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Kantorovich-Stancu type Lototsky-Chlodowsky operators

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In this study, we define Kantorovich-Stancu type Lototsky-Chlodowsky operators and obtain some approximation properties in weighted space of these operators. Furthermore, we prove Voronovskaja type results for the Kantorovich-Stancu type Lototsky-Chlodowsky operators.

Key words and phrases: Kantorovich-Stancu type Lototsky-Chlodowsky operator, Korovkin theorem, rate of convergence, approximation.

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Introduction

In 1966, J.P. King [9] introduced Lototsky-Bernstein operator $L_n : C[0,1] \to C[0,1]$ for $n \in \mathbb{N}$ as follows

$$L_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) a_{n,k}(x), \quad x \in [0,1],$$

with the help of basis function $a_{n,k}(x)$ obtained from the following relation

$$\prod_{i=1}^{n} (h_i(x)y + 1 - h_i(x)) = \sum_{k=0}^{n} a_{n,k}(x) y^k, \quad y \in \mathbb{R},$$

$$a_{n,k}(x) = \sum_{\substack{J \cup \overline{J} = \mathbb{N}_n \\ Card(J)=k}}^{n} \prod_{i \in \overline{J}} (1 - h_i(x)) \prod_{i \in J} h_i(x),$$

where $h_i : \ [0,1] \to [0,1]$ is a sequence of continuous functions and $a_{0,0}(x) = 1$, $a_{0,k}(x) = 0$ for $k > 0$. Recently, these operators have been introduced their approximation properties (see [15, 11, 14, 1]).

The classical Bernstein-Chlodowsky polynomials have the following form

$$C_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n} b_n\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}, \quad 0 \le x \le b_n,$$

where $0 \le x \le b_n$ and b_n is a sequence of positive numbers such that

$$\lim_{n\to\infty} b_n = +\infty, \quad \sum_{n\to\infty} \frac{b_n}{n} = 0.$$

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These polynomials were introduced by I. Chlodowsky in 1932 as a generalization of Bernstein polynomials on an unbounded set. Recently, some authors have studied some Chlodowsky type polynomials. For example, a very fine generalization of the Chlodowsky polynomials is given by A.D. Gadjiev et. al. (see [4]). Some detailed investigations on Bernstein-Chlodowsky polynomials may be found in [3, 6, 5]. Recently, in [10], the authors introduced the Lototsky-Chlodowsky operators $L_n^* : B[0,\infty) \to C[0,\infty)$ for $n \in \mathbb{N}$ given by

$$L_n^*(f;x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}b_n\Big) a_{n,k}\Big(\frac{x}{b_n}\Big), \quad x \in [0, b_n], \tag{1}$$

where $a_{n,k} = a_{n,k}(\frac{x}{b_n})$ are Lototsky-Bernstein basis functions satisfying

$$\prod_{i=1}^{n}\Big(h_i\Big(\frac{x}{b_n}\Big)y + 1 - h_i\Big(\frac{x}{b_n}\Big)\Big) = \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)y^k, \quad y \in \mathbb{R}, \tag{2}$$

where

$$a_{n,k}\Big(\frac{x}{b_n}\Big) = \sum_{\substack{J\cup\bar{J}=\mathbb{N}_n \\ Card(J)=k}}^{n} \prod_{i\in\bar{J}}\Big(1 - h_i\Big(\frac{x}{b_n}\Big)\Big)\prod_{i\in J}^{n} h_i\Big(\frac{x}{b_n}\Big).$$

The approximation properties of the operators (1) can be found in [10]. In this article, the Kantorovich-Stancu type Lototsky-Chlodowsky operators will be defined based on the Lototsky-Chlodowsky basis function and Kantorovich-Stancu type operators and the approximation properties of these operators will be examined.

