ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2024, **16** (2), 638–662
doi:10.15330/cmp.16.2.638-662

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2024, Т.16, №2, С.638–662

Generalized Ricci-Bourguignon flow

Shahroud Azami

In this paper, we consider a kind of generalized Ricci-Bourguignon flow system, which is closely like the Ricci-Bourguignon flow and possesses a gradient form. We establish the existence and uniqueness of the solution to this flow on an n-dimensional closed Riemannian manifold. We introduce generalized Ricci-Bourguignon system soliton and give a condition to a gradient generalized Ricci-Bourguignon system soliton to be isometric to an Euclidean sphere. Then we give the evolution of some geometric structure of manifold along this flow and establish higher-derivative estimates for compact manifolds and the compactness theorem for this general Ricci-Bourguignon flow system on closed Riemannian manifolds.

Key words and phrases: Ricci-Bourguignon flow, gradient Ricci-Bourguignon soliton, gradient estimate, compactness theorem.

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
E-mail: azami@sci.ikiu.ac.ir

1 Introduction

Geometric flows are evolution of geometric structures under differential equations with functionals on a manifold, which play an important role in differential geometry and physics. Let M be an n-dimensional complete Riemannian manifold with Riemannian metric $g = (g_{ij})$. The first important geometric flow is Ricci flow, which is defined by

$$\frac{\partial}{\partial t}g = -2Ric, \qquad g(0) = g_0, \tag{1}$$

where Ric denotes the Ricci curvature of g and Ricci flow evolves a Riemannian metric by its Ricci curvature. The short-time existence and uniqueness for solution of Ricci flow proved by R. Hamilton (see [7]) and D. DeTurck (see [6]) on compact Riemannian manifolds. Also evolution equations for geometric structures dependant to metric investigated by B. Chow and D. Knopf (see [5]).

A generalization of Ricci flow is the Ricci-Bourguignon flow, which is defined as follows

$$\frac{\partial}{\partial t}g = -2Ric + 2\rho Rg = -2(Ric - \rho Rg), \quad g(0) = g_0, \tag{2}$$

where R is the scalar curvature of g and ρ is a real constant. The Ricci-Bourguignon flow was introduced by J.P. Bourguignon for the first time in 1981 (see [3]). Short-time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$ have been shown by G. Catino et. al. in [4] for $\rho < \frac{1}{2(n-1)}$.

УДК 514.7, 517.957
2020 *Mathematics Subject Classification:* 53E20, 53C21, 53C25, 35K55, 35P15.

Recently, researchers started considering the Ricci flow together with some other geometric flows. For instance, Y. Li [9] considered the following generalized Ricci flow on M

$$\begin{cases} \frac{\partial g_{ij}}{\partial t}(x,t) = -2Ric(x,t) + h(x,t), & g(x,0) = g(x), \\ \frac{\partial}{\partial t}H(x,t) = \Delta_{HL,g(x,t)}H(x,t), & H(x,0) = H(x), \end{cases} \tag{3}$$

and J.Y. Wu in [12] studied a generalization of harmonic-Ricci flow as follows

$$\begin{cases} \frac{\partial g_{ij}}{\partial t}(x,t) = -2Ric(x,t) + h(x,t) + 2\tau du \otimes du, & g(x,0) = g(x), \\ \frac{\partial}{\partial t}H(x,t) = \Delta_{HL,g(x,t)}H(x,t), & H(x,0) = H(x), \\ \frac{\partial}{\partial t}u(x,t) = \Delta u(x,t), & u(x,0) = u_0(x), \end{cases}$$

where Ric is the Ricci tensor of the manifold M, h is a two-form with the components $h_{ij} = \frac{1}{2}H_{ikl}H_j^{kl}$, $\Delta_{HL} = -(dd^* + d^*d)$ denotes the Hodge-Laplace operator, and τ is a positive constant. They established the existence and uniqueness for solution of above geometric system flow and higher-derivative estimates for compact manifolds. As an application, they proved the compactness theorem for these flow systems.

Let (M, g) denote an n-dimensional closed Riemannian manifold and $H = \{H_{ijk}\}$ be a three-form on M. Motivated by the above works, in this paper, we consider the following generalized Ricci-Bourguignon flow (GRBF for short) on M

$$\begin{cases} \frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) + 2\rho R g_{ij}(x,t) + \frac{1}{2}H_{ikl}H_j^{kl}(x,t), \\ \frac{\partial}{\partial t}H(x,t) = \Delta_{HL,g(x,t)}H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x), \end{cases} \tag{4}$$

where Ric is the Ricci tensor of the manifold M, h is a two-form with the components $h_{ij} = \frac{1}{2}H_{ikl}H_j^{kl}$, ρ is a real constant, and $\Delta_{HL} = -(dd^* + d^*d)$ denotes the Hodge-Laplace operator. In the above system, if the form H is closed, then the corresponding system is called the refined generalized Ricci-Bourguignon flow (RGRBF for short), namely

$$\begin{cases} \frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) + 2\rho R g_{ij}(x,t) + \frac{1}{2}H_{ikl}H_j^{kl}(x,t), \\ \frac{\partial}{\partial t}H(x,t) = -dd^*_{g(x,t)}H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x), \end{cases}$$

where d^* is the dual operator of d with respect to the metric $g(x,t)$. We prove the existence and uniqueness of the solution to flow (4). We define the generalized Ricci-Bourguignon system soliton and give a condition to a gradient generalized Ricci-Bourguignon system soliton to be isometric to a Euclidean sphere. Then we study the evolution of some geometric structure of manifold along this flow and show higher-derivative estimates for compact manifolds and the compactness theorem for the flow (4).

The rest of this paper is organized as follows. In Section 2, we prove the existence and uniqueness for solution of GRBF system (4). In Section 3, we introduce GRBF system soliton and gradient GRBF system soliton. Then we give a condition to a gradient GRBF system soliton to be isometric to an Euclidean sphere and we show that any complete shrinking GRBF system soliton has finite fundamental group. In Section 4, we find the evolution formula for Riemannain curvature tensor, Ricci curvature tensor, and scalar curvature of manifold along the GRBF system (4). In Section 5, we establish higher-derivative estimates for compact manifolds. Finally, in Section 6, we prove the compactness theorems for the GRBF system.

