ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2024, **16** (2), 391–400
doi:10.15330/cmp.16.2.391-400

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2024, Т.16, №2, С.391–400

On structure of branched continued fractions

Antonova T.M.

The paper provides a survey of various multidimensional generalizations of continued fractions that arose when solving the problem of approximating functions of one or several variables, including some hypergeometric functions. It is shown that all these generalizations can be considered as separate cases of the general concept of a branched continued fraction, the definition of which is given in the work.

Key words and phrases: branched continued fraction, holomorphic function, approximation by rational functions.

Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine
E-mail: tamara.m.antonova@lpnu.ua

Introduction

It is well known that continued fractions have effective applications in the theory of approximation of functions of one variable [17, 28, 36, 43]. V.Ya. Skorobohatko proposed the problem of constructing a multidimensional generalization of the continued fraction, which would play the same role in function theory of several variables as multiple power series.

In 1966, the first work on an infinite branched continued fraction (BCF) of the general form was published, although there were partial cases earlier (for more details we refer the reader to the review paper [16]). BCF is written as follows

$$v_0+\sum_{i_1=1}^{N}\cfrac{u_{i_1}}{v_{i_1}+\sum\limits_{i_2=1}^{N}\cfrac{u_{i_1,i_2}}{v_{i_1,i_2}+\ddots+\sum\limits_{i_k=1}^{N}\cfrac{u_{i_1,i_2,\ldots,i_k}}{v_{i_1,i_2,\ldots,i_k}+\ddots}}}, \tag{1}$$

where $v_0, u_{i_1}, v_{i_1}, \ldots, u_{i_1,\ldots,i_k}, v_{i_1,\ldots,i_k}, \ldots$ are so-called elements of the fraction (they may be numbers, functions, matrices, operators, etc.). V.Ya. Skorobohatko approached the concept of BCF as an expression of the form (1), considering a tree graph of the most general form (see [41]), since the geometric representation of a continued fraction is an oriented tree graph. By analogy with the definition of a continued fraction [43], P.I. Bodnarchuk proposed the definition of BCF based on combinations of multidimensional fractional-linear mappings [11]. D.I. Bodnar

УДК 517.5
2020 *Mathematics Subject Classification:* 32A17, 32A10, 41A20.

later proposed to consider BCF (1) as a sequence of its approximants (see [16])

$$f_0 = v_0, \quad f_1 = v_0 + \sum_{i_1=1}^{N} \frac{u_{i_1}}{v_{i_1}}, \quad f_2 = v_0 + \sum_{i_1=1}^{N} \frac{u_{i_1}}{v_{i_1} + \sum\limits_{i_2=1}^{N} \frac{u_{i_1,i_2}}{v_{i_1,i_2}}}, \ldots,$$

$$f_k = v_0 + \sum_{i_1=1}^{N} \frac{u_{i_1}}{v_{i_1} + \sum\limits_{i_2=1}^{N} \frac{u_{i_1,i_2}}{v_{i_1,i_2} + \ddots + \sum\limits_{i_k=1}^{N} \frac{u_{i_1,i_2,\ldots,i_k}}{v_{i_1,i_2,\ldots,i_k}}}}, \quad \text{etc.}$$

Today, BCFs are used not only to approximate functions of one or several variables [1–3, 5, 22, 25], but also in various problems of applied mathematics, chemistry, mathematical physics, and engineering [29, 30, 35].

One of the approaches to representing analytic functions in the form of continued fractions is the construction of continued fractions corresponding to given formal power series [17, 28]. Another approach used to represent hypergeometric functions by continued fractions is the method of constructing continued fractions based on recurrence relations of hypergeometric series [28, 43]. Both approaches are used in the analytical theory of BCFs. But unlike the one-dimensional case, the problem of constructing BCF (1) corresponding to given formal multiple power series is solved ambiguously. Moreover, the recurrence relations of hypergeometric series also does not always lead to the construction of (1). Solving these and related problems led to the emergence BCFs with various branches [1, 10, 23, 33, 39, 40]. In addition, many different sequences of approximants appeared (see [8, 9]). As a result, in each specific case, a system of notation, a sequence of approximants for a certain BCF is often suggested.

This paper proposes an approach to the definition of BCF based on its structure. All known BCF structures are described.

