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Infinite distributive laws for the lateral operations on a Riesz space

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We prove analogues of the well known infinite distributive laws for the lateral infima and suprema instead of order ones. The proofs are more involved than that for the original laws. We show that one of the laws holds true whenever both sides of the equality are well defined. The other one is false in general, even if both sides are well defined, but true for finite sets. The proofs of two laws are completely different. The question of under what assumptions on the Riesz space and objects involved in, the second distributive law is valid for infinite sets, remains unsolved.

Key words and phrases: Riesz space, lateral order, lateral band.

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Introduction

We continue investigation of the lateral order on Riesz spaces started in [5] and then continued in a number of papers, see survey [8] and references therein. In particular, the interest to the lateral order is due to its great importance in the investigation of linear operators [2] and orthogonally additive operators on Riesz spaces [4, 6].

Let E be a Riesz space. An element $x \in E$ is called a *fragment*[1] of $y \in E$ provided $x \perp y - x$. In this case we write $x \sqsubseteq y$. The set of all fragments of a given element $e \in E$ will be denoted by $\mathfrak{F}_e$. By $x \sqcup y$ we denote the disjoint sum of elements x and y of E, that is, the usual sum $x + y$ under the assumption that $x \perp y$.

Obviously, if $e = x + y$, then the following three conditions are equivalent: $x \sqsubseteq e$, $y \sqsubseteq e$ and $x \perp y$. Hence if $e = \bigsqcup_{k=1}^{m} x_k$ then $(x_k)_{k=1}^{m}$ are disjoint fragments of e.

The binary relation $\sqsubseteq$ possesses the following elementary properties.

Proposition 1 ([5, Proposition 3.1]). *Let E be a Riesz space and $x, y \in E$. Then*

1) $x \sqsubseteq y$ if and only if $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$;

2) if $x \sqsubseteq y$, then

(a) $x^+ \le y^+$, $x^- \le y^-$ and $|x| \le |y|$;

(b) $x^- \perp y^+$ and $x^+ \perp y^-$;

(c) $|x| \sqsubseteq |y|$.

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[1] *Component* in the terminology of [1].

It is an easy exercise to show that $\sqsubseteq$ is a (non-strict) partial order on E (see [3] for the proof), which is called the *lateral order* on E. Since $0 \sqsubseteq x$ for all $x \in E$, every subset of E is laterally bounded from below by zero. The lateral supremum and infimum with respect to the lateral order $\sqsubseteq$ on E are denoted by the bold symbols $\bigcup, \cup$ and $\bigcap, \cap$ respectively, to distinguish them from the set-theoretical operations.

Proposition 2 ([5, Proposition 3.4]). *Let E be a Riesz space and $e \in E$. Then*

1) *the set $\mathfrak{F}_e$ is a Boolean algebra with zero 0, unit e with respect to the operations $\cup$ and $\cap$. Moreover, for every $x, y \in \mathfrak{F}_e$ one has:*
   (a) $x \cup y = (x^+ \vee y^+) - (x^- \vee y^-)$;
   (b) $x \cap y = (x^+ \wedge y^+) - (x^- \wedge y^-)$;
   (c) $x \cap y = 0$ *if and only if* $x \perp y$;
2) *if, moreover, $e \geq 0$, then the lateral order $\sqsubseteq$ on $\mathfrak{F}_e$ coincides with the lattice order $\leq$.*

Given any element e of a Riesz space E and a subset $G \subseteq E$, we use the following notations:

$$e \cap G := \{e \cap x : x \in G\}, \qquad e \cup G := \{e \cup x : x \in G\},$$
$$G^+ := \{x^+ : x \in G\}, \quad G^- := \{x^- : x \in G\}, \quad |G| := \{|x| : x \in G\}.$$

The lateral analogue of the Riesz decomposition property

There are known simple but useful lateral analogues of the decomposition property [1, Theorem 1.13] and the Riesz decomposition property [1, Theorem 1.20], see also [7, Proposition 3.11].

Below we provide short proofs of these results, showing that they are simple consequences of Proposition 2.