2 Short-time existence and uniqueness the GRBF system

In this section, by a similar argument with the existence and uniqueness of geometric flow such as Ricci flow, and Ricci-Bourguignon flow, we establish the short-time existence and uniqueness for the GRBF system on a compact n-dimensional Riemannian manifold. Firstly, we show that if $H(x)$ is closed, then we have the following propositions.

Proposition 1. *Along the RGRBF, the form $H(x,t)$ is closed if the initial value $H(x)$ is closed.*

Proof. The exterior derivative d is independent of the metric, so we get

$$\frac{\partial}{\partial t}dH(x,t) = d\frac{\partial}{\partial t}H(x,t) = d\left(-dd^*_{g(x,t)}H(x,t)\right) = 0.$$

Therefore, $dH(x,t)$ is independent of time variable t and $dH(x,t) = dH(x)$. Since $H(x)$ is closed form, we conclude that $dH(x,t) = 0$. ☐

Proposition 2. *If $\left(g(x,t), H(x,t)\right)$ is a solution to RGRBF and the initial value $H(x)$ is closed form, then $\left(g(x,t), H(x,t)\right)$ is also a solution to GRBF.*

Proof. From Proposition 1, since $H(x)$ is closed form, $H(x,t)$ is closed form under the RGRBF. Therefore,

$$\Delta_{HL,g(x,t)}H(x,t) = -dd^*_{g(x,t)}H(x,t).$$

☐

Theorem 1. *Let $\rho < \frac{1}{2(n-1)}$. Then the evolution equation GRBF has a unique solution for a short time on any smooth, n-dimensional, closed Riemannian manifold M.*

Proof. For the proof of theorem we use the DeTurk trick in Ricci flow to prove its short time existence. Let $\left(g(x,t), H(x,t)\right)$ be the solution of the GRBF and $\phi_t : M \to M$ be a family of smooth diffeomorphisms of M. Suppose $\hat{g}(x,t) = \phi_t^* g(x,t)$ is the pull-back metric of $g(x,t)$. For computing the evolution equation for the metric $\hat{g}(x,t)$, let

$$y(x,t) = \phi_t(x) = \left\{y^1(x,t), y^2(x,t), \dots, y^n(x,t)\right\},$$

in local coordinates system $x = \left\{x^1, x^2, \dots, x^n\right\}$. Following the same calculations as in [10], we get

$$\hat{g}_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}g_{\alpha\beta}(y,t)$$

and

$$\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\left(\frac{\partial}{\partial t}g_{\alpha\beta}(y,t)\right) + \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}g_{\alpha\beta}(y,t) + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)g_{\alpha\beta}(y,t).$$

For a fixed point $p \in M$, we consider a normal coordinate $\left\{x^i\right\}$ around p such that $\frac{\partial \hat{g}_{ij}}{\partial x^k} = 0$ at p. Since $g(x,t)$ is the solution of GRBF, we get

$$\frac{\partial}{\partial t}g_{\alpha\beta}(x,t) = -2R_{\alpha\beta}(x,t) + 2\rho R g_{\alpha\beta}(x,t) + h_{\alpha\beta}(x,t),$$

where $h_{\alpha\beta} = \frac{1}{2} H_{\alpha kl} H^{kl}_{\beta}(x,t)$. Hence,

$$\frac{\partial}{\partial t} g_{\alpha\beta}(y,t) = -2R_{\alpha\beta}(y,t) + 2\rho R g_{\alpha\beta}(y,t) + h_{\alpha\beta}(y,t) + \frac{\partial g_{\alpha\beta}}{\partial y^{\gamma}} \frac{\partial y^{\gamma}}{\partial t},$$

and substituting above equality in (2) we infer

$$\begin{aligned}\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) =& -2\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j} R_{\alpha\beta}(y,t) + 2\rho \frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j} R g_{\alpha\beta}(y,t) + \frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j} h_{\alpha\beta}(y,t) \\ &+ \frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial g_{\alpha\beta}}{\partial y^{\gamma}}\frac{\partial y^{\gamma}}{\partial t} + \frac{\partial}{\partial x^i}\left(\frac{\partial y^{\alpha}}{\partial t}\right)\frac{\partial y^{\beta}}{\partial x^j} g_{\alpha\beta}(y,t) + \frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^{\beta}}{\partial t}\right) g_{\alpha\beta}(y,t).\end{aligned}$$

Since

$$\hat{R}_{ij}(x,t) = \frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j} R_{\alpha\beta}(y,t), \qquad \hat{R}(x,t) = R(y,t), \qquad \hat{h}_{ij}(x,t) = \frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j} h_{\alpha\beta}(y,t),$$

and in the normal coordinate

$$\begin{aligned}\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial t}\frac{\partial g_{\alpha\beta}}{\partial y^{\gamma}} &= \frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial t}\hat{g}_{kl}(x,t)\frac{\partial}{\partial y^{\gamma}}\left(\frac{\partial x^k}{\partial y^{\alpha}}\frac{\partial x^l}{\partial y^{\beta}}\right) \\ &= \frac{\partial y^{\alpha}}{\partial t}\frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial y^{\alpha}}\right)\hat{g}_{jk}(x,t) + \frac{\partial y^{\beta}}{\partial t}\frac{\partial}{\partial x^j}\left(\frac{\partial x^k}{\partial y^{\beta}}\right)\hat{g}_{ik}(x,t) \\ &= \frac{\partial}{\partial x^i}\left(\frac{\partial y^{\alpha}}{\partial t}\frac{\partial x^k}{\partial y^{\alpha}}\hat{g}_{jk}(x,t)\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial y^{\beta}}{\partial t}\frac{\partial x^k}{\partial y^{\beta}}\hat{g}_{ik}(x,t)\right) \\ &\quad - \frac{\partial}{\partial x^i}\left(\frac{\partial y^{\alpha}}{\partial t}\right)\frac{\partial x^k}{\partial y^{\alpha}}\hat{g}_{jk}(x,t) - \frac{\partial}{\partial x^j}\left(\frac{\partial y^{\beta}}{\partial t}\right)\frac{\partial x^k}{\partial y^{\beta}}\hat{g}_{ik}(x,t) \\ &= \frac{\partial}{\partial x^i}\left(\frac{\partial y^{\alpha}}{\partial t}\frac{\partial x^k}{\partial y^{\alpha}}\hat{g}_{jk}(x,t)\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial y^{\beta}}{\partial t}\frac{\partial x^k}{\partial y^{\beta}}\hat{g}_{ik}(x,t)\right) \\ &\quad - \frac{\partial}{\partial x^i}\left(\frac{\partial y^{\alpha}}{\partial t}\right)\frac{\partial y^{\beta}}{\partial x^j} g_{\alpha\beta}(y,t) - \frac{\partial}{\partial x^j}\left(\frac{\partial y^{\beta}}{\partial t}\right)\frac{\partial y^{\alpha}}{\partial x^i} g_{\alpha\beta}(y,t),\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) =& -2\hat{R}_{ij}(x,t) + 2\rho R\hat{g}_{ij}(x,t) + \hat{h}_{ij}(x,t) \\ &+ \nabla_i\left(\frac{\partial y^{\alpha}}{\partial t}\frac{\partial x^k}{\partial y^{\alpha}}\hat{g}_{jk}(x,t)\right) + \nabla_j\left(\frac{\partial y^{\beta}}{\partial t}\frac{\partial x^k}{\partial y^{\beta}}\hat{g}_{ik}(x,t)\right).\end{aligned} \tag{5}$$

