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About properties and the monomiality principle of Bell-based Apostol-Bernoulli-type polynomials

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This article investigates the properties and monomiality principle within Bell-based Apostol-Bernoulli-type polynomials. Beginning with the establishment of a generating function, the study proceeds to derive explicit expressions for these polynomials, providing insight into their structural characteristics. Summation formulae are then derived, facilitating efficient computation and manipulation. Implicit formulae are also examined, revealing underlying patterns and relationships. Through the lens of the monomiality principle, connections between various polynomial aspects are elucidated, uncovering hidden symmetries and algebraic properties. Moreover, connection formulae are derived, enabling seamless transitions between different polynomial representations. This analysis contributes to a comprehensive understanding of Bell-based Apostol-Bernoulli-type polynomials, offering valuable insights into their mathematical nature and applications.

Key words and phrases: special polynomial, monomiality principle, operational connection, symmetric identity, summation formula.

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1 Introduction and preliminaries

Special polynomials are distinguished by their unique properties or particular importance across diverse mathematical domains. Well-recognized examples encompass families like Legendre, Chebyshev, Hermite, Bell, and Touchard polynomials. These polynomial classes frequently emerge in mathematical physics, engineering, computer science, and various scientific fields. Special polynomials of two variables hold significant importance across mathematical disciplines due to their versatile applications and unique properties. These polynomials, often expressed as functions of two variables, play crucial roles in fields such as algebraic geometry, combinatorics, and mathematical physics. They serve as fundamental tools for representing complex surfaces, solving systems of equations, and studying intricate mathematical structures. Examples include bivariate orthogonal polynomials like Jacobi, Hermite, and Legendre polynomials, which find applications in approximation theory, numerical analysis, and probability theory. Moreover, special families of bivariate polynomials, such as Schur poly-

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nomials and symmetric functions, are central in algebraic combinatorics and representation theory, offering insights into symmetric functions, partition theory, and symmetric group representations. Through their rich mathematical properties and diverse applications, special polynomials of two variables continue to contribute significantly to advancing theoretical understanding and practical problem-solving in various mathematical contexts, as evidenced in references such as [1–3, 6, 12, 13, 17–20].

One of the significant classes of two variables, special polynomials, are Bell polynomials. The Bell polynomials are a powerful mathematical tool for describing and analysing combinatorial structures and algebraic relationships. Named after the renowned mathematician Eric Temple Bell, these polynomials play a fundamental role in various areas of mathematics, including combinatorics, number theory, and mathematical physics. Originating from the study of exponential generating functions, Bell polynomials provide a systematic way to express and manipulate certain polynomial sequences, making them invaluable in theoretical and applied contexts. With their ability to encode combinatorial information and generate efficient algorithms for solving combinatorial problems, Bell polynomials have found wide-ranging applications in fields such as probability theory, statistical mechanics, and computer science. In this introduction, we will explore Bell polynomials' key properties and applications, shedding light on their significance in mathematical research and problem-solving. The generating function of these polynomials in two variables (see, [9]) is represented as

$$\sum_{s=0}^{\infty} B_s(\omega,\varpi)\frac{t^s}{s!} = e^{\omega t+\varpi(e^t-1)}. \tag{1}$$

Substituting $\omega = 0$, gives $B_s(0;\varpi) = B_s(\varpi)$, which is known as classical Bell polynomials (or exponential polynomials) and is given by following generating function (see, [2–5]), which is defined as follows

$$\sum_{s=0}^{\infty} B_s(\varpi)\frac{t^s}{s!} = e^{\varpi(e^t-1)}. \tag{2}$$

If we take $\varpi = 1$ in (2) we obtain $B_s(1) = B_s$, known as Bell numbers (see, [2–5])

$$\sum_{s=0}^{\infty} B_s\frac{t^s}{s!} = e^{(e^t-1)}.$$

For $s \in \mathbb{N}_0$ and $\kappa, \beta > -1$, the sth Jacobi polynomial $P_s^{(\kappa,\beta)}(\omega)$ may be defined by means of Rodrigues' formula (see, [11, 16])

$$P_s^{(\kappa,\beta)}(\omega) = (1-\omega)^{-\kappa}(1+\omega)^{-\beta}\frac{(-1)^s}{2^s s!}\frac{d^s}{d\omega^s}\{(1-\omega)^{s+\kappa}(1+\omega)^{s+\kappa}\}, \quad \omega \in \mathbb{C}\setminus\{-1,1\}.$$