In 1930, L.V. Kantorovich [7, 8] introduced a generalization of Bernstein operators as $K_n : L_1[0,1] \to C[0,1]$ for $n \in \mathbb{N}$ defined by

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,$$

where $b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ are Bernstein basis functions. In 1968, D. Stancu [13] constructed positive linear operators $S_n^{\alpha,\beta} : C[0,1] \to C[0,1]$ for $n \in \mathbb{N}$ by

$$S_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\Big(\frac{k+\alpha}{n+\beta}\Big),$$

where α and β real parameters such that $0 \le \alpha \le \beta$. In 2004, D. Barbosu [2] gave a Kantorovich-Stancu form as $K_n^{\alpha,\beta}(f;x) : L_1[0,1] \to C[0,1]$ for $n \in \mathbb{N}$ defined by

$$K_n^{\alpha,\beta}(f;x) = (n+\beta+1)\sum_{k=0}^{n} b_{n,k}(x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\,dt, \quad 0 \le \alpha \le \beta.$$

In the present paper, we consider a Kantorovich-Stancu version of the operators (1) as follows $L_{n,\gamma,\beta}^* : L_1[0,\infty) \to C[0,\infty)$ for $0 \le \gamma \le \beta$ and $n \in \mathbb{N}$ defined by

$$L_{n,\gamma,\beta}^*(f;x) = (n+\beta+1)\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} f(b_n t)\,dt, \tag{3}$$

where $a_{n,k}(\frac{x}{b_n})$ are Lototsky-Bernstein basis functions.

1 Approximation properties of the operators $L^*_{n,\gamma,\beta}$

In this part, we give some lemmas which are necessary to prove our main theorems.

Lemma 1. *Let $x \in [0, b_n]$ and $e_i(t) = t^i$, $i = 0,1,2$. Then the Kantorovich-Stancu type Lototsky-Chlodowsky operators $L^*_{n,\gamma,\beta}$ satisfy*

$$L^*_{n,\gamma,\beta}(e_0; x) = 1, \tag{4}$$

$$L^*_{n,\gamma,\beta}(e_1; x) = \frac{b_n}{(n+\beta+1)}\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \frac{2\gamma+1}{2}\right), \tag{5}$$

and

$$\begin{aligned} L^*_{n,\gamma,\beta}(e_2; x) =& \frac{b_n^2}{(n+\beta+1)^2}\left[\left(\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\right)^2 + \sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right)\left(1 - h_i\left(\frac{x}{b_n}\right)\right)\right. \\ &\left. + (2\gamma+1)\sum_{i=1}^{n} h_i\left(\frac{x}{b_n}\right) + \gamma + \frac{1}{3} + \gamma^2(n+\beta+1)^2\right]. \end{aligned} \tag{6}$$

Proof. We have $L^*_{n,\gamma,\beta}(e_0; x) = \sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)$. Putting $y = 1$ in (2), we get

$$\prod_{i=1}^{n}\left(h_i\left(\frac{x}{b_n}\right) + 1 - h_i\left(\frac{x}{b_n}\right)\right) = \sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right),$$

therefore, (4) follows.

Since

$$\begin{aligned} L^*_{n,\gamma,\beta}(e_1; x) &= (n+\beta+1)\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} b_n t\, dt \\ &= b_n(n+\beta+1)\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{2k+2\gamma+1}{2(n+\beta+1)^2} = b_n\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{2k+2\gamma+1}{2(n+\beta+1)} \\ &= b_n\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{k+\gamma}{(n+\beta+1)} + b_n\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{1}{2(n+\beta+1)} \\ &= b_n\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{k}{(n+\beta+1)} + b_n\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{\gamma}{(n+\beta+1)} \\ &\quad + b_n\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{1}{2(n+\beta+1)} \\ &= \frac{b_n}{(n+\beta+1)}\sum_{k=0}^{n} k a_{n,k}\left(\frac{x}{b_n}\right) + \frac{\gamma b_n}{(n+\beta+1)} + \frac{b_n}{2(n+\beta+1)}, \end{aligned}$$

in view of (2), we get (5).

Finally, since

$$\begin{aligned} L^*_{n,\gamma,\beta}(e_2; x) &= (n+\beta+1)\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} b_n^2 t^2\, dt \\ &= \frac{b_n^2(n+\beta+1)}{3}\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\left(\left(\frac{k+\gamma+1}{n+\beta+1}\right)^3 - \left(\frac{k+\gamma}{n+\beta+1}\right)^3\right) \\ &= b_n^2(n+\beta+1)\sum_{k=0}^{n} a_{n,k}\left(\frac{x}{b_n}\right)\frac{(k+\gamma+1)^3 - (k+\gamma)^3}{3(n+\beta+1)^3} \end{aligned}$$