According to DeTurk trick, if we define $y(x,t) = \phi_t(x)$ by the equations

$$\frac{\partial y^{\alpha}}{\partial t} = \frac{\partial y^{\alpha}}{\partial x^k}\hat{g}^{jl}\left(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}\right), \tag{6}$$

and $V_i = \hat{g}_{ik}\hat{g}^{jl}\left(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}\right)$, then (5) becomes

$$\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = -2\hat{R}_{ij}(x,t) + 2\rho R\hat{g}_{ij}(x,t) + \hat{h}_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i, \quad \hat{g}_{ij}(x,0) = \hat{g}(x).$$

Also,

$$\frac{\partial \hat{H}}{\partial t} = \phi_t^* \left(\frac{\partial H}{\partial t} + L_V H \right) = \Delta_{LB} \hat{H} - d\langle \hat{H}, V \rangle, \quad \hat{H}(x,0) = \hat{H}(x).$$

Since

$$\hat{\Gamma}_{ij}^k = \frac{\partial y^\alpha}{\partial x^j} \frac{\partial y^\beta}{\partial x^i} \frac{\partial x^k}{\partial y^\gamma} \Gamma_{\alpha\beta}^\gamma + \frac{\partial x^k}{\partial y^\alpha} \frac{\partial^2 y^\alpha}{\partial x^j \partial x^i},$$

the initial value problem (6) can be rewritten as

$$\begin{cases} \dfrac{\partial y^\alpha}{\partial t} = \hat{g}^{jl} \left(\Gamma_{\alpha\gamma}^\beta \dfrac{\partial y^\gamma}{\partial x^i} \dfrac{\partial y^\beta}{\partial x^j} + \dfrac{\partial^2 y^\alpha}{\partial x^j \partial x^l} - \mathring{\Gamma}_{jl}^k \dfrac{\partial y^\alpha}{\partial x^k} \right), \\ y^\alpha(x,0) = x^\alpha. \end{cases} \tag{7}$$

Note, that (7) is strictly parabolic system. Since manifold M is compact, it follows from the theory of parabolic equations that the system (7) has a unique smooth solution for a short time. At the same time, we have

$$\begin{aligned} \frac{\partial}{\partial t} \hat{g}_{ij}(x,t) = \hat{g}^{kl} \frac{\partial^2 \hat{g}_{ij}}{\partial x^k \partial x^l}(x,t) &- 2\rho \hat{g}_{ij} \hat{g}^{pq} \hat{g}^{kl} \frac{\partial^2 \hat{g}_{kl}}{\partial x^p \partial x^q}(x,t) \\ &+ 2\rho \hat{g}_{ij} \hat{g}^{pq} \hat{g}^{kl} \frac{\partial^2 \hat{g}_{ql}}{\partial x^p \partial x^k}(x,t) + \text{lower order terms}, \end{aligned} \tag{8}$$

and

$$\frac{\partial}{\partial t} \hat{H}_{ijk}(x,t) = \hat{g}^{rs} \frac{\partial^2 \hat{H}_{ijk}}{\partial x^r \partial x^s}(x,t) + \text{lower order terms}. \tag{9}$$

From [2, 4] it follows that for $\rho < \frac{1}{2(n-1)}$ the equations (8) and (9) form a strictly parabolic system. Since manifold M is compact, by the standard theory of parabolic equations, the system (8)–(9) have a unique smooth solution for a short time. From the solution of (8)–(9) we can obtain a solution of the GRBF (4).

Let us show the uniqueness of the solution. For any two solutions $g_{ij}^{(1)}(x,t)$ and $g_{ij}^{(2)}(x,t)$ of the GRBF system (4) with the same initial data, we solve the initial value problem (7) and find two families $\phi_t^{(1)}$ and $\phi_t^{(2)}$ of diffeomorphisms of M. Therefore we get two solutions

$$\hat{g}_{ij}^{(1)}(x,t) = \left(\phi_t^{(1)}\right)^* g_{ij}^{(1)}(x,t) \quad \text{and} \quad \hat{g}_{ij}^{(2)}(x,t) = \left(\phi_t^{(2)}\right)^* g_{ij}^{(2)}(x,t)$$

to the modified evolution equation (8) with same initial data $\hat{g}_{ij}(x,0) = g_{ij}(x)$. The uniqueness result for the strictly parabolic equation (8) implies that $\hat{g}_{ij}^{(1)}(x,t) = \hat{g}_{ij}^{(2)}(x,t)$ and then by system (7) and the standard uniqueness result of PDE system, the corresponding solutions $\phi_t^{(1)}$ and $\phi_t^{(2)}$ of (7) must agree. Consequently the metrics $g_{ij}^{(1)}(x,t)$ and $g_{ij}^{(2)}(x,t)$ must agree also. Hence we have proved the uniqueness for solution of the GRBF (4). $\square$

3 GRBF system soliton

In this section, we introduce the GRBF system soliton and give some properties of this soliton.