Proposition 3 ([9, Lemma 1.8]). *Let E be a Riesz space, $x, y_1, \ldots, y_n \in E$ and $x \sqsubseteq y_1 \sqcup \ldots \sqcup y_n$. Then there are $x_1, \ldots, x_n$ such that $x = x_1 \sqcup \ldots \sqcup x_n$ and $x_i \sqsubseteq y_i$ for $i = 1, \ldots, n$.*

Proof. Set $e = y_1 \sqcup \ldots \sqcup y_n$. Since $x, y_1, \ldots, y_n$ are elements of the Boolean algebra $\mathfrak{F}_e$, the elements $x_i := x \cap y_i$ are well defined and possess the desired properties. ☐

Proposition 4 ([7, Proposition 3.11]). *Let E be a Riesz space, $u_1, \ldots, u_m$, $v_1, \ldots v_n \in E$ and $e := \bigsqcup_{i=1}^m u_i = \bigsqcup_{k=1}^n v_k$. Then there exists a (disjoint) double sequence $(w_{i,k})_{i=1}^{m}{}_{k=1}^{n}$ in E such that*

1) $u_i = \bigsqcup\limits_{k=1}^{n} w_{i,k}$ *for any* $i \in \{1, \ldots, m\}$;
2) $v_k = \bigsqcup\limits_{i=1}^{m} w_{i,k}$ *for any* $k \in \{1, \ldots, n\}$.

Proof. Since u_i, v_k are elements of the Boolean algebra $\mathfrak{F}_e$, the elements $w_{i,k} := u_i \cap v_k$ are well defined and possess the desired properties. ☐

1 Further properties of the lateral operations

In this section, we prove some auxiliary statements for main results. We begin with an example showing the lack of associativity of the lateral infimum in the full sense.

Recall that a Riesz space E is said to have the *intersection property* provided every two-point subset $\{x,y\}$ of E has a lateral infimum $x \Cap y$. In particular, the principal projection property implies the intersection property, however there exists an Archimedean Riesz space without the intersection property [5]. For more information about the intersection property see [6].

Proposition 5. *Let E be a Riesz space without the intersection property. Then there are elements x,y,z of E such that:*

1) $x \Cap (y \Cap z)$ exists and $(x \Cap y) \Cap z$ does not exist;

2) $\bigcap\{x,y,z\}$ exists and $\bigcap\{x,y\}$ does not exist.

Proof. Let x,y be any elements of E such that $x \Cap y$ does not exist and set $z = 2y$. Then we have $x \Cap (y \Cap z) = x \Cap 0 = 0$ and $\bigcap\{x,y,z\} = 0$. □

However, the associativity holds whenever both sides are well defined.

Proposition 6. *Let x,y,z be elements of a Riesz space E.*

(i) If $x \Cap (y \Cap z)$ exists, then $\bigcap\{x,y,z\}$ exists and $x \Cap (y \Cap z) = \bigcap\{x,y,z\}$.

(ii) If both $x \Cap (y \Cap z)$ and $(x \Cap y) \Cap z$ exist, then

$$x \Cap (y \Cap z) = (x \Cap y) \Cap z = \bigcap\{x,y,z\}.$$

Proof. *(i)* We show that $h := x \Cap (y \Cap z)$ is the lateral infimum of the set $\{x,y,z\}$. Indeed, $h \sqsubseteq x$ and $h \sqsubseteq y \Cap z$. Hence, $h \sqsubseteq y$ and $h \sqsubseteq z$, and so h is a lower lateral bound of $\{x,y,z\}$. Assume that w is any lower lateral bound of $\{x,y,z\}$ and show that $w \sqsubseteq h$. By the assumption, we have

$$w \in \mathfrak{F}_x \cap (\mathfrak{F}_y \cap \mathfrak{F}_z) = \mathfrak{F}_x \cap \mathfrak{F}_{y \Cap z} = \mathfrak{F}_{x \Cap (y \Cap z)},$$

which yields $w \sqsubseteq h$.

Item *(ii)* easily follows from *(i)*. □

The next statement is "almost" known, being a slight strengthening of items (2) and (3) of [5, Corollary 3.6]. However, [5] contains no proof of it. Later it was stated in [6, Proposition 3.4] in another weaker form, but its proof actually covers the following version.