$$
\begin{aligned}
&= \frac{b_n^2}{3(n+\beta+1)^2}\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)(k+\gamma+1)^3-(k+\gamma)^3\\
&= \frac{b_n^2}{3(n+\beta+1)^2}\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)[3(k+\gamma)^2+3(k+\gamma)+1]\\
&= \frac{b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)(k+\gamma)^2+\frac{b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)(k+\gamma)\\
&\quad+\frac{b_n^2}{3(n+\beta+1)^2}\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\\
&= \frac{b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} k^2 a_{n,k}\Big(\frac{x}{b_n}\Big)+\frac{2\gamma b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} k a_{n,k}\Big(\frac{x}{b_n}\Big)\\
&\quad+\gamma^2 b_n^2\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)+\frac{b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} k a_{n,k}\Big(\frac{x}{b_n}\Big)\\
&\quad+\frac{\gamma b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)+\frac{b_n^2}{3(n+\beta+1)^2}\\
&= \frac{b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} k^2 a_{n,k}\Big(\frac{x}{b_n}\Big)+\frac{(2\gamma+1)b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} k a_{n,k}\Big(\frac{x}{b_n}\Big)\\
&\quad+\gamma^2 b_n^2+\frac{\gamma b_n^2}{(n+\beta+1)^2}+\frac{b_n^2}{3(n+\beta+1)^2},
\end{aligned}
$$

in view of (2), we get (6). □

Lemma 2. *Let $x \in [0, b_n]$. For the operators $L^*_{n,\gamma,\beta}(f;x)$, we have*

$$
L^*_{n,\gamma,\beta}((t-x);x)=\frac{b_n}{(n+\beta+1)}\Big(\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big)+\frac{2\gamma+1}{2}\Big)-x
$$

and

$$
\begin{aligned}
L^*_{n,\gamma,\beta}((t-x)^2;x) &= L^*_{n,\gamma,\beta}(t^2;x)-2xL^*_{n,\gamma,\beta}(t;x)+x^2L^*_{n,\gamma,\beta}(1;x)\\
&= \frac{b_n^2}{(n+\beta+1)^2}\bigg[\Big(\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big)\Big)^2+\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big)\Big(1-h_i\Big(\frac{x}{b_n}\Big)\Big)\bigg]\\
&\quad+\frac{b_n^2}{(n+\beta+1)^2}\bigg[(2\gamma+1)\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big)+\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\bigg]\\
&\quad-\frac{2xb_n}{(n+\beta+1)}\Big(\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big)+\frac{2\gamma+1}{2}\Big)+x^2.
\end{aligned}
$$

Lemma 3 ([11]). *For every $\alpha > 0$, let*

$$
h_i\Big(\frac{x}{b_n}\Big)=\frac{i^\alpha \frac{x}{b_n}}{i^\alpha+\frac{x}{b_n}}.
$$

Since $0 < \alpha < 1$, we have

$$
\sum_{i=1}^{n}\left(\frac{1}{i^\alpha+\frac{x}{b_n}}\right)\cong\frac{n^{1-\alpha}}{b_n(1-\alpha)},\quad \text{and thus}\quad \sum_{i=1}^{n}\Big(h_i\Big(\frac{x}{b_n}\Big)-\frac{x}{b_n}\Big)\cong-\frac{x^2}{b_n^3}\frac{n^{1-\alpha}}{1-\alpha}.
$$

Lemma 4 ([12]). *For $q \in \mathbb{N}$ and fixed $x \in I$, let $L^*_{n,\gamma,\beta} : L_\infty(I) \to C(I)$ be a sequence of positive linear operators with the property*

$$(A_n(t-x)^p; x) = O(n^{-[(p+1)/2]}) \ \text{ as } \ n \to +\infty, \quad p = 0, 1, \dots, 2q+2.$$

Then we have for each $f \in L_\infty(I)$, which is $2q$ times differentiable at x, the asymptotic relation

$$(A_n f)(x) = \sum_{p=0}^{2q} \frac{1}{p!}(A_n(t-x)^p; x) f^{(p)}(x) + O(n^{-q}) \ \text{ as } \ n \to +\infty. \tag{7}$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $O(n^{-q})$ in (7) can be replaced by $O(n^{-(q+1)})$.

2 Convergence results

In this part, we study some approximation properties of the operator $L^*_{n,\gamma,\beta}(f; x)$ defined by (3). Let $[a, b]$ be a compact subset of $[0, \infty)$ and consider the following type lattice homomorphism

$$H : \ C[0, \infty) \to C[a, b],$$

defined by $H(f) := f \mid_{[a,b]}$ for every $f \in C[0, \infty)$, where $f \mid_{[a,b]}$ is the restriction of the domain of f to the interval $[a, b]$. Clearly we have

$$\lim_{n \to +\infty} H(L^*_{n,\gamma,\beta}(e_i, x)) = H(e_i) \ \text{ uniformly on } \ [a, b]$$

for each $i = 0, 1, 2$. Owing to the Korovkin property we have the following Korovkin type approximation result related to the uniform convergence.