1 Definition of BCF

Let r be a natural number and $\mathcal{G} = \{1, 2, \ldots, N\}$. For each $k \geq 0$ and $i_k^{(p)} \in \mathcal{G}$, $1 \leq p \leq r$, let $I_k^{(r)} = (i_k^{(1)}, i_k^{(2)}, \ldots, i_k^{(r)})$ be a multiple index. Next, let $G\big(I_0^{(r)}\big)$ be a subset of $\mathcal{G}^r$, where $\mathcal{G}^r = \underbrace{\mathcal{G} \times \mathcal{G} \times \ldots \times \mathcal{G}}_{r}$. For each $r \geq 2$, $k \geq 1$ and $1 \leq p \leq k$, let

$$\mathcal{I}_{(k)}^{(r)} = \big(I_0^{(r)}, I_1^{(r)}, \ldots, I_k^{(r)}\big) = \big(i_0^{(1)}, i_0^{(2)}, \ldots, i_0^{(r)}, i_1^{(1)}, i_1^{(2)}, \ldots, i_1^{(r)}, \ldots, i_k^{(1)}, i_k^{(2)} \ldots, i_k^{(r)}\big)$$

be a multiple multiindex, $G\big(\mathcal{I}_{(p-1)}^{(r)}\big)$ be a subset of $\mathcal{G}^r$ herewith $G\big(\mathcal{I}_{(0)}^{(r)}\big) = G\big(I_0^{(r)}\big)$, and let

$$\mathcal{J}_0^{(r)} = \big\{\mathcal{I}_{(k)}^{(r)} : \ I_p^{(r)} \in G\big(\mathcal{I}_{(p-1)}^{(r)}\big), \ G\big(\mathcal{I}_{(p-1)}^{(r)}\big) \subseteq \mathcal{G}^r, \ 1 \leq p \leq k, \ k \geq 1\big\}$$

be a set of multiple multiindices.

A *branched continued fraction* is called an expression of the form

$$v_{I_0^{(r)}} + \sum_{I_1^{(r)} \in G(\mathcal{I}_{(0)}^{(r)})} \frac{u_{\mathcal{I}_{(1)}^{(r)}}}{v_{\mathcal{I}_{(1)}^{(r)}} + \sum\limits_{I_2^{(r)} \in G(\mathcal{I}_{(1)}^{(r)})} \frac{u_{\mathcal{I}_{(2)}^{(r)}}}{v_{\mathcal{I}_{(2)}^{(r)}} + \ddots + \sum\limits_{I_k^{(r)} \in G(\mathcal{I}_{(k-1)}^{(r)})} \frac{u_{\mathcal{I}_{(k)}^{(r)}}}{v_{\mathcal{I}_{(k)}^{(r)}} + \ddots}}}, \tag{2}$$

where $v_{I_0^{(r)}}, u_{\mathcal{I}_{(1)}^{(r)}}, v_{\mathcal{I}_{(1)}^{(r)}}, \ldots, u_{\mathcal{I}_{(k)}^{(r)}}, v_{\mathcal{I}_{(k)}^{(r)}}, \ldots$ are called elements (they may be numbers, functions, matrices, operators, etc.).

Note that if $r = 1$, then $I_0^{(1)} = i_0 \in \mathcal{G}$ and for given i_0 and for each $k \geq 1$, $\mathcal{I}_{(k)}^{(1)}$ can be written as follows

$$\mathcal{I}_{(k)}^{(1)} = i(k) = (i_0, i_1, \ldots, i_k), \quad k \geq 1,$$

and

$$\mathcal{J}_0^{(1)} = \mathcal{J}_0 = \{i(k) : \ i_p \in G(i(p-1)),\ G(i(p-1)) \subseteq \mathcal{G},\ 1 \leq p \leq k,\ k \geq 1\}, \quad i(0) = i_0,$$

be a set of simple multiindex.

Thus, (2) can be written as

$$v_{i_0} + \sum_{i_1 \in G(i(0))} \cfrac{u_{i(1)}}{v_{i(1)} + \sum\limits_{i_2 \in G(i(1))} \cfrac{u_{i(2)}}{v_{i(2)} + \ddots + \sum\limits_{i_k \in G(i(k-1))} \cfrac{u_{i(k)}}{v_{i(k)} + \ddots}}}. \tag{3}$$

2 Structures of BCF

We begin our description of BCF structures with a general BCF.

2.1 BCF with N branching branches

This is a BCF of the form (see [12])

$$v_{i_0} + \sum_{i_1=1}^{N} \cfrac{u_{i(1)}}{v_{i(1)} + \sum\limits_{i_2=1}^{N} \cfrac{u_{i(2)}}{v_{i(2)} + \ddots}}. \tag{4}$$

Here $r = 1$, $\mathcal{G} = \{1, 2, \ldots, N\}$, $G(i(k-1)) = \mathcal{G}$ for $k \geq 1$, $i(0) = i_0$, and

$$\mathcal{J}_0 = \big\{i(k) : \ 1 \leq i_p \leq N,\ 1 \leq p \leq k,\ k \geq 1\big\}.$$

Such BCFs arose, in particular, when constructing of BCF expansions for ratios of Appell's hypergeometric functions F_1 [27], F_3 [37], F_4 [15], Horn's H_3 [2], Lauricella-Saran's F_S [26], Lauricella's $F_D^{(N)}$ [7], and its confluent form $\Phi_D^{(N)}$ [3]. BCFs of the form (4) were also obtained during the construction of algorithms for expansion of formal multiple power series into a multidimensional C-fraction [15] and a multidimensional g-fraction [24].