Proposition 7 ([6, Proposition 3.4 (3)])**.** *Let G be a subset of a Riesz space E.*

1. *If $\bigcup G$ exists, then $\bigcup G^+, \bigcup G^-$ and $\bigcup |G|$ exist and moreover*
   (a) $(\bigcup G)^+ = \bigcup G^+ = \sup G^+$, $(\bigcup G)^- = \bigcup G^- = \sup G^-$,
   (b) $|\bigcup G| = \bigcup |G| = \sup |G|$.
2. *If $\bigcap G$ exists, then $\bigcap G^+, \bigcap G^-$ and $\bigcap |G|$ exist and, moreover,*

$$\bigcap G^+ = \Big(\bigcap G\Big)^+, \quad \bigcap G^- = \Big(\bigcap G\Big)^-, \quad \bigcap |G| = \Big|\bigcap G\Big|.$$

Now we point out an important consequence of Proposition 7, which was implicitly used without a proof in [10] in the definition of a horizontally convergent net, where the lateral supremum of a laterally increasing net was used in place of its order convergence.

Corollary 1. *Let (x_α) be a net in a Riesz space E such that $x_\alpha \sqsubseteq x_\beta$ as $\alpha < \beta$ and $x \in E$. Then the following assertions are equivalent.*

(i) $x = \bigsqcup_\alpha x_\alpha$.

(ii) $|x - x_\alpha| \downarrow 0$.

Proof. By Proposition 7, we have

$$x = \bigsqcup_\alpha x_\alpha \quad \Leftrightarrow \quad x^+ = \sup_\alpha x_\alpha^+ \text{ and } x^- = \sup_\alpha x_\alpha^- \quad \Leftrightarrow \quad |x - x_\alpha| = x^+ - x_\alpha^+ + x^- - x_\alpha^- \downarrow 0.$$

□

2 Distributive laws and related properties

Since the set $\mathfrak{F}_f$ of all fragments of an element f of a Riesz space is a Boolean algebra with respect to the lateral operations, the finite distributive laws for elements and subsets of $\mathfrak{F}_f$ are beyond doubt. So for every $e, x, y, z \in \mathfrak{F}_f$ one has

$$(e \sqcap x) \sqcup (e \sqcap y) = e \sqcap (x \sqcup y) \tag{1}$$

and

$$(e \sqcup x) \sqcap (e \sqcup y) = e \sqcup (x \sqcap y). \tag{2}$$

There were no investigation of whether the above two equalities are true if the terms are not laterally bounded.

2.1 The first infinite distributive law for lateral operations

Theorem 1. *Let G be a nonempty subset of a Riesz space E and $e \in E$. Suppose that $\bigsqcup G$ and $e \sqcap (\bigsqcup G)$ exist. Then $\bigsqcup (e \sqcap G)$ exists and*

$$\bigsqcup (e \sqcap G) = e \sqcap \left(\bigsqcup G \right). \tag{3}$$

For the proof, we need some statements which may be of interest by themselves. The first one asserts that (1) holds under a weaker assumption.

Lemma 1. *Assume that e, x, y, f are elements of a Riesz space E, $x \sqsubseteq f$, $y \sqsubseteq f$, and the lateral infima $e \sqcap x$ and $e \sqcap y$ exist. Then $e \sqcap (x \sqcup y)$ exists and (1) holds.*

Proof. Since $e \sqcap x \sqsubseteq f$ and $e \sqcap y \sqsubseteq f$, the following vector $w := (e \sqcap x) \sqcup (e \sqcap y)$ is well defined. Show that w is the maximal common fragment of e and $x \sqcup y$. Indeed, $e \sqcap x \sqsubseteq e$ with $e \sqcap y \sqsubseteq e$ imply $w \sqsubseteq e$, and $e \sqcap x \sqsubseteq x \sqsubseteq x \sqcup y$ with $e \sqcap y \sqsubseteq y \sqsubseteq x \sqcup y$ imply $w \sqsubseteq x \sqcup y$.

Now suppose that u is any common fragment of e and $x \sqcup y$ and prove that $u \sqsubseteq w$. Since u, $e \sqcap x$, $e \sqcap y$, w are elements of f, by (1) and Proposition 6 we obtain

$$\begin{aligned} u \sqcap w &= (u \sqcap (e \sqcap x)) \sqcup (u \sqcap (e \sqcap y)) \\ &= \left(\bigsqcap \{u, e, x\} \right) \sqcup \left(\bigsqcap \{u, e, y\} \right) \\ &= (u \sqcap x) \sqcup (u \sqcap y) = u \sqcap (x \sqcap y) = u, \end{aligned}$$

which yields $u \sqsubseteq w$. □

Using induction, we obtain the following consequence, which is a version of Theorem 1 for finite G (it is stated separately to point out that the lateral boundedness of a finite subset G guarantees the existence of $\bigsqcup G$).