Theorem 1. *For every $f \in C[0, \infty)$, let*

$$\lim_{n \to \infty} f(x) = k_f < +\infty \quad \text{and} \quad \lim_{n \to +\infty} \frac{b_n^2}{n} = 0.$$

Then

$$\lim_{n \to +\infty} \| L^*_{n,\gamma,\beta} f - f \|_{C[0,\infty)} = 0.$$

Now, in order to get on uniform convergence result on the positive real axis $[0, +\infty)$, we consider the following subspace

$$C_\rho([0, \infty)) = \{ f \in C_\rho([0, \infty)) : \ \forall \, x \in [0, \infty), \ |f(x)| \le M_f \rho(x) \}$$

endowed with the sup-norm. We have the following result.

Theorem 2. *Let $\rho(x) = 1 + x^4$. Then for every $f \in C_\rho([0, \infty))$*

$$\lim_{n \to +\infty} \| L^*_{n,\gamma,\beta} f - f \|_\rho = 0.$$

Proof. For $0 \le x \le b_n$, by using the equality (4), we may write

$$\| L^*_{n,\gamma,\beta}(1; x) - 1 \|_\rho = \sup_{0 \le x \le b_n} \frac{| L^*_{n,\gamma,\beta}(1; x) - 1 |}{\rho(x)} = \sup_{0 \le x \le b_n} \frac{|1 - 1|}{\rho(x)} = 0.$$

Thus

$$\lim_{n\to+\infty}\|L^*_{n,\gamma,\beta}(1;x)-1\|_\rho=0.$$

Using the equality (5), we can write

$$\begin{aligned}
\|L^*_{n,\gamma,\beta}(t;x)-x\|_\rho &= \sup_{0\le x\le b_n}\frac{|L^*_{n,\gamma,\beta}(t;x)-x|}{\rho(x)}\\
&= \sup_{0\le x\le b_n}\frac{1}{1+x^4}\left|\frac{b_n}{(n+\beta+1)}\left(\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)+\frac{2\gamma+1}{2}\right)-x\right|\\
&= \sup_{0\le x\le b_n}\frac{1}{1+x^4}\left|\frac{b_n}{(n+\beta+1)}\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)+\frac{(2\gamma+1)b_n}{2(n+\beta+1)}-x\right|\\
&= \sup_{0\le x\le b_n}\frac{1}{1+x^4}\left|\frac{nx}{n+\beta+1}-x-\frac{x^2n^{1-\alpha}}{b_n^2(1-\alpha)(n+\beta+1)}+\frac{(2\gamma+1)b_n}{2(n+\beta+1)}\right|\\
&\le \left|\frac{n}{n+\beta+1}-1\right|+\left|\frac{(2\gamma+1)b_n}{2(n+\beta+1)}\right|+\left|\frac{n^{1-\alpha}}{b_n^2(1-\alpha)(n+\beta+1)}\right|.
\end{aligned}$$

Since

$$\lim_{n\to+\infty}\left|\frac{n}{n+\beta+1}-1\right|+\left|\frac{(2\gamma+1)b_n}{2(n+\beta+1)}\right|+\left|\frac{n^{1-\alpha}}{b_n^2(1-\alpha)(n+\beta+1)}\right|=0,$$