The connection between the sth monomial ω^s and the sth Jacobi polynomial $P_s^{(\kappa,\beta)}(\omega)$ may be written as follows (see, [11, equation (2), p. 262])

$$\omega^s = s!\sum_{k=0}^{s}\binom{s+\kappa}{s-k}(-1)^k\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{s+1}}P_k^{(\kappa,\beta)}(1-2\omega). \tag{3}$$

For $s \in \mathbb{N}_0$ and $\omega \in \mathbb{C}$, the Stirling numbers of second kind $S(s,k)$ are defined by means of the following expansion (see, [7, Theorem B, p. 207])

$$\omega^s = \sum_{k=0}^{s}\binom{\omega}{k}k!S(s,k), \tag{4}$$

so that $S(s,k) = 0$ if $1 \leqslant s < k$. We put $S(0,0) = 1$ and $S(0,k) = 0$ for $k \geqslant 1$.

Proposition 1. *For $m \in \mathbb{N}$, let $\{B_s^{[m-1]}(\omega)\}_{s\geq 0}$ be the sequence of generalized Bernoulli polynomials of level m. Then, the following identities are satisfied (see, [10, equation (4)])*

$$\omega^s = \sum_{k=0}^{s} \binom{s}{k} \frac{k!}{(k+m)!} B_{s-k}^{[m-1]}(\omega). \tag{5}$$

The paper enhances our understanding of Bell-based Apostol-Bernoulli-type polynomials, providing valuable insights into their mathematical structure and properties. In Section 2, the study establishes the domain of Bell-based Apostol-Bernoulli-type polynomials and derives connection formulae to facilitate transitions between different representations and formulations of these polynomials. Additionally, explicit forms of these polynomials are derived, offering a clear expression of their structure and characteristics. Section 3 scrutinizes implicit formulae, revealing underlying patterns and relationships that contribute to a deeper understanding of their properties. In Section 4, the paper rigorously analyzes and explores the monomiality principle for these polynomials. Finally, the paper concludes with a summary in the conclusion section.

2 Bell-based Apostol-Bernoulli-type polynomials

In this section, we explore the generating function of Bell-based Apostol-Bernoulli-type polynomials of order α and investigate their various relationships, including correlation formulae, implicit summation formulae, and partial derivative formulae.

Definition 1. *For any $\alpha \in \mathbb{C}$ and $s \in \mathbb{N}_0$, the Bell-based Apostol-Bernoulli-type polynomial of order α is defined as*

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!} = \left(\frac{t^2}{2\lambda e^t - 2}\right)^{\alpha} e^{\omega t+\varpi(e^t-1)}, \quad |t| < |\log\lambda|, \quad 1^{\alpha} := 1. \tag{6}$$

Substituting $\omega = 0$ and $\varpi = 1$ in (6), let us define a Bell-based Apostol-Bernoulli-type number of order α, as follows

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(0,1;\lambda)\frac{t^s}{s!} = \left(\frac{t^2}{2\lambda e^t - 2}\right)^{\alpha} e^{(e^t-1)}.$$

Remark 1. *If we choose $\alpha = 0$ in* (6)*, we have to reduce Bell-based Apostol-Bernoulli-type polynomials of order α into bivariate Bell polynomials defined in* (1) *as follows*

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(0)}(\omega,\varpi;\lambda)\frac{t^s}{s!} = e^{\omega t+\varpi(e^t-1)} = \sum_{s=0}^{\infty} B_s(\omega,\varpi)\frac{t^s}{s!}.$$

Remark 2. *If we choose $\varpi = 0$ and $\lambda = 1$ in* (6)*, we obtain familiar generalized Bernoulli-type polynomials $R_s^{(\alpha)}(\omega)$ (see, [14])*

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,0;1)\frac{t^s}{s!} = \left(\frac{t^2}{2e^t - 2}\right)^{\alpha} e^{\omega t} = \sum_{s=0}^{\infty} R_s^{(\alpha)}(\omega)\frac{t^s}{s!}.$$

Remark 3. *If we choose $\varpi = 0$, $\lambda = 1$, and $\alpha = 1$ in* (6) *the Bell-based Apostol-Bernoulli-type polynomials ${}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)$ reduces to usual Bernoulli-type polynomials $R_s(\omega)$ (see, [14])*

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(1)}(\omega,0;1)\frac{t^s}{s!} = \left(\frac{t^2}{2e^t - 2}\right)e^{\omega t} = \sum_{s=0}^{\infty} R_s(\omega)\frac{t^s}{s!}.$$

Remark 4. *If we choose $\lambda = 1$, then* (6) *reduces to Bell-based Bernoulli-type polynomials of order α*

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;1)\frac{t^s}{s!} = \left(\frac{t^2}{2e^t-2}\right)^{\alpha} e^{\omega t+\varpi(e^t-1)} = \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi)\frac{t^s}{s!}.$$

Below we show particular examples of these polynomials.