Sometimes it is convenient to write BCF as follows

$$v_{i_0} + \sum_{i_1=L_{i(0)}}^{N_{i(0)}} \cfrac{u_{i(1)}}{v_{i(1)} + \sum\limits_{i_2=L_{i(1)}}^{N_{i(1)}} \cfrac{u_{i(2)}}{v_{i(2)} + \ddots}},$$

for instance, the BCF expansions of Horn's confluent hypergeometric function H_6 ratios have the form (see [6])

$$v_{i_0} + \sum_{i_1=2-[(i_0-1)/2]}^{3-[(i_0-1)/2]} \cfrac{u_{i(1)}}{v_{i(1)} + \sum\limits_{i_2=2-[(i_1-1)/2]}^{3-[(i_1-1)/2]} \cfrac{u_{i(2)}}{v_{i(2)} + \ddots}}, \quad i_0 \in \{1, 2, 3\},$$

where $[\,\cdot\,]$ denotes an integer part. It is clear that

$$\mathcal{G} = \{1, 2, 3\}, \quad G(i(k-1)) = \{2 - [(i_{k-1}-1)/2], 3 - [(i_{k-1}-1)/2]\}, \quad k \geq 1, \quad i(0) = i_0.$$

2.2 BCF with different number of branching branches

A BCF structure

$$1+\sum_{i_1=1}^{2}\cfrac{u_{i(1)}}{1+\cfrac{v_{i(1)}}{1+\sum\limits_{i_2=1}^{2}\cfrac{u_{i(2)}}{1+\cfrac{v_{i(2)}}{1+\ddots}}}}$$

was obtained for Appell's hypergeometric functions F_2 in [13] and F_3 in [14], which can be rewritten as

$$1+\sum_{i_1\in G(i(0))}\cfrac{w_{i(1)}}{1+\sum\limits_{i_2\in G(i(1))}\cfrac{w_{i(2)}}{1+\ddots}}$$

where, for $k\geq 1$,

$$\mathcal{G}=G(i(2k-2))=\{1,2\},\quad G(i(2k-1))=\{i_{2k-1}\},\quad w_{i(2k-1)}=u_{i(k)},\quad w_{i(2k)}=v_{i(k)}.$$

For the Appel's hypergeometric function F_4, the following BCF was considered (see [25])

$$\cfrac{1}{1+\sum\limits_{i_1\in G(i(0))}\cfrac{u_{i(1)}}{1+\sum\limits_{i_2\in G(i(1))}\cfrac{u_{i(2)}}{1+\ddots}}},$$

where

$$\mathcal{G}=G(i(0))=G(i(2k))=\{1,2\},\quad G(i(2k-1))=\{3-i_{2k-1}\},\quad k\geq 1.$$

In works [10,18,21,23], a BCF of the structure

$$v_{i_0}+\sum_{i_1=1}^{N}\cfrac{u_{i(1)}}{v_{i(1)}+\sum\limits_{i_2=1}^{i_1}\cfrac{u_{i(2)}}{v_{i(2)}+\sum\limits_{i_3=1}^{i_2}\cfrac{u_{i(3)}}{v_{i(3)}+\ddots}}} \tag{5}$$

was considered, that is

$$G(i(0))=\mathcal{G}=\{1,2,\ldots,N\},\quad G(i(k))=\{1,2,\ldots,i_k\},\quad k\geq 1.$$

For the ratios of Lauricella-Saran's hypergeometric function F_K with certain values of parameters the following BCFs

$$1-z_1+\cfrac{u_{i(1)}z_3}{1-z_2+\cfrac{u_{i(2)}z_3}{1-z_1+\cfrac{u_{i(3)}z_3}{1-\ddots}}} \tag{6}$$

and

$$1 - z_2 + \cfrac{u_{i(1)}z_3}{1 - z_1 + \cfrac{u_{i(2)}z_3}{1 - z_2 + \cfrac{u_{i(3)}z_3}{1 - \ddots}}} \tag{7}$$

were constructed (see [1]).