Definition 1. *A solution $\big(g(t), H(t)\big)$, $t \in [0, T)$, of the GRBF system on manifold M^n with initial data $\big(g(0), H(0)\big)$ is called a GRBF system soliton if there exists a one-parameter family of diffeomorphisms $\psi_t : M \to M$ with $\psi_0 = id_M$ and a scaling function $c : [0, T) \to \mathbb{R}_+$ such that*

$$\begin{cases} g(t) = c(t)\psi_t^* g(0), \\ H(t) = \psi_t^* H(0). \end{cases} \tag{10}$$

The cases $\dot{c} = \frac{\partial}{\partial t} c(t) < 0$, $\dot{c} = 0$ and $\dot{c} > 0$ correspond to shrinking, steady and expanding solitons, respectively. If the diffeomorphisms ψ_t are generated by a vector field $X(t) = \nabla f(t)$ for some function $f(t)$ on M, then the soliton called gradient soliton and f is called the potential of the soliton.

Lemma 1. *Let M^n be a Riemannain manifold and $\big(g(t), H(t)\big)$, $t \in [0, T)$, be a GRBF system soliton. Then there exists a vector field X on M^n such that*

$$\begin{cases} -Ric\big(g(0)\big) + \rho R_{g(0)} g(0) + \frac{1}{2} h(0) = \frac{1}{2} \mathcal{L}_X g(0) + \lambda g(0), \\ \Delta_{HL, g(0)} H(0) = \mathcal{L}_X H(0), \end{cases} \tag{11}$$

where $\lambda = \frac{1}{2}\dot{c}(0)$ and $\mathcal{L}_X g(0)$ denotes the Lie derivative of the metric $g(0)$ with respect to the vector field X.

Conversely, given a vector field X on M and a solution of (11)*, there exist one-parameter families of scalars $c(t)$ and diffeomorphisms $\psi_t : M \to M$ with $\psi_0 = id_M$ such that $\big(g(t), H(t)\big)$, $t \in [0, T)$, becomes a solution of the GRBF system, when $\big(g(t), H(t)\big)$ is defined by* (10).

Proof. First suppose that $\big(g(t), H(t)\big)$, $t \in [0, T)$, is a GRBF system soliton. Without loss of generality we assume that $c(0) = 1$ and $\psi_0 = id_M$. Then we infer

$$\begin{aligned} -2Ric\big(g(0)\big) + 2\rho R_{g(0)} g(0) + h(0) &= \frac{\partial}{\partial t} g(t)|_{t=0} \\ &= \frac{\partial}{\partial t} \left(c(t)\psi_t^* g(0)\right)|_{t=0} = \dot{c}(0) g(0) + \mathcal{L}_{Y(0)} g(0), \end{aligned}$$

and

$$\Delta_{HL, g(0)} H(0) = \frac{\partial}{\partial t} H(t)|_{t=0} = \frac{\partial}{\partial t} \left(\psi_t^* H(0)\right)|_{t=0} = \mathcal{L}_{Y(0)} H(0),$$

where $Y(t)$ is the family of vector fields generating the diffeomorphisms ψ_t. This implies that $\big(g(0), H(0)\big)$ satisfies (11) with $\lambda = \frac{1}{2}\dot{c}(0)$ and $X = Y(0)$.

Conversely, suppose that $\big(g(0), H(0)\big)$ satisfies (11) for some vector field X on M. Define $c(t) := 1 + 2\lambda t$ and define an one-parameter family of vector fields $Y(t)$ on M by $Y(t) := \frac{1}{c(t)} X$. Suppose that ψ_t are the diffeomorphisms generated by the family $Y(t)$, where $\psi_0 = id_M$ and define $\big(g(t), \phi(t)\big)$ as in (10). The computation

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= \dot{c}(t)\psi_t^*\big(g(0)\big) + c(t)\psi_t^*\big(\mathcal{L}_{Y(t)} g(0)\big) = \psi_t^*\big(2\lambda g(0) + \mathcal{L}_X g(0)\big) \\ &= \psi_t^*\Big(-2Ric\big(g(0)\big) + 2\rho R_{g(0)} g(0) + h(0)\Big) = -2Ric\big(g(t)\big) + 2\rho R_{g(t)} g(t) + h(t), \end{aligned}$$

and

$$\frac{\partial}{\partial t}H(t)=\psi_t^*\big(\mathcal{L}_{Y(t)}H(0)\big)=\frac{1}{c(t)}\psi_t^*\big(\mathcal{L}_X H(0)\big)=\frac{1}{c(t)}\psi_t^*\big(\Delta_{HL,g(0)}H(0)\big)=\Delta_{HL,g(t)}H(t),$$

imply that $\big(g(t),H(t)\big)$ is a solution of the GRBF system. □

After then we say that (M^n,g,H,X,λ,ρ) is a GRBF system soliton whenever it satisfies (11) and it is shrinking, steady and expanding, if $\lambda<0$, $\lambda=0$ and $\lambda>0$, respectively. Also, we say that $(M^n,g,H,\nabla f,\lambda,\rho)$ is a gradient GRBF system soliton if it satisfies

$$\begin{cases}-Ric+\rho Rg+\frac{1}{2}h=\mathrm{Hess}f+\lambda g,\\ \Delta_{HL,g}H=d\langle H,\nabla f\rangle.\end{cases}\tag{12}$$

In the following, we provide some equation of structure for gradient GRBF system soliton.

Proposition 3. *For a gradient GRBF system soliton $(M^n,g,H,\nabla f,\lambda,\rho)$, the following identities hold*

(i) $(1-n\rho)R+\Delta f-\frac{1}{4}|H|^2+n\lambda=0,$

(ii) $\big(1-2(n-1)\rho\big)\nabla_iR=-\nabla^jh_{ij}+2R_{il}\nabla^lf+\frac{1}{2}\nabla_i|H|^2,$

(iii) $\big(1-2\rho(n-1)\big)\nabla_iR=2\rho hR\nabla_if+h_{il}\nabla^lf-\nabla_i|\nabla f|^2-2\lambda\nabla_if-\nabla^jh_{ij}+\frac{1}{2}\nabla_i|H|^2.$