Example 1. *For $\alpha = 1$ and $\lambda = 3$ we have the following polynomials:*

$$\begin{aligned}&{}_B\mathfrak{R}_0^{(1)}(\omega,\varpi;3) = 0, \quad {}_B\mathfrak{R}_1^{(1)}(\omega,\varpi;3) = 0, \quad {}_B\mathfrak{R}_2^{(1)}(\omega,\varpi;3) = 6,\\ &{}_B\mathfrak{R}_3^{(1)}(\omega,\varpi;3) = 30\omega + 30\varpi - 45, \quad {}_B\mathfrak{R}_4^{(1)}(\omega,\varpi;3) = \frac{(\omega+\varpi)^2 - 3\omega + 3}{4} - \frac{\varpi}{2}.\end{aligned}$$

Example 2. *For $\alpha = 2$ and $\lambda = 2$ we have the following polynomials:*

$$\begin{aligned}&{}_B\mathfrak{R}_0^{(2)}(\omega,\varpi;2) = {}_B\mathfrak{R}_1^{(2)}(\omega,\varpi;2) = {}_B\mathfrak{R}_2^{(2)}(\omega,\varpi;2) = {}_B\mathfrak{R}_3^{(2)}(\omega,\varpi;2) = 0,\\ &{}_B\mathfrak{R}_4^{(2)}(\omega,\varpi;2) = 6, \quad {}_B\mathfrak{R}_5^{(2)}(\omega,\varpi;2) = 30\omega + 30\varpi - 120.\end{aligned}$$

Theorem 1. *For $\alpha \in \mathbb{C}$ and $s \in \mathbb{N}_0$, the following relation*

$${}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda) = \sum_{k=0}^{s}\binom{s}{k} R_k^{(\alpha)}(\omega;\lambda)B_{s-k}(\varpi) \tag{7}$$

holds, where $R_k^{(\alpha)}(\omega;\lambda)$ are referred to as generalized Apostol-Bernoulli-type polynomials.

Proof. By using the relation (6), we have

$$\begin{aligned}\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!} &= \left(\frac{t^2}{2\lambda e^t-2}\right)^{\alpha} e^{\omega t+\varpi(e^t-1)}\\ &= \left(\frac{t^2}{2\lambda e^t-2}\right)^{\alpha} e^{\omega t}e^{\varpi(e^t-1)} = \sum_{k\geq 0} R_k^{(\alpha)}(\omega;\lambda)\frac{t^k}{k!}\sum_{s=0}^{\infty} B_s(\varpi)\frac{t^s}{s!}.\end{aligned}$$

Applying the series rearrangement, we obtain

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!} = \sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k} R_k^{(\alpha)}(\omega;\lambda)B_{s-k}(\varpi)\frac{t^s}{s!}.$$

After simplification by using series rearrangement, we obtain the result (7). □

Theorem 2. *For any $\alpha \in \mathbb{C}$ and $s \in \mathbb{N}_0$, the following relation holds true*

$${}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda) = \sum_{k=0}^{s}\binom{s}{k} {}_B\mathfrak{R}_s^{(\alpha)}(\lambda)B_{s-k}(\omega,\varpi). \tag{8}$$

Proof. By using result (6), we obtain

$$\begin{aligned}\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!} &= \left(\frac{t^2}{2\lambda e^t-2}\right)^{\alpha} e^{\omega t+\varpi(e^t-1)}\\ &= \left(\frac{t^2}{2\lambda e^t-2}\right)^{\alpha} e^{\omega t}e^{\varpi(e^t-1)} = \sum_{k\geq 0} R_k^{(\alpha)}(\lambda)\frac{t^k}{k!}\sum_{s=0}^{\infty} B_s(\omega,\varpi)\frac{t^s}{s!}.\end{aligned}$$

After simplification by using series rearrangement, we obtain the result (8). □

Theorem 3. *For any $s \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, the Bell-based Apostol-Bernoulli-type polynomials of order α satisfies the relation*

$$ {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda) = \sum_{k=0}^{s}\binom{s}{k}\,{}_B\mathfrak{R}_k^{(\alpha)}(\varpi;\lambda)\omega^{s-k}. \tag{9} $$