then

$$\lim_{n\to+\infty}\|L^*_{n,\gamma,\beta}(t;x)-x\|_\rho=0.$$

Using the equality (6), we can write

$$\begin{aligned}
\|L^*_{n,\gamma,\beta}(t^2;x)-x^2\|_\rho &= \sup_{0\le x\le b_n}\frac{1}{\rho(x)}\left|\frac{b_n^2}{(n+\beta+1)^2}\left[\left(\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)\right)^2+\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)\left(1-h_i\left(\frac{x}{b_n}\right)\right)\right]\right.\\
&\quad\left.+\frac{b_n^2}{(n+\beta+1)^2}\left[(2\gamma+1)\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)+\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\right]-x^2\right|\\
&\le \sup_{0\le x\le b_n}\frac{1}{\rho(x)}\left|\frac{b_n^2}{(n+\beta+1)^2}\left(\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)\right)^2+\frac{b_n^2}{(n+\beta+1)^2}\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)\right.\\
&\quad\left.+\frac{b_n^2}{(n+\beta+1)^2}\left[(2\gamma+1)\sum_{i=1}^n h_i\left(\frac{x}{b_n}\right)+\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\right]-x^2\right|\\
&\le \sup_{0\le x\le b_n}\frac{1}{1+x^4}\left|\frac{b_n^2}{(n+\beta+1)^2}\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)^2+\frac{b_n^2}{(n+\beta+1)^2}\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)\right.\\
&\quad\left.+\frac{b_n^2}{(n+\beta+1)^2}\left[(2\gamma+1)\left(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^3(1-\alpha)}\right)+\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\right]-x^2\right|\\
&\le \sup_{0\le x\le b_n}\frac{1}{1+x^4}\left|\frac{b_n^2}{(n+\beta+1)^2}\left(\frac{n^2x^2}{b_n^2}-\frac{2x^3n^{2-\alpha}}{b_n^4(1-\alpha)}+\frac{x^4n^{2-2\alpha}}{b_n^6(1-\alpha)^2}\right)+\frac{b_nnx}{(n+\beta+1)^2}\right.\\
&\quad-\frac{x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2}+\frac{(2\gamma+1)b_nnx}{(n+\beta+1)^2}-\frac{(2\gamma+1)x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2}\\
&\quad\left.+\frac{b_n^2}{(n+\beta+1)^2}\left(\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\right)-x^2\right|
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\le x\le b_n} \frac{1}{1+x^4}\bigg| \frac{n^2x^2}{(n+\beta+1)^2} - \frac{2x^3n^{2-\alpha}}{b_n^2(n+\beta+1)^2(1-\alpha)} + \frac{x^4n^{2-2\alpha}}{b_n^4(n+\beta+1)^2(1-\alpha)^2} \\
&\quad + \frac{b_n n x}{(n+\beta+1)^2} - \frac{x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2} + \frac{(2\gamma+1)b_n n x}{(n+\beta+1)^2} - \frac{(2\gamma+1)x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2} \\
&\quad + \frac{b_n^2}{(n+\beta+1)^2}\Big(\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\Big) - x^2\bigg| \\
&= \sup_{0\le x\le b_n} \frac{1}{1+x^4}\bigg| \frac{x^4n^{2-2\alpha}}{b_n^4(n+\beta+1)^2(1-\alpha)^2} - \frac{2x^3n^{2-\alpha}}{b_n^2(n+\beta+1)^2(1-\alpha)} \\
&\quad + \Big(\frac{n^2}{(n+\beta+1)^2} - \frac{n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2} - \frac{(2\gamma+1)n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2} - 1\Big)x^2 \\
&\quad + \Big(\frac{b_n n}{(n+\beta+1)^2} + \frac{(2\gamma+1)b_n n}{(n+\beta+1)^2}\Big)x + \frac{b_n^2}{(n+\beta+1)^2}\Big(\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\Big)\bigg| \\
&\le \bigg|\frac{n^{2-2\alpha}}{b_n^4(n+\beta+1)^2(1-\alpha)^2}\bigg| + \bigg|\frac{n^{2-\alpha}}{b_n^2(n+\beta+1)^2(1-\alpha)}\bigg| \\
&\quad + \bigg|\frac{n^2}{(n+\beta+1)^2} - \frac{n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2} - \frac{(2\gamma+1)n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)^2} - 1\bigg| \\
&\quad + \bigg|\frac{b_n n}{(n+\beta+1)^2} + \frac{(2\gamma+1)b_n n}{(n+\beta+1)^2}\bigg| + \bigg|\frac{b_n^2}{(n+\beta+1)^2}\Big(\gamma+\frac{1}{3}+\gamma^2(n+\beta+1)^2\Big)\bigg|.
\end{aligned}
$$

Thus

$$\lim_{n\to+\infty} \|L^*_{n,\gamma,\beta}f - f\|_\rho = 0.$$

□

Now, we compute the rates of convergence of the operators $L^*_{n,\gamma,\beta}(f)$ to f by means of a classical approach, the modulus of continuity.

