $\varphi \in H$ satisfying (1) *and any sequence $\zeta \in \mathcal{Z}$ satisfying* (48) *for all $r \geq r_0$ there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1,+\infty)$ of finite logarithmic measure such that*

$$\ln_2 M_f(r) = o\left(\exp(A\ln_2 n_\zeta(r)/\ln_3^{\delta-1} n_\zeta(r))\ln n_\zeta(r)\varphi(\ln_3^\delta n_\zeta(r))\right), \quad r\notin E,\ r\to+\infty;$$

(ii) there exists a sequence $\zeta \in \mathcal{Z}$ such that inequality (48) *holds for all $r \geq r_0$, the equality $\ln A + \ln^{1/\delta} n_\zeta(r-0) - (\delta-1)\ln_2 n_\zeta(r-0)/\delta = \ln_3 r$ holds on an unbounded from above set of values of r, and, for every function $\varphi \in H$ satisfying* (2) *and any function $f \in \mathcal{E}_\zeta$, we have*

$$\exp(A\ln_2 n_\zeta(r)/\ln_3^{\delta-1} n_\zeta(r))\ln n_\zeta(r)\varphi(\ln_3^\delta n_\zeta(r)) = o(\ln T_f(r)), \quad r\in F(\varphi,f),\ r\to+\infty,$$

where $F(\varphi,f) \subset [1,+\infty)$ is a set of infinite logarithmic measure.

Case 4: $\alpha(x) = \ln_2 x$ for all $x \geq a$. Then for each $c > 0$ we have

$$\alpha^{-1}(\alpha(x)+c)/(x\alpha(x)) = \exp\left(e^c\ln x - \ln x - \ln_3 x\right) = x^{(1+o(1))(e^c-1)}, \quad x\to+\infty. \tag{49}$$

Consider an arbitrary function $l \in L$ for which there exist positive constants d_1 and d_2 such that $y^{d_1} \leq l(y) \leq y^{d_2}$, $y \geq y_0$. Putting $b_1 = 1/d_2$ and $b_2 = 1/d_1$, from the latter inequalities we have

$$x^{b_1} \leq l^{-1}(x) \leq x^{b_2}, \quad x \geq x_0. \tag{50}$$

Using (50), for each $c > 0$ we get

$$\begin{aligned}\ln l^{-1}(\alpha^{-1}(y+c)) - \ln l^{-1}(\alpha^{-1}(y)) &\geq b_1\ln\alpha^{-1}(y+c) - b_2\ln\alpha^{-1}(y)\\ &= e^y(b_1e^c - b_2), \quad y\geq y_1.\end{aligned} \tag{51}$$

From (49), (50), and (51) we see that if some constants c_1 and c_2 satisfy the inequalities $0 < c_1 < \ln(1+b_1)$ and $c_2 > \max\{\ln(1+b_2), \ln(b_2/b_1)\}$, then all the conditions of Theorems 1 and 2 are satisfied, and therefore we have the following corollary of these theorems.

Corollary 4. *Let $l \in L$ be a function satisfying* (50) *with some positive constants b_1 and b_2. Then:*

(i) for every function $\varphi \in H$ satisfying (1) *and any sequence $\zeta \in \mathcal{Z}$ satisfying* (6) *there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1, +\infty)$ of finite logarithmic measure such that*

$$\ln_2 M_f(r) = o\big(l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r) \varphi(\ln_3 n_\zeta(r))\big), \quad r \notin E,\ r \to +\infty;$$

(ii) there exists a sequence $\zeta \in \mathcal{Z}$ such that (6) *holds, $\ln n_\zeta(r-0) = l(\ln r)$ on an unbounded from above set of values of r, and, for every function $\varphi \in H$ satisfying* (2) *and any function $f \in \mathcal{E}_\zeta$, we have*

$$l^{-1}(\ln n_\zeta(r)) \ln n_\zeta(r) \varphi(\ln_3 n_\zeta(r)) = o(\ln T_f(r)), \quad r \in F(\varphi, f),\ r \to +\infty,$$

where $F(\varphi, f) \subset [1, +\infty)$ is a set of infinite logarithmic measure.

Finally, we consider a rather general case, which concerns functions α from a wide sub-class of the class L.

We recall (see, for example, [23]) that a positive measurable function β, defined on $[a, +\infty)$, is called slowly varying if for every fixed $c > 0$ we have $\beta(ct) \sim \beta(t)$ as $t \to +\infty$. It is well known that for every slowly varying function β we have $\ln \beta(t) = o(\ln t)$, $t \to +\infty$. Moreover, if a function β is positive continuously differentiable on $[a, +\infty)$, then this function is slowly varying if and only if

$$t\beta'(t)/\beta(t) \to 0, \quad t \to +\infty. \tag{52}$$

Consider an arbitrary function $\alpha \in L$ and put

$$\beta(t) = \exp(\alpha(e^t)), \quad t \in \mathbb{R}. \tag{53}$$

If β is slowly varying, then relation $\ln \beta(t) = o(\ln t)$, $t \to +\infty$, holds, and this relation can be rewritten in the form

$$\alpha(x) = o(\ln_2 x), \quad x \to +\infty. \tag{54}$$

Note that in all the cases considered earlier, condition (54) is not satisfied.