Corollary 2. *Assume that e is an element of a Riesz space E, G is a finite laterally bounded subset of E, and for every $g \in G$ the lateral infimum $e \sqcap g$ exists. Then $e \sqcap (\bigsqcup G)$ exists and* (3) *holds.*

Lemma 2. *Let E be a Riesz space and $e, g \in E$. If $w := e \sqcap g$ exists, then $e \sqcap g'$ and $w \sqcap g'$ exist for every $g' \sqsubseteq g$ and, moreover, $e \sqcap g' = w \sqcap g'$.*

Proof. Since $w, g' \in \mathfrak{F}_g$, by Proposition 2, $w \sqcap g'$ exists. Then the conditions $w \sqcap g' \sqsubseteq w \sqsubseteq e$ imply $w \sqcap g' \in \mathfrak{F}_e \cap \mathfrak{F}_{g'}$, that is, $w \sqcap g'$ is a $\sqsubseteq$-lower bound of $\{e, g'\}$. Let z be any $\sqsubseteq$-lower bound of $\{e, g'\}$, that is, $z \in \mathfrak{F}_e \cap \mathfrak{F}_{g'}$. Then $z \in \mathfrak{F}_e \cap \mathfrak{F}_g$ and hence $z \sqsubseteq e \sqcap g = w$. Now since $z \in \mathfrak{F}_w \cap \mathfrak{F}_{g'}$, we obtain that $z \sqsubseteq w \sqcap g'$ and therefore $w \sqcap g'$ is the maximal $\sqsubseteq$-lower bound of $\{e, g'\}$. ☐

By Lemma 2 and [9, Lemma 1.10], we obtain the following assertion, which strengthens item (v) of [5, Proposition 3.21].

Corollary 3. *Let E be a Riesz space, $x, y, e \in E$ and $x \perp y$. Then the following conditions are equivalent:*

1) *$(x \sqcup y) \sqcap e$ exists;*

2) *$x \sqcap e$ and $y \sqcap e$ exist.*

Moreover, $(x \sqcup y) \sqcap e = (x \sqcap e) \sqcup (y \sqcap e)$ whenever the conditions hold.

Proof of Theorem 1. Set $g = \bigsqcup G$ and $w := e \sqcap g$, and prove that $w = \bigsqcup (e \sqcap G)$. Obviously, w is an upper lateral bound of $e \sqcap G$. Now let z be any upper lateral bound of $e \sqcap G$ and show that $w \sqsubseteq z$. Since G is laterally bounded, every finite subset of G has a lateral supremum by [5, Corollary 3.7]. So, let $G^{<\omega}$ be the set of all finite subsets of G, which is directed by inclusion: $\alpha \le \beta$ if and only if $\alpha \subseteq \beta$, $\alpha, \beta \in G^{<\omega}$. For every $\alpha \in G^{<\omega}$ we set $g_\alpha := \bigsqcup \alpha$. Then $(g_\alpha)_{\alpha \in G^{<\omega}}$ is a laterally increasing net in E with $\bigsqcup_\alpha g_\alpha = g$. By Corollary 1, $g_\alpha \xrightarrow{o} g$. Since $g_\alpha \sqsubseteq g$, by Lemma 2, $e \sqcap g_\alpha$ is well defined for all α. Then for every $\alpha \in G^{<\omega}$ one has $g = g_\alpha \sqcup (g - g_\alpha)$ and hence, by Corollary 3,

$$e \sqcap g = (e \sqcap g_\alpha) \sqcup (e \sqcap (g - g_\alpha)).$$

Therefore,

$$(e \sqcap g) - (e \sqcap g_\alpha) = e \sqcap (g - g_\alpha) \sqsubseteq g - g_\alpha.$$