Proof. Using the relation (6), we get

$$ \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!} = \left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{\omega t+\varpi(e^t-1)} = \left(\frac{t^2}{2\lambda e^t-2}\right)^{\alpha} e^{\varpi(e^t-1)}e^{\omega t} $$
$$ = \sum_{k\geq 0} {}_B\mathfrak{R}_k^{(\alpha)}(\varpi;\lambda)\frac{t^k}{k!}\sum_{s=0}^{\infty}\frac{(\omega t)^s}{s!} = \sum_{s=0}^{\infty}\sum_{k\geq 0} {}_B\mathfrak{R}_k^{(\alpha)}(\varpi;\lambda)\frac{\omega^s}{s!}\frac{t^{s+k}}{k!}. $$

After, applying series rearrangement, we obtain

$$ \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!} = \sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k}\,{}_B\mathfrak{R}_k^{(\alpha)}(\varpi;\lambda)\omega^{s-k}\frac{t^s}{s!}. $$

After simplification by using series rearrangement, we obtain the result (9). □

Theorem 4. *For any $\alpha \in \mathbb{C}$ and $s \in \mathbb{N}_0$, the following relation holds true*

$$ {}_B\mathfrak{R}_s^{(\alpha)}(\omega+\varpi,z;\lambda) = \sum_{k=0}^{s}\binom{s}{k}\mathfrak{R}_k^{(\alpha)}(\omega;\lambda)B_{s-k}(\varpi,z). \tag{10} $$

Proof. By using the result (6), we get

$$ \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega+\varpi,z;\lambda)\frac{t^s}{s!} = \left(\frac{t^2}{2\lambda e^t-2}\right)^{\alpha} e^{(\omega+\varpi)t+z(e^t-1)} $$
$$ = \left(\frac{t^2}{2\lambda e^t-2}\right)^{\alpha} e^{\omega t}e^{\varpi t+z(e^t-1)} = \sum_{s\geq 0} R_k^{(\alpha)}(\omega;\lambda)\frac{t^k}{k!}\sum_{s=0}^{\infty}B_s(\varpi,z)\frac{t^s}{s!}. $$

After simplification by using series rearrangement, we obtain the result (10). □

From the identity (3) and Proposition 1, we can derive several intriguing algebraic relations that link the polynomials ${}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)$ with other polynomial families, including Jacobi polynomials, generalized Bernoulli polynomials of level m, and Genocchi polynomials.

Theorem 5. *For $\alpha \in \mathbb{C}$, the Bell-based Apostol-Bernoulli-type polynomials of order α, are related with the Jacobi polynomials $P_s^{(\kappa,\beta)}(\omega)$, by means of the following identity*

$$ {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda) $$
$$ = \sum_{k=0}^{s}\sum_{j=k}^{s}(-1)^j\binom{s}{j}\binom{j+\kappa}{j-k}\,{}_B\mathfrak{R}_{s-j}^{(\alpha)}(\varpi;\lambda)(s-j)!\frac{(1+\kappa+\beta+2j)}{(1+\kappa+\beta+j)_{s-j+1}}P_k^{(\kappa,\beta)}(1-2\omega). \tag{11} $$

Proof. By substituting (3) into the right-hand side of (9), we have the following result.

$$
\begin{aligned}
&{}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\\
&=\sum_{j=0}^{s}\binom{s}{j}{}_B\mathfrak{R}_j^{(\alpha)}(\varpi;\lambda)(s-j)!\sum_{k=0}^{s-j}\binom{s-j+\kappa}{s-j-k}(-1)^k\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{s-j+1}}P_k^{(\kappa,\beta)}(1-2\omega)\\
&=\sum_{j=0}^{s}\sum_{k=0}^{s-j}\binom{s}{j}{}_B\mathfrak{R}_j^{(\alpha)}(\varpi;\lambda)(s-j)!\binom{s-j+\kappa}{s-j-k}(-1)^k\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{s-j+1}}P_k^{(\kappa,\beta)}(1-2\omega)\\
&=\sum_{k=0}^{s}\sum_{j=0}^{s-k}\binom{s}{j}\binom{n-j+\kappa}{s-j-k}{}_B\mathfrak{R}_j^{(\alpha)}(\varpi;\lambda)(s-j)!(-1)^j\frac{(1+\kappa+\beta+2j)}{(1+\kappa+\beta+j)_{s-j+1}}P_k^{(\kappa,\beta)}(1-2\omega)\\
&=\sum_{k=0}^{s}\sum_{j=k}^{s}(-1)^j\binom{s}{j}\binom{j+\kappa}{j-k}{}_B\mathfrak{R}_{s-j}^{(\alpha)}(\varpi;\lambda)(s-j)!\frac{(1+\kappa+\beta+2j)}{(1+\kappa+\beta+j)_{s-j+1}}P_k^{(\kappa,\beta)}(1-2\omega).
\end{aligned}
$$