Let $f \in C[0,1]$. Then for $\delta > 0$, the modulus of continuity of f, which is denoted by $\omega_f(\delta)$, is defined by

$$\omega_f(\delta) = \sup_{\substack{x,y\in[0,1] \\ |x-y|\le\delta}} |f(x) - f(y)|.$$

Then, for any $\delta > 0$ and each $x \in [0,1]$, the inequality

$$|f(x) - f(y)| \le \omega_f(\delta)\Big(\frac{|x-y|}{\delta} + 1\Big) \tag{8}$$

holds.

One can estimate the rate of convergence of the sequence $L^*_{n,\gamma,\beta}(f)$ to f via the modulus of continuity as follows.

Theorem 3. *Let $f \in C[0,\infty)$. Then we have*

$$|L^*_{n,\gamma,\beta}f(x) - f(x)| \le 2\omega(\delta_{n,\gamma,\beta}),$$

where

$$\delta = \delta_{n,\gamma,\beta} = \sqrt{(L^*_{n,\gamma,\beta}(t-x)^2; x)}.$$

Proof. Using linearity of the operators $L^*_{n,\gamma,\beta}$, (4) and (8), we get

$$\begin{aligned}
|L^*_{n,\gamma,\beta} f(x) - f(x)| &\le (n+\beta+1)\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} |f(b_n t) - f(x)|\,dt \\
&\le (n+\beta+1)\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} \omega(f;|b_n t - x|)\,dt \\
&\le (n+\beta+1)\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} \Big[1+\Big(\frac{b_n t - x}{\delta}\Big)^2\Big]\omega(f;\delta)\,dt \\
&= (n+\beta+1)\omega(f;\delta)\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\Bigg\{\int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} dt + \int_{\frac{k+\gamma}{n+\beta+1}}^{\frac{k+\gamma+1}{n+\beta+1}} \frac{(b_n t - x)^2}{\delta^2}\,dt\Bigg\} \\
&= (n+\beta+1)\omega(f;\delta)\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\Bigg\{\frac{1}{n+\beta+1} + \frac{1}{\delta^2}\Bigg[b_n^2\frac{k^2+2\gamma k+\gamma^2}{(n+\beta+1)^3} + b_n^2\frac{k+\gamma}{(n+\beta+1)^3} \\
&\quad + b_n^2\frac{1}{3(n+\beta+1)^3} - \frac{2xb_n k + 2xb_n\gamma}{(n+\beta+1)^2} - \frac{b_n x}{(n+\beta+1)^2} + \frac{x^2}{n+\beta+1}\Bigg]\Bigg\}.
\end{aligned}$$

Now, we can write

$$\begin{aligned}
|L^*_{n,\gamma,\beta} f(x) - f(x)| &\le \omega(f;\delta)\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big) + \frac{\omega(f;\delta)}{\delta^2}\Bigg\{\sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{b_n^2 k^2}{(n+\beta+1)^3} \\
&\quad + \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{b_n^2 2\gamma k}{(n+\beta+1)^2} + \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{b_n^2\gamma^2}{(n+\beta+1)^2} + \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{b_n^2 k}{(n+\beta+1)^2} \\
&\quad + \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{b_n^2\gamma}{(n+\beta+1)^2} + \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{b_n^2}{3(n+\beta+1)^2} - \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{2xb_n k}{(n+\beta+1)} \\
&\quad - \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{2xb_n\gamma}{(n+\beta+1)} + \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)\frac{b_n x}{(n+\beta+1)} + \sum_{k=0}^{n} a_{n,k}\Big(\frac{x}{b_n}\Big)x^2\Bigg\} \\
&= \omega(f;\delta) + \frac{\omega(f;\delta)}{\delta^2}\Bigg\{\frac{b_n^2}{(n+\beta+1)^3}\sum_{k=0}^{n} k^2 a_{n,k}\Big(\frac{x}{b_n}\Big) + \frac{b_n^2 2\gamma}{(n+\beta+1)^2}\sum_{k=0}^{n} k a_{n,k}\Big(\frac{x}{b_n}\Big) \\
&\quad + \frac{b_n^2\gamma^2}{(n+\beta+1)^2} + \frac{b_n^2}{(n+\beta+1)^2}\sum_{k=0}^{n} k a_{n,k}\Big(\frac{x}{b_n}\Big) + \frac{b_n^2\gamma}{(n+\beta+1)^2} + \frac{b_n^2}{3(n+\beta+1)^2} \\
&\quad - \frac{2xb_n}{(n+\beta+1)}\sum_{k=0}^{n} k a_{n,k}\Big(\frac{x}{b_n}\Big) - \frac{2xb_n\gamma}{(n+\beta+1)} + \frac{b_n x}{(n+\beta+1)} + x^2\Bigg\}.
\end{aligned}$$