By Proposition 1,

$$|(e \sqcap g) - (e \sqcap g_\alpha)| \le |g - g_\alpha|,$$

which yields $e \sqcap g_\alpha \xrightarrow{o} e \sqcap g = w$. By Corollary 2,

$$e \sqcap g_\alpha = \bigsqcup_{f \in \alpha} (e \sqcap f) \sqsubseteq z.$$

By the order closedness of $\mathfrak{F}_z$ [6, Proposition 2.2], $w \sqsubseteq z$ and so (3) is proved. ☐

Remark 1. *The existence of the left-hand side of* (3) *does not guarantee the existence of the right-hand side. For instance, if e is any nonzero element of a Riesz space E and $G = \{2e, 3e\}$ then $e \sqcap G = \{0\}$ and $\bigsqcup (e \sqcap G) = 0$, however, $\bigsqcup G$ does not exist.*

2.2 The second infinite distributive law for lateral operations

Now we consider the second distributive law for the lateral order, namely

$$\bigcap(e \cup G) = e \cup \Big(\bigcap G\Big). \tag{4}$$

Below we show that (4) does not hold in general.

Proposition 8.

(i) For every nontrivial Riesz space E, there are $e \in E$ and a two-point subset G of E such that the right-hand side of (4) *exists, and the left-hand side does not.*

(ii) There are a Riesz space E, $e \in E$ and $G \subseteq E$ such that the left-hand side of (4) *exists, and $\bigcap G$ does not exist.*

(iii) There are a Riesz space E, $e \in E$ and $G \subseteq E$ such that both sides of (4) *are well defined, however* (4) *is false.*

Proof. (i) If e is any nonzero element of a Riesz space E and $G = \{e, 2e\}$, then $\bigcap G = 0$ and $e \cup (\bigcap G) = e$, however, $e \cup G$ does not exist.

(ii) Consider the Riesz space $C_0^1[1/2, 1]$ of all functions $x\colon [0,1] \to \mathbb{R}$, that are continuous on $[1/2, 1]$, with the pointwise order. We set $e := \mathbf{1}_{[0,1]}$ and $G := \{\mathbf{1}_{[t,1]} : 0 \le t < 1/2\}$, where $\mathbf{1}_A$ denotes the characteristic function of a subset $A \subseteq [0,1]$. Then $e \cup G = \{e\}$ and $\bigcap(e \cup G) = e$, however $\bigcap G$ does not exist.

(iii) Let E be the Riesz space of all functions $x\colon [0,1] \to \mathbb{R}$, that are continuous on $(1/2, 1]$ and left-continuous at $1/2$, with the pointwise order. Set $e := \mathbf{1}_{[0,1/2]}$ and $G := \{\mathbf{1}_{[t,1]} : 0 \le t < 1/2\}$. Then $\bigcap G = 0$ and hence the right-hand side of (4) equals e. On the other hand, $e \cup G = \{\mathbf{1}_{[0,1]}\}$ and hence, $\bigcap(e \cup G) = \mathbf{1}_{[0,1]}$. ☐

However, (4) holds true for a finite G under some existence assumptions. First we prove (2) under a weaker assumption than the lateral boundedness of all its terms.

Theorem 2. *Let e be an element of a Riesz space E and G a finite subset of E such that $\bigcap(e \cup G)$ and $\bigcap G$ exist. Then $e \cup (\bigcap G)$ exists and* (4) *holds.*

To prove Theorem 2, we need auxiliary lemmas. Following [5], if $u \cap v$ exists for elements u, v of a Riesz space E, then we denote $u \setminus v := u - u \cap v$.

Lemma 3 ([5, Proposition 3.21 (iii)]). *Let u, v be elements of a Riesz space E such that $u \cap v$ exists. Then*

$$(u \setminus v) \cap v = 0.$$

Next lemma is a consequence of Corollary 3.

Lemma 4. *Let $e, e_1, \dots, e_n, f, f_1, \dots, e_m$ be elements of a Riesz space E with $e = e_1 \sqcup \dots \sqcup e_n$ and $f = f_1 \sqcup \dots \sqcup f_m$, $n, m \in \mathbb{N}$. Then the following assertions are equivalent:*

1) $e \cap f$ exists;

2) $e_i \cap f_j$ exists for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.

Moreover,

$$e \cap f = \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{m} (e_i \cap f_j)$$

in case of existence.