In works [1,19], these BCFs are considered as a confluent BCF of the form (5) with $N = 3$. Indeed, for (6) we have $\mathcal{G} = \{1,2,3\}$, $G(i(0)) = \{1,3\}$,

$$G(i(2k-1)) = \begin{cases} \varnothing, & \text{if } i_{2k-1} = 1, \\ \{2,3\}, & \text{if } i_{2k-1} = 3, \end{cases}$$

$$G(i(2k)) = \begin{cases} \varnothing, & \text{if } i_{2k} = 2, \\ \{1,3\}, & \text{if } i_{2k} = 3, \end{cases}$$

with $k \geq 1$, and for (7) we have $\mathcal{G} = \{1,2,3\}$, $G(i(0)) = \{2,3\}$,

$$G(i(2k-1)) = \begin{cases} \varnothing, & \text{if } i_{2k-1} = 2, \\ \{1,3\}, & \text{if } i_{2k-1} = 3, \end{cases}$$

$$G(i(2k)) = \begin{cases} \varnothing, & \text{if } i_{2k} = 1, \\ \{2,3\}, & \text{if } i_{2k} = 3, \end{cases}$$

with $k \geq 1$.

BCF of the form

$$c_0 + \Phi_0 + \cfrac{c_{1,1}}{1 + \Phi_1 + \cfrac{c_{2,2}}{1 + \Phi_2 + \ddots}}, \tag{8}$$

where

$$\Phi_k = \cfrac{c_{k+1,k}}{1 + \cfrac{c_{k+2,k}}{1 + \dots}} + \cfrac{c_{k,k+1}}{1 + \cfrac{c_{k,k+2}}{1 + \ddots}}, \quad k \geq 0,$$

was studied in [31,33,38,39,42]. It is easy to show that (8) can be written in the form (3), where $\mathcal{G} = G(i(0)) = \{1,2,3\}$,

$$G(i(k-1)) = \begin{cases} \{1\}, & \text{if } i_{k-1} = 1, \\ \{2\}, & \text{if } i_{k-1} = 2, \\ \{1,2,3\}, & \text{if } i_{k-1} = 3, \end{cases}$$

for $k \geq 2$, and

$$u_{i(k)} = \begin{cases} c_{k,0}, & \text{if } i_1 = 1,\ k \geq 1, \\ c_{0,k}, & \text{if } i_1 = 2,\ k \geq 1, \\ c_{k,k}, & \text{if } i_k = 3,\ k \geq 1, \\ c_{k,p}, & \text{if } i_p = 3,\ i_{p+1} = 1,\ 1 \leq p \leq k-1,\ k \geq 2, \\ c_{p,k}, & \text{if } i_p = 3,\ i_{p+1} = 2,\ 1 \leq p \leq k-1,\ k \geq 2. \end{cases}$$

There are few works devoted to BCF of type (8) with N branching branches (see [32,34,39]). In particular, when $N = 3$ one has

$$c_{0,0,0} + \Psi_0 + \cfrac{c_{1,1,1}}{\Psi_1 + \cfrac{c_{2,2,2}}{\Psi_2 + \ddots}},$$

where

$$\begin{aligned}\Psi_i = 1 &+ \cfrac{c_{i+1,i,i}}{1 + \cfrac{c_{i+2,i,i}}{1 + \ddots}} + \cfrac{c_{i,i+1,i}}{1 + \cfrac{c_{i,i+2,i}}{1 + \ddots}} + \cfrac{c_{i,i,i+1}}{1 + \cfrac{c_{i,i,i+2}}{1 + \ddots}} + \cfrac{c_{i+1,i+1,i}}{\Phi_{i+1,i+1,i} + \cfrac{c_{i+2,i+2,i}}{\Phi_{i+2,i+2,i} + \ddots}} \\ &+ \cfrac{c_{i+1,i,i+1}}{\Phi_{i+1,i,i+1} + \cfrac{c_{i+2,i,i+2}}{\Phi_{i+2,i,i+2} + \ddots}} + \cfrac{c_{i,i+1,i+1}}{\Phi_{i,i+1,i+1} + \cfrac{c_{i,i+2,i+2}}{\Phi_{i,i+2,i+2} + \ddots}}\end{aligned}$$

for $i \geq 0$, and

$$\Phi_{k,k,i} = 1 + \cfrac{c_{k+1,k,i}}{1 + \cfrac{c_{k+2,k,i}}{1 + \ddots}} + \cfrac{c_{k,k+1,i}}{1 + \cfrac{c_{k,k+2,i}}{1 + \ddots}},$$

$$\Phi_{k,i,k} = 1 + \cfrac{c_{k+1,i,k}}{1 + \cfrac{c_{k+2,i,k}}{1 + \ddots}} + \cfrac{c_{k,i,k+1}}{1 + \cfrac{c_{k,i,k+2}}{1 + \ddots}},$$