Proof. For a gradient GRBF system soliton the identity (12) holds. Taking trace of equation (12) yields (i). In order to obtain equation (ii), take the covariant derivative of (i) in an orthonormal frame, this gives

$$(1-n\rho)\nabla_iR+\nabla_i\nabla^j\nabla_jf-\frac{1}{4}\nabla_i|H|^2=0.$$

Using the Bianchi identity and the contracted second Bianchi identity, we get

$$\begin{aligned}(1-n\rho)\nabla_iR&=-\nabla^j\nabla_i\nabla_jf+R_{il}\nabla^lf+\frac{1}{4}\nabla_i|H|^2\\&=-\nabla^j\left(-R_{ij}-\lambda g_{ij}+\rho Rg_{ij}+\frac{1}{2}h_{ij}\right)+R_{il}\nabla^lf+\frac{1}{4}\nabla_i|H|^2\\&=\frac{1}{2}\nabla_iR-\rho\nabla_iR-\frac{1}{2}\nabla^jh_{ij}+R_{il}\nabla^lf+\frac{1}{4}\nabla_i|H|^2,\end{aligned}$$

which proves (ii). Now, to prove equation (iii), from (ii) we obtain

$$\begin{aligned}\big(1-2\rho(n-1)\big)\nabla_iR&=2\nabla^lf\left(-\nabla_i\nabla_lf-\lambda g_{il}+\rho Rg_{il}+\frac{1}{2}h_{il}\right)-\nabla^jh_{ij}+\frac{1}{2}\nabla_i|H|^2\\&=-\nabla_i|\nabla f|^2-2\lambda\nabla_if+2\rho R\nabla_if+h_{il}\nabla^lf-\nabla^jh_{ij}+\frac{1}{2}\nabla_i|H|^2.\end{aligned}$$

Hence the proof of proposition complete. □

Theorem 2. *Let $(M^n,g,H,\nabla f,\lambda,\rho)$ be a compact gradient GRBF system soliton. Then*

$$\begin{aligned}\int_M2\left|\nabla^2f-\frac{\Delta f}{n}g+\frac{1}{4n}|H|^2g\right|^2d\mu=\frac{n-2}{n}\int_Mg(\nabla R,\nabla f)d\mu-\frac{1}{2n}\int_M\Delta f|H|^2d\mu\\+\int_M(f-1)\nabla^i\nabla^jh_{ij}d\mu+\frac{1}{8n}\int_M|H|^4d\mu,\end{aligned}\tag{13}$$

$$\int_M\left|Ric-\frac{R}{n}g-\frac{1}{2}h\right|^2d\mu=\int_M\left|\nabla^2f-\frac{\Delta f}{n}g+\frac{1}{4n}|H|^2g\right|^2d\mu.\tag{14}$$

Proof. Taking the divergence of equation (iii) from Proposition 3, we get

$$\begin{aligned}\big(1-2\rho(n-1)\big)\Delta R &= -\Delta|\nabla f|^2 - 2\lambda\Delta f + 2\rho\nabla^i R\nabla_i f + 2\rho R\Delta f \\ &\quad + \nabla^i h_{il}\nabla^l f + h_{il}\nabla^i\nabla^l f - \nabla^i\nabla^j h_{ij} + \frac{1}{2}\Delta|H|^2.\end{aligned} \tag{15}$$

On the other hand we have

$$\Delta|\nabla f|^2 = 2\nabla_i f\nabla^i\Delta f + 2R_{ij}\nabla^i f\nabla^j f + 2|\nabla^2 f|^2.$$

Taking covariant derivative of equation (i) from Proposition 3 and using (12), we conclude

$$\begin{aligned}0 = \nabla_i\Delta f + (1-n\rho)\nabla_i R - \frac{1}{4}\nabla_i|H|^2 &= (1-n\rho)\nabla_i R - \frac{1}{4}\nabla_i|H|^2 \\ &\quad + \nabla^j\left(-R_{ij} - \lambda g_{ij} + \rho R g_{ij} + \frac{1}{2}h_{ij}\right) - hR_{il}\nabla^l f \\ &= \left(\frac{1}{2} - \rho(n-1)\right)\nabla_i R - \frac{1}{4}\nabla_i|H|^2 - R_{il}\nabla^l f + \frac{1}{2}\nabla^j h_{ij}.\end{aligned}$$

Therefore

$$0 = \big(1-2\rho(n-1)\big)\nabla_i R\nabla^i f - \frac{1}{2}\nabla^i\nabla_i|H|^2 + \nabla^j h_{ij}\nabla^i f - 2R_{il}\nabla^l f\nabla^i f,$$

and it implies that

$$\Delta|\nabla f|^2 = 2\nabla_i f\nabla^i\Delta f + \big(1-2\rho(n-1)\big)\nabla_i R\nabla^i f - \frac{1}{2}\nabla^i\nabla_i|H|^2 + \nabla^j h_{ij}\nabla^i f + 2|\nabla^2 f|^2. \tag{16}$$

Identity $(1-n\rho)\nabla_i R + \nabla_i\Delta f - \frac{1}{4}\nabla^i\nabla_i|H|^2 = 0$ implies

$$(1-n\rho)\nabla_i R\nabla^i f + \nabla_i\Delta f\nabla^i f - \frac{1}{4}\nabla^i\nabla_i|H|^2\nabla^i f = 0.$$

Substituting the above equality into (16), we infer

$$\Delta|\nabla f|^2 = (2\rho-1)\nabla_i R\nabla^i f + \nabla^j h_{ij}\nabla^i f + 2|\nabla^2 f|^2.$$

Therefore we can write (15) as follows

$$\big(1-2\rho(n-1)\big)\Delta R = \nabla^i R\nabla_i f + 2\rho R\Delta f + h_{il}\nabla^i\nabla^l f - 2|\nabla^2 f|^2 - 2\lambda\Delta f - \nabla^i\nabla^j h_{ij} + \frac{1}{2}\Delta|H|^2.$$

Then

$$\begin{aligned}\big(1-2\rho(n-1)\big)\Delta R + 2\lambda\Delta f - \frac{1}{2}\Delta|H|^2 &= 2\rho R\Delta f + \nabla^i R\nabla_i f + h_{il}\nabla^i\nabla^l f - \nabla^i\nabla^j h_{ij} \\ &\quad - 2\left|\nabla^2 f - \frac{\Delta f}{n}g + \frac{1}{4n}|H|^2 g\right|^2 - \frac{1}{8n}|H|^4 \\ &\quad - 2\frac{\Delta f}{n}\left((n\rho-1)R - n\lambda + \frac{1}{4}|H|^2\right).\end{aligned}$$