Consequently, we obtain identity (11). □

Theorem 6. *For $\alpha\in\mathbb{C}$, the Bell-based Apostol-Bernoulli-type polynomials of order α, are related to the generalized Bernoulli polynomials of level m $B_s^{[m-1]}(\omega)$, by means of the following identity*

$$
{}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)=\sum_{k=0}^{s}\sum_{j=k}^{s}\binom{s}{j}\binom{j}{k}{}_B\mathfrak{R}_{s-j}^{(\alpha)}(\varpi;\lambda)\frac{k!}{(k+m)!}B_{j-k}^{[m-1]}(\omega). \tag{12}
$$

Proof. By substituting (5) into the right-hand side of (9), we get the following result

$$
\begin{aligned}
{}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)&=\sum_{j=0}^{s}\binom{s}{j}{}_B\mathfrak{R}_j^{(\alpha)}(\varpi;\lambda)\sum_{k=0}^{s-j}\binom{s-j}{k}\frac{k!}{(k+m)!}B_{s-j-k}^{[m-1]}(\omega)\\
&=\sum_{j=0}^{s}\sum_{k=0}^{s-j}\binom{s}{j}{}_B\mathfrak{R}_j^{(\alpha)}(\varpi;\lambda)\binom{s-j}{k}\frac{k!}{(k+m)!}B_{s-j-k}^{[m-1]}(\omega)\\
&=\sum_{k=0}^{s}\sum_{j=0}^{s-k}\binom{s}{j}\binom{s-j}{k}{}_B\mathfrak{R}_j^{(\alpha)}(\varpi;\lambda)\frac{k!}{(k+m)!}B_{s-j-k}^{[m-1]}(\omega)\\
&=\sum_{k=0}^{s}\sum_{j=k}^{s}\binom{s}{j}\binom{j}{k}{}_B\mathfrak{R}_{s-j}^{(\alpha)}(\varpi;\lambda)\frac{k!}{(k+m)!}B_{j-k}^{[m-1]}(\omega).
\end{aligned}
$$

Consequently, we obtain identity (12). □

Theorem 7. *For $\alpha\in\mathbb{C}$, the Bell-based Apostol-Bernoulli-type polynomials of order α are related to the Stirling numbers of the second kind, using the following identity*

$$
{}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)=\sum_{k=0}^{s}\binom{s}{k}{}_B\mathfrak{R}_k^{(\alpha)}(\varpi;\lambda)\sum_{k=0}^{s}\binom{\omega}{k}k!S(s-j,k).
$$

Proof. After replacing (4) in the right-hand side of (9), we can proceed by applying the proof provided in Theorem 5, with appropriate adaptations. □

3 Implicit summation formulae

This section discusses useful identities such as the implicit summation formula for the Bell-based Apostol-Bernoulli-type polynomials of order α, which is defined in the following theorems.

Theorem 8. *For arbitrary $s \in \mathbb{N}_0$ and $\alpha_1, \alpha_2 \in \mathbb{N}$ the following relation holds true*

$$_B\mathfrak{R}_s^{(\alpha_1+\alpha_2)}(\omega_1+\omega_2, \varpi_1+\varpi_2; \lambda) = \sum_{k=0}^{s} \binom{s}{k} {}_B\mathfrak{R}_k^{(\alpha_1)}(\omega_1, \varpi_1; \lambda) {}_B\mathfrak{R}_{s-k}^{(\alpha_2)}(\omega_2, \varpi_2; \lambda). \tag{13}$$

Proof. From the relation (6) we get

$$\begin{aligned}\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha_1+\alpha_2)}(\omega_1+\omega_2, \varpi_1+\varpi_2; \lambda)\frac{t^s}{s!} &= \frac{t^{2\alpha_1}e^{\omega_1 t+\varpi_1(e^t-1)}}{(2\lambda e^t-2)^{\alpha_1}} \frac{t^{2\alpha_2}e^{\omega_2 t+\varpi_2(e^t-1)}}{(2\lambda e^t-2)^{\alpha_2}} \\ &= \sum_{k\geq 0} {}_B\mathfrak{R}_k^{(\alpha_1)}(\omega_1, \varpi_1; \lambda)\frac{t^k}{k!} \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha_2)}(\omega_2, \varpi_2; \lambda)\frac{t^s}{s!} \\ &= \sum_{s=0}^{\infty}\sum_{k\geq 0} {}_B\mathfrak{R}_k^{(\alpha_1)}(\omega_1, \varpi_1; \lambda){}_B\mathfrak{R}_s^{(\alpha_2)}(\omega_2, \varpi_2; \lambda)\frac{t^{s+k}}{s!k!}.\end{aligned}$$