$$\Phi_{i,k,k} = 1 + \cfrac{c_{i,k+1,k}}{1 + \cfrac{c_{i,k+2,k}}{1 + \ddots}} + \cfrac{c_{i,k,k+1}}{1 + \cfrac{c_{i,k,k+2}}{1 + \ddots}}$$

for $i \geq 0$ and $k \geq 0$. For this BCF one has $\mathcal{G} = G(i(0)) = \{1, 2, 3, 4, 5, 6, 7\}$, and

$$G(i(k-1)) = \begin{cases} \{1\}, & \text{if } i_{k-1} = 1, \\ \{2\}, & \text{if } i_{k-1} = 2, \\ \{3\}, & \text{if } i_{k-1} = 3, \\ \{1,2,4\}, & \text{if } i_{k-1} = 4, \\ \{1,3,5\}, & \text{if } i_{k-1} = 5, \\ \{2,3,6\}, & \text{if } i_{k-1} = 6, \\ \{1,2,3,4,5,6,7\}, & \text{if } i_{k-1} = 7, \end{cases} \quad k \geq 2.$$

One more BCF with two branching branches

$$d_0 + F_{0,0} + \cfrac{d_{1,0}}{1 + F_{1,0} + \cfrac{d_{2,0}}{1 + F_{2,0} + \ddots}} + \cfrac{d_{0,1}}{1 + F_{0,1} + \cfrac{d_{0,2}}{1 + F_{0,2} + \ddots}},$$

where

$$F_{i,0} = \cfrac{d_{1+i,1}}{1 + \cfrac{d_{2+i,2}}{1 + \ddots}}, \quad F_{0,i} = \cfrac{d_{1,1+i}}{1 + \cfrac{d_{2,2+i}}{1 + \ddots}}, \quad i \geq 0,$$

can be found in [40].

For this BCF we have $\mathcal{G} = G(i(0)) = \{1,2,3\}$,

$$G(i(k-1)) = \begin{cases} \{1,3\}, & \text{if } i_{k-1} = 1, \\ \{2,3\}, & \text{if } i_{k-1} = 2, \quad k \geq 2, \\ \{3\}, & \text{if } i_{k-1} = 3, \end{cases}$$

and

$$u_{i(k)} = \begin{cases} d_{k,0}, & \text{if } i_k = 1,\ k \geq 1, \\ d_{0,k}, & \text{if } i_k = 2,\ k \geq 1, \\ d_{k,k}, & \text{if } i_1 = 3,\ k \geq 1, \\ d_{k,k-p}, & \text{if } i_p = 1,\ i_{p+1} = 3,\ 1 \leq p \leq k-1,\ k \geq 2, \\ d_{k-p,k}, & \text{if } i_p = 2,\ i_{p+1} = 3,\ 1 \leq p \leq k-1,\ k \geq 2. \end{cases}$$

2.3 BCF with multiple multiindex and different number of branching branches

Finally, we consider two BCFs, which arose during the construction of expansion for ratios of hypergeometric functions ${}_3F_2$ (see [5]) and H_4 (see [4,20]), that is

$$1 + \sum_{\substack{i_1 \in \{1+\delta^1_{i_0}, 2\},\ j_1 \in \{1,2\} \\ |i_1 - j_1| \neq |i_0 - j_0|}} \cfrac{c^{(ij)_0}_{(ij)_1} z}{1 + \sum\limits_{\substack{i_2 \in \{1+\delta^1_{i_1}, 2\},\ j_2 \in \{1,2\} \\ |i_2 - j_2| \neq |i_1 - j_1|}} \cfrac{c^{(ij)_0}_{(ij)_2} z}{1 + \ddots}}, \tag{9}$$

where $(ij)_0 = (i_0, j_0)$, $(ij)_0 \in \{(1,1); (1,2); (2,1); (2,2)\}$, $(ij)_k = (i_1, j_1, i_2, j_2, \ldots, i_k, j_k)$,

$$1 + \delta^1_{i_{k-1}} \leq i_k \leq 2, \quad j_k \in \{1,2\}, \quad |i_k - j_k| \neq |i_{k-1} - j_{k-1}|, \quad k \geq 1,$$

and

$$1 + \sum_{\substack{i_1 \in \{1, 2-\delta^2_{i_0}\} \\ j_1 = i_1 + \delta^2_{i_0}}} \cfrac{h^{(ij)_0}_{(ij)_1} z_{j_1}}{1 + \sum\limits_{\substack{i_2 \in \{1, 2-\delta^2_{i_1}\} \\ j_2 = i_2 + \delta^2_{i_1}}} \cfrac{h^{(ij)_0}_{(ij)_2} z_{j_2}}{1 + \ddots}}, \tag{10}$$

where $(ij)_0 = (i_0, j_0)$, $(ij)_0 \in \{(1,1); (1,2); (2,2)\}$, $(ij)_k = (i_1, j_1, i_2, j_2, \ldots, i_k, j_k)$,

$$j_k = i_k + \delta^2_{i_{k-1}}, \quad 1 \leq i_k \leq 2 - \delta^2_{i_{k-1}}, \quad k \geq 1,$$

respectively.