By integrating of both sides of the above identity on closed Riemannian manifold M, we obtain (13). Since

$$\begin{aligned}Ric - \frac{R}{n}g - \frac{1}{2}h = -\nabla^2 f - \lambda g + \rho R g - \frac{R}{n}g &= -\nabla^2 f + \left(-\lambda + \rho R - \frac{R}{n}\right)g \\ &= -\nabla^2 f + \left(\Delta f - \frac{1}{4}|H|^2\right)\frac{g}{n},\end{aligned}$$

we conclude (14). □

Corollary 1. *In a nontrivial compact gradient GRBF system soliton $(M^n, g, H, \nabla f, \lambda, \rho)$ with $n \geq 3$, vector field ∇f is a nontrivial conformal vector field, if the following condition holds*

$$\frac{n-2}{n}\int_M g(\nabla R, \nabla f)d\mu - \frac{1}{2n}\int_M \Delta f|H|^2 d\mu + \int_M (f-1)\nabla^i\nabla^j h_{ij}d\mu + \frac{1}{8n}\int_M |H|^4 d\mu \leq 0.$$

Proof. The assumptions of Corollary conclude that the right hand side of (13) is less that or equal to zero, but left hand side of (13) is greater than or equal to zero, hence $Ric = \frac{R}{n}g + \frac{1}{2}h$. So, $\nabla\nabla f = \left(-\lambda + R\left(\rho - \frac{1}{n}\right)\right)g$. Therefore ∇f is a nontrivial conformal vector field. □

Theorem 3. *Let (M, g) be a complete Riemannian manifold with a 3-form $H = \{H_{ijk}\}$ satisfying*

$$Ric - \rho R g - \frac{1}{2}h + \frac{1}{2}\mathcal{L}_X g \geq \sigma g \tag{17}$$

for some smooth vector field X on M and some constant $\sigma > 0$. Let $h \geq 0$ and $\rho R \geq a$ for some constant a such that $a + \sigma > 0$. Then M is compact if and only if $\|X\|$ is bounded on (M, g) by a constant K. Moreover, in this case, we have

$$diam(M) \leq \frac{2\pi}{a+\sigma}\left(K + \sqrt{K^2 + (n-1)\frac{a+\sigma}{2}}\right).$$

Proof. If manifold M is compact, then it obvious that $\|X\|$ is bounded.

Conversely, let $\|X\|$ be bounded by a constant K and p be a point in M. Consider any geodesic $\gamma : [0, +\infty) \to M$ emanating p and parameterized by arc length t. Along geodesic γ we have

$$\mathcal{L}_X g\left(\gamma'(t), \gamma'(t)\right) = 2g\left(\nabla_{\gamma'(t)}X, \gamma'(t)\right) = 2\frac{d}{dt}g\left(X, \gamma'(t)\right). \tag{18}$$

Multiplying the both sides of (17) by $\gamma'^i\gamma'^j$ and using (18) we obtain

$$\gamma'^i\gamma'^j R_{ij} \geq \gamma'^i\gamma'^j\sigma g_{ij} + \gamma'^i\gamma'^j\rho R g_{ij} + \frac{1}{2}\gamma'^i\gamma'^j h_{ij} - \frac{d}{dt}\left(\gamma'^k X_k\right) = a + \sigma + \frac{d}{dt}\left(-\gamma'^k X_k\right).$$

The Cauchy-Schwaz inequality implies that

$$\begin{aligned}\left|-\gamma'^k X_k\right| = \left|g_{kl}\left(\gamma(t), \gamma'(t)\right)\gamma'^k X_k\right| &\leq \left|g_{kl}\left(\gamma(t), \gamma'(t)\right)X^k X^l\right|^{\frac{1}{2}} \\ &\leq \max\left|g_{kl}\left(\gamma(t), \gamma'(t)\right)X^k X^l\right|^{\frac{1}{2}} = \|X\|_{\gamma(t)} \leq K\end{aligned}$$

and the results follow from [1, Lemma 1]. □

Corollary 2. *Let $(M, g, H, X, \lambda, \rho)$ be a complete shrinking GRBF system soliton. Let $h \geq 0$ and $\rho R \geq a$ for some constant a such that $a + \lambda > 0$. Then M is compact if and only if $\|X\|$ is bounded on (M, g) by a constant K. Moreover, in this case, we have*

$$diam(M) \leq \frac{2\pi}{a+\lambda}\left(K + \sqrt{K^2 + (n-1)\frac{a+\lambda}{2}}\right).$$

Remark 1. *In Theorem 3, the assumption of boundedness of $\|X\|$ is necessary to show that M is compact. For instance, Euclidean space with the vector field $X(v) = v, \forall v \in \mathbb{R}^n$, and $H = 0$ satisfies* (17).

Theorem 4. *Let (M, g) be a complete Riemannian manifold with a vector field X and 3-form $H = \{H_{ijk}\}$ satisfying $h \geq 0$ and (17) for some $\sigma > 0$. Then for any $p, q \in M$ and $\rho \geq 0$ we have*

$$d(p,q) \leq \max\left\{1, \frac{1}{\sigma+\rho\Lambda}\Big(2(n-1)+G_p+G_q+2\|X_p\|+\|X_q\|\Big)\right\}, \tag{19}$$

where $\sigma + \rho\Lambda \geq 0$, $R \geq \Lambda$, and

$$G_p = \max\Big\{0, \sup\left\{Ric_y(v,v) : y \in B(p,1),\ \|v\| = 1\right\}\Big\}.$$

Proof. Assume that $r = d(p,q) > 1$ and let γ be the minimal geodesic from p to q parameterized by arc length. Using equations (17) and $\mathcal{L}_X\big(\gamma'(s), \gamma'(s)\big) = 2\frac{d}{ds}g\big(X, \gamma'(s)\big)$, we get

$$\begin{aligned}\int_0^r Ric\big(\gamma'(s),\gamma'(s)\big)ds &\geq \int_0^r \Big[\sigma g\big(\gamma'(s),\gamma'(s)\big) - \frac{1}{2}\mathcal{L}_X g\big(\gamma'(s),\gamma'(s)\big) \\ &\qquad + \rho R g\big(\gamma'(s),\gamma'(s)\big)\Big]ds + \frac{1}{2}\int_0^r h\big(\gamma'(s),\gamma'(s)\big)ds \\ &\geq \sigma d(p,q) + g_p\big(X,\gamma'(0)\big) - g_q\big(X,\gamma'(s)\big) + \rho\int_0^r R ds \\ &\geq \sigma d(p,q) + g_p\big(X,\gamma'(0)\big) - g_q\big(X,\gamma'(s)\big) + \rho\Lambda d(p,q).\end{aligned} \tag{20}$$