Case 5: $\alpha \in L$ is such that the function β defined by (53) is slowly varying. Note that if α is a continuously differentiable function on $[b, +\infty)$, then condition (52) is equivalent to the condition $\alpha'(x) x \ln x \to 0$ as $x \to +\infty$. Using this fact, it is easy to show that our case includes, for example, functions $\alpha \in L$ such that for all $x \geq x_0$ we have $\alpha(x) = \ln_2^\delta x$, where $\delta \in (0, 1)$ is a fixed number, or $\alpha(x) = \ln_k^\delta x$, where δ is a fixed positive number and $k \geq 3$ is a fixed integer.

Let $\varepsilon > 0$ and $q > 1$ be arbitrary fixed numbers. Then $\ln \beta(qt) \leq \ln \beta(t) + \varepsilon$ for all $t \geq t_0$, i.e. $\alpha^{-1}(y + \varepsilon) \geq (\alpha^{-1}(y))^q$ for all $y \geq y_0$. Hence, if $c > d > 0$, then, in view of (54), for all sufficiently large y we have

$$\alpha^{-1}(y + c)/(\alpha^{-1}(y) y) \geq \alpha^{-1}(y + c)/(\alpha^{-1}(y))^2 \geq \alpha^{-1}(y + d).$$

This shows, that if $l \in L$, then (7) and (9) are satisfied simultaneously with some positive constants c_1 and c_2, respectively, if and only if there exist positive constants d_1 and d_2 such that

$$\alpha^{-1}(\alpha(x) + d_1) \leq l^{-1}(x) \leq \alpha^{-1}(\alpha(x) + d_2), \quad x \geq x_0. \tag{55}$$

In particular, if (55) holds and $c_2 > d_2$, then (9) holds and, in addition,

$$\varliminf_{y\to+\infty} \frac{l^{-1}(\alpha^{-1}(y+c_2))}{l^{-1}(\alpha^{-1}(y))} \geq \varliminf_{y\to+\infty} \frac{\alpha^{-1}(y+c_2+d_1)}{\alpha^{-1}(y+d_2)} = +\infty,$$

that is, (10) is satisfied. Therefore, since (55) can be rewritten as

$$\alpha^{-1}(\alpha(y)-d_2) \leq l(y) \leq \alpha^{-1}(\alpha(y)-d_1), \quad y \geq y_0, \tag{56}$$

we have the following corollary of Theorems 1 and 2.

Corollary 5. *Let $\alpha, l \in L$ be functions such that the function β defined by* (53) *is slowly varying and* (56) *holds with some positive constants d_1 and d_2. Then:*

(i) for every function $\varphi \in H$ satisfying (1) *and any sequence $\zeta \in \mathcal{Z}$ satisfying* (6) *there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset [1,+\infty)$ of finite logarithmic measure such that* (8) *holds;*

(ii) there exists a sequence $\zeta \in \mathcal{Z}$ such that (6) *holds, $\ln n_\zeta(r-0) = l(\ln r)$ on an unbounded from above set of values of r, and, for every function $\varphi \in H$ satisfying* (2) *and any function $f \in \mathcal{E}_\zeta$, we have* (11)*, where $F(\varphi, f) \subset [1,+\infty)$ is a set of infinite logarithmic measure.*

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Нехай l — додатна, неперервна, зростаюча до $+\infty$ на $\mathbb{R}$ функція. Знайдено достатні та необхідні умови на додатну, неспадну на $\mathbb{R}$ функцію h, за яких для довільної комплексної послідовності (ζ_n) такої, що $\zeta_n \to \infty$, якщо $n \to \infty$, і $\ln n(r) \geq l(\ln r)$ для всіх достатньо великих r, існує ціла функція f з нулями в точках ζ_n і лише в них (з урахуванням кратності), для якої маємо

$$\ln\ln M(r) = o\big(l^{-1}(\ln n(r)) \ln n(r) h(\ln n(r))\big), \quad r \notin E,\ r \to +\infty,$$

де $E \subset [1, +\infty)$ — множина скінченої логарифмічної міри. Тут $n(r)$ — лічильна функція послідовності (ζ_n), а $M(r)$ — максимум модуля функції f.

Ключові слова і фрази: ціла функція, максимум модуля, характеристика Неванлінни, нуль, лічильна функція.