Proof. First we generalize Corollary 3 from two to an arbitrary finite number of summands. Then we obtain that for every $j \in \{1, \dots, m\}$, the following assertions are equivalent:

1_j) $e \cap f_j$ exists,

2_j) $e_i \cap f_j$ exists for all $i \in \{1, \dots, n\}$,

and $e \cap f = \bigsqcup_{i=1}^{n} (e_i \cap f_j)$ in case of existence. Finally we use the same argument to get the lemma. ☐

The following lemma is a consequence of [5, Proposition 3.18].

Lemma 5. *Let $\{x, y\}$ be a laterally bounded subset of a Riesz space E. Then*

$$x \cup y = (x \setminus y) \sqcup (x \cap y) \sqcup (y \setminus x).$$

Proof of Theorem 2. Observe that it is enough to prove the theorem for any two-point subset G of E and then use the induction. So let e, x, y be elements of E such that $e \cup x$, $e \cup y$, $(e \cup x) \cap (e \cup y)$ and $x \cap y$ exist. We prove that $e \cup (x \cap y)$ exists and (2) holds, which is exactly (4) for $G = \{x, y\}$.

Since e and $x \cap y$ are laterally bounded by $e \cup x$, the element $e \cup (x \cap y)$ is well defined by Proposition 2. The relation

$$e \cup (x \cap y) \sqsubseteq (e \cup x) \cap (e \cup y)$$

is obvious. Our goal is to show that

$$(e \cup x) \cap (e \cup y) \sqsubseteq e \cup (x \cap y). \tag{5}$$

By Lemma 5,

$$e \cup x = (e \setminus x) \sqcup (e \cap x) \sqcup (x \setminus e), \quad e \cup y = (e \setminus y) \sqcup (e \cap y) \sqcup (y \setminus e).$$

Hence, by Lemma 4,

$$(e \cup x) \cap (e \cup y) = u_1 \sqcup \dots \sqcup u_9,$$

where

$$u_1 = ((e \setminus x) \cap (e \setminus y)), \quad u_2 = ((e \setminus x) \cap (e \cap y)), \quad u_3 = ((e \setminus x) \cap (y \setminus e)),$$
$$u_4 = ((e \cap x) \cap (e \setminus y)), \quad u_5 = ((e \cap x) \cap (e \cap y)), \quad u_6 = ((e \cap x) \cap (y \setminus e)),$$
$$u_7 = ((x \setminus e) \cap (e \setminus y)), \quad u_8 = ((x \setminus e) \cap (e \cap y)), \quad u_9 = ((x \setminus e) \cap (y \setminus e)).$$

Now to prove (5), it is enough to show that $\forall i \in \{1, \dots, 9\}$, $u_i \sqsubseteq e \cup (x \cap y)$.

The relation is clear for $i \in \{1, 2, 4, 5\}$, because $u_i \sqsubseteq e$ for that indices. For $i \in \{3, 6\}$ one has $u_i \sqsubseteq e \cap (y \setminus e) = 0$ by Lemma 3. Analogously, for $i \in \{7, 8\}$ one has $u_i \sqsubseteq (x \setminus e) \cap e = 0$. Hence, $u_i = 0$ for $i \in \{3, 6, 7, 8\}$. Finally, $u_9 \sqsubseteq x \cap y \sqsubseteq e \cup (x \cap y)$. So (5) is proved. ☐

By Proposition 8 *(iii)*, the finiteness assumption on G is essential in Theorem 2. However, we do not know any partial positive result in this direction.

Problem. *Under what assumptions on a Riesz space E, element e of E and an infinite subset G of E, the second distributive law* (4) *holds?*

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Ми доводимо аналоги відомих нескінченних розподільних законів для латеральних інфімумів та супремумів замість порядкових. Доведення є більш складними, ніж для оригінальних законів. Ми показуємо, що один із законів справедливий, коли обидві сторони рівності коректно визначені. Інший закон загалом невірний, навіть якщо обидві сторони є коректно визначеними, але завжди вірний для скінченних множин. Доведення двох законів абсолютно різні. Залишається невирішеним питання про те, за яких припущень щодо векторної ґратки та об'єктів, що беруть участь у ньому, другий розподільний закон має місце для нескінченних множин.

Ключові слова і фрази: векторна ґратка, латеральний порядок, латеральна смуга.