By (4), (5) and (6), we have

$$\begin{aligned}
|L^*_{n,\gamma,\beta} f(x) - f(x)| &\le \omega(f;\delta) + \frac{\omega(f;\delta)}{\delta^2}\Bigg\{\frac{b_n^2}{(n+\beta+1)^3}\Bigg[\Bigg(\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big)\Bigg)^2 \\
&\quad + \sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big)\Big(1-h_i\Big(\frac{x}{b_n}\Big)\Big)\Bigg] + \frac{b_n^2(2\gamma+1)}{(n+\beta+1)^2}\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big) + \frac{b_n^2\gamma^2}{(n+\beta+1)^2} + \frac{b_n x}{(n+\beta+1)} \\
&\quad - \frac{2xb_n}{(n+\beta+1)}\sum_{i=1}^{n} h_i\Big(\frac{x}{b_n}\Big) + \frac{b_n^2\gamma}{(n+\beta+1)^2} + \frac{b_n^2}{3(n+\beta+1)^2} - \frac{2xb_n\gamma}{(n+\beta+1)} + x^2\Bigg\}
\end{aligned}$$

$$\leq \omega(f;\delta)\Bigg(1+\frac{1}{\delta^2}\Bigg\{\frac{b_n^2}{(n+\beta+1)^3}\Bigg[\Bigg(\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)\Bigg)^2+\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)\Bigg]$$
$$+\frac{b_n^2(2\gamma+1)}{(n+\beta+1)^2}\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)+\frac{b_n^2\gamma^2}{(n+\beta+1)^2}-\frac{2xb_n}{(n+\beta+1)}\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)$$
$$+\frac{b_n^2\gamma}{(n+\beta+1)^2}+\frac{b_n^2}{3(n+\beta+1)^2}-\frac{2xb_n\gamma}{(n+\beta+1)}+\frac{b_nx}{(n+\beta+1)}+x^2\Bigg\}\Bigg).$$

Taking into account this inequality in (8) and Lemma 2, we obtain

$$|L^*_{n,\gamma,\beta}f(x)-f(x)|\leq \omega(f;\delta)(1+\delta^{-2}L^*_{n,\gamma,\beta}((t-x)^2;x)).$$

If we get $\delta=\delta_{n,\gamma,\beta}(x)=\sqrt{(L^*_{n,\gamma,\beta}(t-x)^2;x)}$, we obtain the desired. □

3 Voronovskaja type theorem

Now, we obtain a Voronovskaja-type asymptotic estimate of the operators (3).

Theorem 4. *Let $f\in L_1[0,\infty)$ and f', f'' exist at a fixed point $x\in(0,\infty)$. Then we have*

$$\lim_{n\to+\infty}\frac{n}{b_n}(L^*_{n,\gamma,\beta}f(x)-f(x))=\frac{f''(x)}{2!}x.$$

Proof. By the Taylor expansion, we may write

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+(t-x)^2\delta(t;x),$$

where $\delta(t;x)\to 0$ as $t\to x$ and is a continuous function on $[0,\infty)$. Taking into account the linearity of the operators $L^*_{n,\gamma,\beta}f$ and applying it to both sides of the above Taylor expansion with a simple calculations, we obtain

$$\frac{n}{b_n}(L^*_{n,\gamma,\beta}f(x)-f(x))=\frac{n}{b_n}L^*_{n,\gamma,\beta}((t-x);x)f'(x)$$
$$+\frac{n}{b_n}L^*_{n,\gamma,\beta}((t-x)^2;x)\frac{f''(x)}{2!}+\frac{n}{b_n}L^*_{n,\gamma,\beta}((t-x)^2\delta(t;x);x).$$