Here and subsequently, δ^q_p denotes the Kronecker delta.

Thus, for both (9) and (10) we have $r = 2$, $\mathcal{G} = \{1; 2\}$, $I^{(2)}_0 = (i^{(1)}_0, i^{(2)}_0)$, $I^{(2)}_0 \in \mathcal{G}^2$, and

$$\mathcal{I}^{(2)}_{(k)} = \big(I^{(2)}_0, \ldots, I^{(2)}_k\big) = \big(i^{(1)}_0, i^{(2)}_0, \ldots, i^{(1)}_k, i^{(2)}_k\big), \quad k \geq 1.$$

Furthermore, for (9) we obtain $(ij)_0 \in \{(1,1); (1,2); (2,1); (2,2)\}$ and

$$G\big(\mathcal{I}^{(2)}_{(k-1)}\big) = \big\{\mathcal{I}^{(2)}_{(k)} :\ 1 + \delta^1_{i_{k-1}} \leq i_k \leq 2,\ 1 \leq j_k \leq 2,\ |i_k - j_k| \neq |i_{k-1} - j_{k-1}|,\ k \geq 1\big\},$$

and for (10) we have $(ij)_0 \in \{(1,1); (1,2); (2,2)\}$,

$$G\big(\mathcal{I}^{(2)}_{(k-1)}\big) = \big\{\mathcal{I}^{(2)}_{(k)} :\ 1 \leq i_k \leq 2 - \delta^2_{i_{k-1}},\ j_k = i_k + \delta^2_{i_{k-1}},\ k \geq 1\big\}.$$

Conclusions

This paper proposes an approach to definiting the concept of a branched continued fraction based on its structure. This approach is a development of Skorobohatko's idea of representing continued fractions and their multidimensional generalizations in the form of tree graphs. It is shown that BCFs that arise in various problems can be interpreted as separate cases of the proposed concept. In all cases considered here, the number of branching branches is finite, however our approach can be generalized to an infinite (countable) set of branching branches.

The future direction of the research consists in the construction and description of the methodology for the study of sequences of BCF approximants.

References

[1] Antonova T., Dmytryshyn R., Goran V. *On the analytic continuation of Lauricella-Saran hypergeometric function* $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$. Math. 2023, **11** (21), 4487. doi:10.3390/math11214487

[2] Antonova T., Dmytryshyn R., Kravtsiv V. *Branched continued fraction expansions of Horn's hypergeometric function* H_3 *ratios*. Math. 2021, **9** (2), 148. doi:10.3390/math9020148

[3] Antonova T., Dmytryshyn R., Kurka R. *Approximation for the ratios of the confluent hypergeometric function* $\Phi_D^{(N)}$ *by the branched continued fractions*. Axioms 2022, **11** (9), 426. doi:10.3390/axioms11090426

[4] Antonova T., Dmytryshyn R., Lutsiv I.-A., Sharyn S. *On some branched continued fraction expansions for Horn's hypergeometric function* $H_4(a, b; c, d; z_1, z_2)$ *ratios*. Axioms 2023, **12** (3), 299. doi:10.3390/axioms12030299

[5] Antonova T., Dmytryshyn R., Sharyn S. *Generalized hypergeometric function* ${}_3F_2$ *ratios and branched continued fraction expansions*. Axioms 2021, **10** (4), 310. doi:10.3390/axioms10040310.

[6] Antonova T., Dmytryshyn R., Sharyn S. *Branched continued fraction representations of ratios of Horn's confluent function* H_6. Constr. Math. Anal. 2023, **6** (1), 22–37. doi:10.33205/cma.1243021

[7] Antonova T.M., Hoyenko N.P. *Approximation of Lauricella's functions* F_D *ratio by Nörlund's branched continued fraction in the complex domain*. Mat. Metody Fiz.-Mekh. Polya 2004, **47** (2), 7–15. (in Ukrainian)

[8] Antonova T.M., Sus' O.M. *Sufficient conditions for the equivalent convergence of sequences of different approximants for two-dimensional continued fractions*. J. Math. Sci. 2018, **228** (1), 1–10. doi:10.1007/s10958-017-3601-3 (translation of Mat. Metody Fiz.-Mekh. Polya 2015, **58** (4), 7–14. (in Ukrainian))