On the other hand, [13, Lemma 2.2] implies that

$$\int_0^r Ric\big(\gamma'(s),\gamma'(s)\big)ds \leq 2(n-1) + G_p + G_q. \tag{21}$$

Combining (20) and (21) and solving for $d(p,q)$ gives (19). □

Theorem 5. *Let M be a complete Riemannian manifold satisfying* (17), *where $h \geq 0$, $R \geq \Lambda$, $\rho > 0$, $\sigma > 0$ and $\sigma + \rho\Lambda \geq 0$. Then M has finite fundamental group.*

Proof. Let $f : \tilde{M} \to M$ be the universal covering manifold of M. Notice that the fundamental group of M is in one-to-one corresponding with discrete counterimage of a base point $p \in M$ and

$$f^*g = \tilde{g}, \quad f^*H = \tilde{H}, \quad f^*Ric = \tilde{Ric}, \quad f^*\mathcal{L}_X g = \mathcal{L}_{\tilde{X}}\tilde{g}.$$

Inequality (17) implies that

$$\tilde{Ric} - \rho\tilde{R}\tilde{g} - \frac{1}{2}\tilde{h} + \frac{1}{2}\mathcal{L}_{\tilde{X}}\tilde{g} \geq \sigma\tilde{g}.$$

Fix $\tilde{p}$ in $\tilde{M}$. Let $\alpha \in \pi_1(M)$ identifies a deck transformation on $\tilde{M}$. A deck transformation on the universal covering manifold $\tilde{M}$ is an isometry. Also, $B\big(\tilde{p}, 1\big)$ and $B\big(\alpha(\tilde{p}), 1\big)$ are isometric. Hence, $G_p = G_{\alpha(p)}$ and $\|\tilde{X}_{\tilde{p}}\| = \|\tilde{X}_{\alpha(\tilde{p})}\|$. Then, by applying Theorem 4 to the point $\tilde{p}$ and $\alpha(\tilde{p})$, we get

$$d\big(\tilde{p}, \alpha(\tilde{p})\big) \leq \max\left\{1, \frac{2}{\sigma+\rho\Lambda}\Big(n-1+G_{\tilde{p}}+\|\tilde{X}_{\tilde{p}}\|\Big)\right\}$$

for any deck transformation α. Thus the set $\alpha^{-1}(p)$ is bounded, where $p = \alpha(\tilde{p})$. By applying the geodesically completeness and the Hopf-Rinow's theorem, the closed and bounded subset $\alpha^{-1}(p)$ of $\tilde{M}$ is compact and being discrete is finite. Since M is connected and $\pi_1(M,p)$ is in bijective relation with $\alpha^{-1}(p)$, we imply that $\pi_1(M)$ is finite. □

Corollary 3. *Let $(M, g, H, X, u, \lambda, \rho)$ be a complete shrinking GRBF system soliton. If $h \geq 0$, $R \geq \Lambda$, $\rho > 0$ and $-\lambda + \rho\Lambda \geq 0$, then M has finite fundamental group.*

4 Evolution of the curvatures

In this section, we compute evolution equations for curvature tensors under the GRBF system (4). As the metric tensor evolves by

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + 2\rho R g_{ij} + h_{ij},$$

where $h_{ij} = \frac{1}{2}H_{ikl}H_j^{kl}$, we get

$$\frac{\partial}{\partial t}g^{ij} = 2R^{ij} - 2\rho R g^{ij} - h^{ij},$$

and the canonical volume measure $d\mu$ evolves by

$$\frac{\partial d\mu}{\partial t} = \frac{1}{2}tr_g\left(\frac{\partial g}{\partial t}\right)d\mu = \left[(n\rho - 1)R + \frac{1}{4}|H|^2\right]d\mu.$$

To compute evolution equation for curvature tensors, we need the following results for a general flow (see [5, Lemma 6.5], [11]).

Lemma 2. *Let $(M^n, g(t))$ be a Riemannian manifold with $\frac{\partial}{\partial t}g_{ij} = v_{ij}$, then*

$$\begin{aligned}
&\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right),\\
&\frac{\partial}{\partial t}R_{ijkl} = \frac{1}{2}\left[\nabla_i\nabla_l v_{jk} + \nabla_j\nabla_k v_{il} - \nabla_i\nabla_k v_{jl} - \nabla_j\nabla_l v_{ik}\right] + \frac{1}{2}g^{pq}(R_{ijpl}v_{kq} - R_{ijpk}v_{lq}),\\
&\frac{\partial}{\partial t}R_{ij} = -\frac{1}{2}\left[\Delta_L v_{ij} + \nabla_i\nabla_j(trv) - g^{pq}\left(\nabla_i\nabla_p v_{jq} + \nabla_j\nabla_p v_{iq}\right)\right],\\
&\frac{\partial R}{\partial t} = -\Delta v + g^{pq}g^{rs}\left(\nabla_p\nabla_r v_{qs} - R_{pr}v_{qs}\right),
\end{aligned}$$

where $\Delta_L v_{ij} := \Delta v_{ij} + 2R_{iljp}v^{lp} - R_i^p v_{jp} - R_j^p v_{ip}$.

By computing in a normal coordinates system, the evolution equation for the Christoffel symbols is given by

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = -\nabla_j R_i^k - \nabla_i R_j^k - \nabla^k R_{jk} + \rho\left(\nabla_j R\delta_i^k + \nabla_i R\delta_j^k + \nabla^k R g_{ij}\right) + \frac{1}{2}\left(\nabla_i h_j^k + \nabla_j h_i^k - \nabla^k h_{ij}\right).$$

Proposition 4. *Under the GRBF system* (4)*, the Riemannian curvature tensor R_{ijkl} of $(M^n, g(t))$ satisfies the evolution equation*

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ijkl} = {}& \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right) - R_{pjkl}R_i^p - R_{ipkl}R_j^p - R_{ijpl}R_k^p - R_{ijkp}R_l^p\\
&+ \frac{1}{2}\left(R_{ijpl}h_k^p - R_{ijpk}h_l^p\right) + \frac{1}{2}\left(\nabla_i\nabla_l h_{jk} + \nabla_j\nabla_k h_{il} - \nabla_i\nabla_k h_{jl} - \nabla_j\nabla_l h_{ik}\right)\\
&- \rho\left(\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}\right) + 2\rho R R_{ijkl},
\end{aligned} \tag{22}$$

where $B_{ijkl} = g^{pq}g^{rs}R_{ipjr}R_{kqls}$.