Therefore, using Lemmas 2, 3, 4, we get

$$f'(x)L^*_{n,\gamma,\beta}((t-x);x)=f'(x)\Bigg(\frac{b_n}{(n+\beta+1)}\Bigg(\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)+\frac{2\gamma+1}{2}\Bigg)-x\Bigg)$$
$$=f'(x)\Bigg(\frac{b_n}{(n+\beta+1)}\Bigg(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^2(1-\alpha)}\Bigg)+\frac{b_n(2\gamma+1)}{2(n+\beta+1)}-x\Bigg)$$
$$=f'(x)\Bigg(\frac{nx}{(n+\beta+1)}-\frac{x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)}-x\Bigg)$$
$$=f'(x)\Bigg(\Big(\frac{n}{(n+\beta+1)}-1\Big)x-\frac{x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)}\Bigg)$$

and

$$\begin{aligned}
\frac{f''(x)}{2!}L^*_{n,\gamma,\beta}((t-x)^2;x) &= \frac{f''(x)}{2!}\Bigg[\Big(\frac{b_n}{n+\beta+1}\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)+\frac{b_n(2\gamma+1)}{2(n+\beta+1)}-x\Big)^2 \\
&\quad+\frac{b_n^2}{(n+\beta+1)^2}\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)\Big(1-h_i\Big(\frac{x}{b_n}\Big)\Big)\Bigg] \\
&\leq \frac{f''(x)}{2!}\Bigg[(L^*_{n,\gamma,\beta}((t-x);x))^2+\frac{b_n^2}{(n+\beta+1)^2}\sum_{i=1}^{n}h_i\Big(\frac{x}{b_n}\Big)\Bigg] \\
&= \frac{f''(x)}{2!}\Bigg[\Big(\frac{b_n}{n+\beta+1}\Big(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^2(1-\alpha)}\Big)+\frac{b_n(2\gamma+1)}{2(n+\beta+1)}-x\Big)^2 \\
&\quad+\frac{b_n^2}{(n+\beta+1)^2}\Big(\frac{nx}{b_n}-\frac{x^2n^{1-\alpha}}{b_n^2(1-\alpha)}\Big)\Bigg] \\
&= \frac{f''(x)}{2!}\Bigg[\Big(\frac{nx}{n+\beta+1}-\frac{x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)}+\frac{b_n(2\gamma+1)}{2(n+\beta+1)}-x\Big)^2 \\
&\quad+\frac{b_nnx}{(n+\beta+1)^2}-\frac{x^2n^{1-\alpha}}{(1-\alpha)(n+\beta+1)^2}\Bigg] \\
&= \frac{f''(x)}{2!}\Bigg[\Big(\Big(\frac{n}{n+\beta+1}-1\Big)x-\frac{x^2n^{1-\alpha}}{b_n(1-\alpha)(n+\beta+1)}+\frac{b_n(2\gamma+1)}{2(n+\beta+1)}\Big)^2 \\
&\quad+\frac{b_nnx}{(n+\beta+1)^2}-\frac{x^2n^{1-\alpha}}{(1-\alpha)(n+\beta+1)^2}\Bigg].
\end{aligned}$$

Thus, we can write

$$\lim_{n\to+\infty}\frac{n}{b_n}(L^*_{n,\gamma,\beta}f(x)-f(x))=\frac{f''(x)}{2!}x+\lim_{n\to+\infty}\frac{n}{b_n}L^*_{n,\gamma,\beta}((t-x)^2\delta(t;x);x). \tag{9}$$

Hence, by the Cauchy-Schwarz inequality, we have

$$\frac{n}{b_n}|L^*_{n,\gamma,\beta}(\delta(t;x)(t-x)^2;x)|\leq\sqrt[2]{L^*_{n,\gamma,\beta}(\delta^2(t;x);x)}\sqrt[2]{\Big(\frac{n}{b_n}\Big)^2L^*_{n,\gamma,\beta}((t-x)^4;x)}. \tag{10}$$

Since

$$\lim_{t\to x}\Big(\frac{n}{b_n}\Big)^2L^*_{n,\gamma,\beta}((t-x)^4;x)$$

is bounded by Lemma 4 and $\delta^2(x;x)=0$, by considering (10) in (9), we obtain

$$\lim_{n\to+\infty}\frac{n}{b_n}L^*_{n,\gamma,\beta}(\delta(t;x)(t-x)^2;x)=0,$$

which completes the proof. □

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У цьому дослідженні ми визначаємо оператори Лотоцького-Хлодовського типу Канторовича-Станку та отримуємо деякі властивості апроксимації у зваженому просторі цих операторів. Крім того, ми доводимо результати типу Вороновської для операторів Лотоцького-Хлодовського типу Канторовича-Станку.

Ключові слова і фрази: оператор Лотоцького-Хлодовського типу Канторовича-Станку, теорема Коровкіна, швидкість збіжності, апроксимація.