[9] Antonova T.M., Sus' O.M, Vozna S.M. *Convergence and estimation of the truncation error for the corresponding two-dimensional continued fractions*. Ukrainian Math. J. 2022, **74** (4), 501–518. doi:10.1007/s11253-022-02079-1 (translation of Ukrain. Mat. Zh. 2022, **74** (4), 443–457. doi:10.37863/umzh.v74i4.7031(in Ukrainian))

[10] Baran O.E. Approximation of functions of multiple variables branched continued fractions with independent variables. Cand. Phys.-Math. Sc. (Ph.D.) Thesis in Math. Anal. Pidstryhach IAPMM NASU, Lviv, 2014. (in Ukrainian)

[11] Bodnarchuk P.I., Skorobohatko V.Ya. Branched Continued Fractions and Their Applications. Naukova Dumka, Kyiv, 1974. (in Ukrainian)

[12] Bodnar D.I. Branched Continued Fractions. Naukova Dumka, Kyiv, 1986. (in Russian)

[13] Bodnar D.I. *Expansion of a ratio of hypergeometric functions of two variables in branching continued fractions*. J. Math. Sci. 1993, 64, 1155–1158. doi:10.1007/BF01098839 (translation of Mat. Metody Fiz.-Mekh. Polya 1990, **32**, 40–44. (in Russian))

[14] Bodnar D.I., Manzii O.S. *Expansion of the ratio of Appel hypergeometric functions* F_3 *into a branching continued fraction and its limit behavior*. J. Math. Sci. 2001, **107**, 3550–3554. doi:10.1023/A:1011977720316 (translation of Mat. Metody Fiz.-Mekh. Polya 1998, **41** (4), 12–16. (in Ukrainian))

[15] Bodnar D.I. *Multidimensional C-factions*. J. Math. Sci. 1998, **90**, 2352–2359. doi:10.1007/BF02433965 (translation of Mat. Metody Fiz.-Mekh. Polya 1996, **39** (2), 39–46. (in Ukrainian))

[16] Bodnar D.I. *Multidimensional generalizations of continued fraction*. Mat. Metody Fiz.-Mekh. Polya 2003, **46** (3), 32–39. (in Ukrainian)

[17] Cuyt A.A.M., Petersen V., Verdonk B., Waadeland H., Jones W.B. Handbook of Continued Fractions for Special Functions. Springer, Dordrecht, 2008.

[18] Dmytryshyn R.I. *Convergence of multidimensional A- and J-fractions with independent variables*. Comput. Methods Funct. Theory 2022, **22** (2), 229–242. doi:10.1007/s40315-021-00377-6

[19] Dmytryshyn R., Goran V. *On the analytic extension of Lauricella-Saran's hypergeometric function F_K to symmetric domains*. Sym. 2024, **16** (02), 220. doi:10.3390/sym16020220.

[20] Dmytryshyn R., Lutsiv I.-A., Bodnar O. *On the domains of convergence of the branched continued fraction expansion of ratio $H_4(a, d+1; c, d; \mathbf{z}) / H_4(a, d+2; c, d+1; \mathbf{z})$*. Res. Math. 2023, **31** (2), 19–26. doi:10.15421/242311

[21] Dmytryshyn R.I. *Multidimensional regular C-fraction with independent variables corresponding to formal multiple power series*. Proc. Roy. Soc. Edinburgh Sect. A 2020, **150** (4), 153–1870. doi:10.1017/prm.2019.2

[22] Dmytryshyn R.I., Sharyn S.V. *Approximation of functions of several variables by multidimensional S-fractions with independent variables*. Carpathian Math. Publ. 2021, **13** (3), 592–607. doi:10.15330/cmp.13.3.592-607

[23] Dmytryshyn R.I. Some Classes of Functional Branched Continued Fractions with Independent Variables and Multiple Power Series. Dr. Phys.-Math. Sc. Thesis in Math. Anal. Vasyl Stefanyk PNU, Ivano-Frankivsk, 2018. (In Ukrainian)

[24] Dmytryshyn R.I. *The multidimensional generalization of g-fractions and their application*. J. Comput. Appl. Math. 2004, **164–165**, 265–284. doi:10.1016/S0377-0427(03)00642-3

[25] Hladun V.R., Hoyenko N.P., Manzij O.S., Ventyk L.S. *On convergence of function $F_4(1, 2; 2, 2; z_1, z_2)$ expansion into a branched continued fraction*. Math. Model. Comput. 2022, **9** (3), 767–778. doi:1023939/mmc2022.03.767