Using the series rearrangement technique, we obtain

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha_1+\alpha_2)}(\omega_1+\omega_2, \varpi_1+\varpi_2; \lambda)\frac{t^s}{s!} = \sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k} {}_B\mathfrak{R}_k^{(\alpha_1)}(\omega_1, \varpi_1; \lambda){}_B\mathfrak{R}_{s-k}^{(\alpha_2)}(\omega_2, \varpi_2; \lambda)\frac{t^s}{s!}.$$

Now, equating both sides, we obtained the result (13). □

Theorem 9. *For any arbitrary $\alpha \in \mathbb{N}$ and $s \in \mathbb{N}_0$, the following relation holds true*

$$_B\mathfrak{R}_{s+1}^{(\alpha)}(\omega+1, \varpi; \lambda) - {}_B\mathfrak{R}_{s+1}^{(\alpha)}(\omega, \varpi; \lambda) = \sum_{k=0}^{s}\binom{s+1}{k} {}_B\mathfrak{R}_k^{(\alpha)}(\omega, \varpi; \lambda). \tag{14}$$

Proof. Using the relation (6), we get

$$\begin{aligned}&\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega+1, \varpi; \lambda)\frac{t^s}{s!} - \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{s!} \\ &\qquad = \frac{t^{2\alpha}e^{(\omega+1)t+\varpi(e^t-1)}}{(2\lambda e^t-2)^{\alpha}} - \frac{t^{2\alpha}e^{\omega t+\varpi(e^t-1)}}{(2\lambda e^t-2)^{\alpha}} \\ &\qquad = \frac{t^{2\alpha}e^{\omega t+\varpi(e^t-1)}(e^t-1)}{(2\lambda e^t-2)^{\alpha}} = \sum_{k\geq 0} {}_B\mathfrak{R}_k^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^k}{k!}\sum_{s=0}^{\infty}\frac{t^{s+1}}{(s+1)!}.\end{aligned}$$

Applying the series rearrangement technique implies the desired result (14) □

Theorem 10. *For any arbitrary $\alpha \in \mathbb{N}$ and $s, r \in \mathbb{N}_0$, the following relation holds true*

$$_B\mathfrak{R}_{s+r}^{(\alpha)}(\omega, \varpi; \lambda) = \sum_{n,m=0}^{s,r}\binom{s}{n}\binom{r}{m}(\rho-\omega)^{n+m} {}_B\mathfrak{R}_{s+r-j-k}^{(\alpha)}(\omega, \varpi; \lambda). \tag{15}$$

Proof. Replacing t by $t+\eta$ in expression (6), it follows that

$$\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{(t+\eta)^s}{s!} = \frac{(t+\eta)^{2\alpha}}{(2\lambda e^{t+\eta}-2)^{\alpha}} e^{\omega(t+\eta)+\varpi(e^{t+\eta}-1)}.$$

Substituting the first part of the exponential term from the right hand side to the left hand side in the preceding expression, we have

$$e^{-\omega(t+\eta)} \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega,\varpi;\lambda)\frac{(t+\eta)^s}{s!} = \frac{(t+\eta)^{2\alpha}}{(2\lambda e^{t+\eta}-2)^{\alpha}} e^{\varpi(e^{t+\eta}-1)},$$

which, in view of well well-known series manipulation formula

$$\sum_{M=0}^{\infty} \mathfrak{L}(M)\frac{(t+\eta)^M}{M!} = \sum_{s,r=0}^{\infty} \mathfrak{L}(s+r)\frac{t^s}{s!}\frac{\eta^r}{r!},$$

becomes

$$e^{-\omega(t+\eta)} \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!} = \frac{(t+\eta)^{2\alpha}}{(2\lambda e^{t+\eta}-2)^{\alpha}} e^{\varpi(e^{t+\eta}-1)}. \tag{16}$$

Replacing ω by ρ in the previous expression (16), it follows that

$$e^{-\rho(t+\eta)} \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\rho,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!} = \frac{(t+\eta)^{2\alpha}}{(2\lambda e^{t+\eta}-2)^{\alpha}} e^{\varpi(e^{t+\eta}-1)}. \tag{17}$$

Comparing expressions (16) and (17), we find

$$e^{-\omega(t+\eta)} \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\omega,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!} = e^{-\rho(t+\eta)} \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\rho,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!}.$$

Substituting the first part of the exponential term from the right hand side to the left hand side in the preceding expression, we have

$$e^{(\rho-\omega)(t+\eta)} \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\rho,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!} = \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\rho,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!}.$$

Thus, the preceding expression can further be simplified as

$$\sum_{n,m=0}^{\infty} (\rho-\omega)^{n+m}\frac{t^n}{n!}\frac{\eta^m}{m!} \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\rho,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!} = \sum_{s,r=0}^{\infty} {}_B\mathfrak{R}_{s+r}^{(\alpha)}(\rho,\varpi;\lambda)\frac{t^s}{s!}\frac{\eta^r}{r!}.$$

Using series rearrangement in the left hand side of the previous expression and comparing the like exponents of t and η on both sides, assertion (15) is established. □

4 Monomiality principle

The concept of monomiality traces back to 1941, with J.F. Steffenson introducing the poweroid notion [15], later refined by G. Dattoli [8]. The operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ serve as both multiplicative and derivative operators for a polynomial set $\{b_s(u)\}_{s\in\mathbb{N}}$, satisfying the following expressions

$$b_{s+1}(u) = \hat{\mathcal{M}}\{b_s(u)\} \tag{18}$$

and

$$s\, b_{s-1}(u) = \hat{\mathcal{D}}\{b_s(u)\}. \tag{19}$$

The set $\{b_s(u)\}_{s\in\mathbb{N}}$ manipulated by these operators is termed a quasi-monomial and must adhere to the formula

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1},$$

displaying a Weyl group structure. The properties of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ determine the characteristics of the quasi-monomial set $\{b_s(u)\}_{s\in\mathbb{N}}$:

(i) $b_s(u)$ satisfies the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_s(u)\} = s\, b_s(u), \tag{20}$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ have differential realizations;

(ii) the explicit form of $b_s(u)$ is given by

$$b_s(u) = \hat{\mathcal{M}}^s\, \{1\}, \tag{21}$$

with $b_0(u) = 1$;

(iii) the generating relation in exponential form for $b_s(u)$ can be expressed as

$$e^{t\hat{\mathcal{M}}}\{1\} = \sum_{s=0}^{\infty} b_s(u)\frac{t^s}{s!}, \quad |t| < \infty,$$

using identity (21).

The primary objective of the monomiality principle is to identify operators for multiplication and differentiation. Additionally, in the context of the monomiality principle, we establish the following outcomes to characterize the Bell-based Apostol-Bernoulli-type polynomials ${}_{B}\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)$.

Theorem 11. *For $\alpha \in \mathbb{C}$ and $s \in \mathbb{N}$, the following multiplicative and derivative operators*

$${}_{B}\mathfrak{R}_{s+1}^{(\alpha)}(\omega, \varpi; \lambda) = \hat{M}_{{}_{B}\mathfrak{R}_s^{(\alpha)}} = \omega + \varpi e^{\partial_w} + \alpha(2\lambda e^{\partial_w} - 2)\Big(2\frac{2\lambda e^{\partial_w} - 2}{\partial_w} - 2\lambda e^{\partial_w}\Big) \tag{22}$$

and

$${}_{B}\mathfrak{R}_{s-1}^{(\alpha)}(\omega, \varpi; \lambda) = \hat{D}_{{}_{B}\mathfrak{R}_s^{(\alpha)}} = \partial_w, \tag{23}$$

respectively, hold true.

Proof. Taking the derivatives of the relation (6) with respect to t on both sides, we have

$$\partial_t\Big[\frac{t^{2\alpha}}{(2\lambda e^t - 2)^\alpha} e^{\omega t + \varpi(e^t - 1)}\Big] = \partial_t\Big[\sum_{s=0}^{\infty} {}_{B}\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{s!}\Big].$$

The preceding expression can further be simplified as

$$\Big[\omega + \varpi e^t + \alpha(2\lambda e^t - 2)\Big(2\frac{2\lambda e^t - 2}{t} - 2\lambda e^t\Big)\Big]\frac{t^{2\alpha} e^{\omega t + \varpi(e^t - 1)}}{(2\lambda e^t - 2)^\alpha} = \sum_{s=0}^{\infty} s\, {}_{B}\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^{s-1}}{s!}.$$

Inserting the left hand side of expression (6) in the left hand side of previous expression, it follows that

$$\Big[\omega + \varpi e^t + \alpha(2\lambda e^t - 2)\Big(2\frac{2\lambda e^t - 2}{t} - 2\lambda e^t\Big)\Big] \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{s!} = \sum_{s=0}^{\infty} s \; {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^{s-1}}{s!}. \tag{24}$$