[26] Hoyenko N., Antonova T., Rakintsev S. *Approximation for ratios of Lauricella-Saran fuctions F_S with real parameters by a branched continued fractions*. Math. Bul. Shevchenko Sci. Soc. 2011, **8**, 28–42. (In Ukrainian)

[27] Hoyenko N.P., Hladun V.R., Manzij O.S. *On the infinite remains of the Norlund branched continued fraction for Appell hypergeometric functions*. Carpathian Math. Publ. 2014, **6** (1), 11–25. doi:10.15330/cmp.6.1.11-25 (in Ukrainian)

[28] Jones W.B., Thron W.J. Continued Fractions: Analytic Theory and Applications. Addison-Wesley Pub. Co., Reading, 1980.

[29] Kaliuzhnyi-Verbovetskyi D., Pivovarchik V. *Recovering the shape of a quantum caterpillar tree by two spectra.* Mech. Math. Methods 2023, **5**, 14–24. doi:10.31650/2618-0650-2023-5-1-14-24

[30] Kaminsky A.A., Selivanov M.F. *On the application of branched operator continued fractions for a boundary problem of linear viscoelasticity.* Int. Appl. Mech. 2006, **42**, 115–126. doi:10.1007/s10778-006-0066-3

[31] Kuchminska Kh.Yo. *Corresponding and associated branched continued fractions for double power series*. Dop. AN UkrSSR. Ser. A. 1978, **7**, 614–618. (in Ukrainian)

[32] Kuchminska Kh.Yo. *On the Sleszynsky-Pringsheim Theorem for the three-dimensional generalization of continued fractions*. J. Math. Sci. 2022, **265** (3), 408–422. doi:10.1007/s10958-022-06061-x (translation of Mat. Metody Fiz.-Mekh. Polya 2019, **62** (4), 60–71. (in Ukrainian))

[33] Kuchminska Kh.Yo. Two-dimensional Continued Fractions. Pidstryhach IAPMM NASU, Lviv, 2010. (in Ukrainian)

[34] Kuchminska Kh.Yo., Vozna S.M. *Developent of an N-multiple power series into N-dimensional regular C-fraction*. J. Math. Sci. 2020, **246** (2), 201–208. doi:10.1007/s10958-020-04730-3 (translation of Mat. Metody Fiz.-Mekh. Polya 2017, **60** (3), 70–75. (in Ukrainian))

[35] Komatsu T. *Asymmetric circular graph with Hosoya index and negative continued fractions.* Carpathian Math. Publ. 2021, **13** (3), 608–618. doi:10.15330/cmp.13.3.608-618

[36] Lorentzen L., Waadeland H. Continued Fractions with Applications. Noth Holland, Amsterdam, 1992.

[37] Manzii O.S. *Investigation of expansion of the ratio of Appel hypergeometric functions F_3 into a branching continued fraction*. Approx. Theor. and its Appl.: Pr. Inst. Math. NAS Ukr. 2000, **31**, 344–353. (in Ukrainian)

[38] Murphy J., O'Donohoe M.R. *A two-variable generalization of the Stieltjes-type continued fractions*. J. Comp. Appl. Math. 1978, **4** (3), 181–190. doi:10.1016/0771-050x(78)90002-5

[39] O'Donohoe M.R. Application of Continued Fractions in One and More Variables. Ph.D. Thesis. Brunel University, London, 1974.

[40] Siemaszko W. *Branched continued fractions for double power series*. J. Comp. Appl. Math. 1980, **6** (2), 121–125. doi:10.1016/0771-050x(80)90005-4

[41] Skorobohatko V.Ya. Theory of Branched Continued Fractions and Its Applications in Computational Mathematics. Nauka, Moscow, 1983. (In Russian)

[42] Wang R., Qian, J. *On branched continued fractions rational interpolation over pyramid-typed grids.* Numer. Algor. 2010, **54**, 47–72. doi:10.1007/s11075-009-9322-z

[43] Wall H.S. Analytic Theory of Continued Fractions. Van Nostrand, New York, 1948.

Received 20.03.2024

Антонова Т.М. *Про структуру гіллястих ланцюгових дробів* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 391–400.

У статті наведено огляд різних багатовимірних узагальнень неперервних дробів, які виникли при розв'язуванні задачі наближення функцій однієї чи багатьох змінних, включно з деякими гіпергеометричними функціями. Показано, що всі ці узагальнення можна розглядати як окремі випадки загального поняття гіллястого ланцюгового дробу, означення якого наведено у роботі.

Ключові слова і фрази: гіллястий ланцюговий дріб, голоморфна функція, наближення раціональними функціями.