Further, differentiating expression (6) with respect to w, it follows that

$$\partial_w \Big[\Big(\frac{t^2}{2\lambda e^t - 2}\Big)^{\alpha} e^{\omega t + \varpi(e^t - 1)}\Big] = t\Big(\frac{t^2}{2\lambda e^t - 2}\Big)^{\alpha} e^{\omega t + \varpi(e^t - 1)}.$$

Inserting left part of expression (6), it follows the identity expression

$$\partial_w \Big[\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{s!}\Big] = t \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{s!}. \tag{25}$$

Further replacing s with $s + 1$ in the right hand side of expression (24), we find

$$\Big[\omega + \varpi e^t + \alpha(2\lambda e^t - 2)\Big(2\frac{2\lambda e^t - 2}{t} - 2\lambda e^t\Big)\Big] \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{s!} = \sum_{s=0}^{\infty} (s+1) \; {}_B\mathfrak{R}_{s+1}^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{(s+1)\; s!}.$$

Therefore, in view of (18) and identity expression (25) in the resultant equation, the assertion (22) is proved.

The expression (25) can further be written as

$$\partial_w \Big[\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^n}{n!}\Big] = \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^{s+1}}{s!}.$$

On substituting s with $s - 1$ in the right hand side of above equation, we find

$$\partial_w \Big[\sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^n}{n!}\Big] = \sum_{s=0}^{\infty} {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)\frac{t^s}{(s-1)!}.$$

Thus, in view of expression (19), the assertion (23) follows. □

Next, we find the differential equation satisfied by these polynomials.

Theorem 12. *The Bell-based Apostol-Bernoulli-type polynomials of order* α ${}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda)$ *satisfy the succeeding differential equation*

$$\Big[\omega\, \partial_w + \varpi e^{\partial_w}\, \partial_w + \alpha(2\lambda e^{\partial_w} - 2)\Big(2\frac{2\lambda e^{\partial_w} - 2}{\partial_w} - 2\lambda e^{\partial_w}\Big)\partial_w - s\Big] {}_B\mathfrak{R}_s^{(\alpha)}(\omega, \varpi; \lambda) = 0. \tag{26}$$

Proof. Inserting expression (22) and (23) in expression (20), we obtain assertion (26). □

5 Conclusion

This article has thoroughly investigated the properties and monomiality principle inherent in Bell-based Apostol-Bernoulli-type polynomials. Commencing with the establishment of a generating function, the study progressed to derive explicit expressions for these polynomials, shedding light on their structural characteristics. The derivation of summation formulae further enhanced the efficiency of computation and manipulation. Additionally, the examination of implicit formulae unveiled underlying patterns and relationships, providing deeper insights into the nature of these polynomials. Through the application of the monomiality principle, connections between various aspects of the polynomials were elucidated, revealing hidden symmetries and algebraic properties. Furthermore, the derivation of connection formulae facilitated seamless transitions between different polynomial representations, contributing significantly to our comprehensive understanding of Bell-based Apostol-Bernoulli-type polynomials and their mathematical applications.

Therefore, this study has made significant strides in unravelling the intricacies of Bell-based Apostol-Bernoulli-type polynomials. By systematically exploring their properties, employing the monomiality principle, and deriving essential formulae, this research has provided valuable insights into these polynomials' mathematical nature and applications. The findings presented herein offer a foundation for further investigations into these polynomials' theoretical and practical aspects, potentially opening avenues for advancements in various mathematical disciplines. Overall, this study contributes to the broader body of knowledge surrounding polynomial theory and its applications.

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Рамірез В., Чезарано К., Вані Ш.А., Юсуф С., Бедоя Д. *Про властивості та принцип мономіальності поліномів типу Апостола-Бернуллі на основі Белла* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 379–390.

У цій статті досліджуються властивості та принцип мономіальності поліномів типу Апостола-Бернуллі на основі Белла. Починаючи з встановлення твірної функції, дослідження продовжується до отримання явних виразів для цих поліномів, що дає змогу зрозуміти їхні структурні характеристики. Потім виводяться формули підсумовування, що полегшує ефективні обчислення та маніпуляції. Неявні формули також перевіряються, виявляючи базові закономірності та зв'язки. Через призму принципу мономіальності з'ясовуються зв'язки між різними поліноміальними аспектами, відкриваючи приховані симетрії та алгебраїчні властивості. Крім того, отримано формули зв'язку, що забезпечує неперервний перехід між різними представленнями поліномів. Цей аналіз сприяє всебічному розумінню поліномів типу Апостола-Бернуллі на основі Белла, пропонуючи цінну інформацію про їх математичну природу та застосування.

Ключові слова і фрази: спеціальний поліном, принцип мономіальності, операційний зв'язок, симетрична тотожність, формула підсумовування.