Proof. Since the quantities $-2R_{ij} + 2\rho R g_{ij}$ and h_{ij} are independent, we can compute the evolution of Riemannian curvature tensor along the metric evolving by those two quantities separately. In [4], for $v_{ij} = -2R_{ij} + 2\rho R g_{ij}$ it has been shown that

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ijkl} = {}& \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right) - R_{pjkl}R_i^p - R_{ipkl}R_j^p - R_{ijpl}R_k^p - R_{ijkp}R_l^p\\
&- \rho\left(\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}\right) + 2\rho R R_{ijkl}.
\end{aligned}$$

If $v_{ij} = h_{ij}$, using Lemma 2, we have that

$$\frac{\partial}{\partial t}R_{ijkl} = \frac{1}{2}\left(R_{ijpl}h_k^p - R_{ijpk}h_l^p\right) + \frac{1}{2}\left(\nabla_i\nabla_l h_{jk} + \nabla_j\nabla_k h_{il} - \nabla_i\nabla_k h_{jl} - \nabla_j\nabla_l h_{ik}\right).$$

Therefore (22) follows by adding the above two evolution formulas. □

Definition 2. *Suppose that A and B are two tensorial quantities on a Riemannian manifold (M, g). We denote by $A * B$ any tensor quantity obtained from $A \otimes B$ by summation over pairs of matching indices, contractions on lower indices with respect to the dual metric, contractions on upper indices with respect to the metric, and multiplication with constants depending only on $n = \dim M$ and ranks of A and B. We also write $A^{*1} := 1 * A$ and $A^{*2} := A * A$.*

Corollary 4. *Under the GRBF system* (4) *we have*

$$\frac{\partial}{\partial t}Rm = \Delta Rm + Rm * Rm + \rho\nabla^2 R * g + \rho R Rm + H * H * Rm + \sum_{i=0}^{2}\nabla^i H * \nabla^{2-i}H. \tag{23}$$

Proof. From Proposition 4 we obtain

$$\frac{\partial}{\partial t}Rm = \Delta Rm + Rm * Rm + \rho\nabla^2 R * g + \rho R Rm + \nabla^2 h + h * Rm. \tag{24}$$

Since $h = H * H$, we get

$$\nabla^2 h = \nabla\big(\nabla(H * H)\big) = \nabla\big(\nabla H * H\big) = \nabla^2 H * H + \nabla H * \nabla H. \tag{25}$$

Substituting (25) into (24), we obtain the result. □

Proposition 5. *The evolution equation of Ricci curvature tensor under the GRBF system* (4) *is as follow*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ik} = {} & \Delta R_{ik} + 2g^{pq}g^{rs}R_{pirk}R_{qs} - 2g^{pq}R_{pi}R_{qk} - (n-2)\rho\nabla_i\nabla_k R - \rho\Delta R g_{ik} \\ & + \frac{1}{2}\left(R_{ip}h_k^p - g^{jl}R_{ijpk}h_l^p\right) - h^{jl}R_{ijkl} \\ & + \frac{1}{2}\left(g^{jl}\nabla_i\nabla_l h_{jk} + g^{jl}\nabla_j\nabla_k h_{il} - \nabla_i\nabla_k|H|^2 - \Delta h_{ik}\right).\end{aligned} \tag{26}$$

Proof. We have

$$\frac{\partial}{\partial t}g^{jl} = 2R^{jl} - 2\rho R g^{jl} - h^{jl} \tag{27}$$

and

$$g^{jl}h_{jl} = g^{jl}H_{jpq}H_l^{pq} = |H|^2.$$

Since

$$\frac{\partial}{\partial t}R_{ik} = \frac{\partial}{\partial t}\left(g^{jl}R_{ijkl}\right) = g^{jl}\frac{\partial}{\partial t}R_{ijkl} + R_{ijkl}\frac{\partial}{\partial t}g^{jl}, \tag{28}$$

by replacing (27) and (28) in (22) we get the result. □

Corollary 5. *Under the GRBF system* (4)*, the evolution equation of the scalar curvature satisfies*

$$\frac{\partial}{\partial t}R = \big(1 - 2(n-1)\rho\big)\Delta R + 2|Ric|^2 - 2\rho R^2 - \frac{3}{4}\Delta|H|^2 + g^{ik}g^{jl}\nabla_i\nabla_j h_{kl} - h^{ik}R_{ik}. \tag{29}$$

Proof. We have

$$\frac{\partial}{\partial t}R = \frac{\partial}{\partial t}\left(g^{ikl}R_{ik}\right) = g^{ik}\frac{\partial}{\partial t}R_{ik} + R_{ik}\frac{\partial}{\partial t}g^{ik} \tag{30}$$

and

$$g^{ik}\left[\frac{1}{2}\left(R_{ip}h_k^p - g^{jl}R_{ijpk}h_l^p\right) - h^{jl}R_{ijkl}\right] = 0.$$

By replacing (26) and $\frac{\partial}{\partial t}g^{jl} = 2R^{jl} - 2\rho R g^{jl} - h^{jl}$ in (30), we obtain (29). □

5 Derivative estimate

At first, from [5], we recall several basic identities of commutators $[\Delta, \nabla]$ and $[\frac{\partial}{\partial t}, \nabla]$. For any t-dependency tensor $A = A(t)$ under a general geometric flow $\frac{\partial}{\partial t} g_{ij} = v_{ij}$ we have

$$\frac{\partial}{\partial t}\nabla A = \nabla \frac{\partial}{\partial t} A + A * \nabla v$$

and

$$[\nabla, \Delta]A = \nabla\Delta A - \Delta\nabla A = Rm * \nabla A + \nabla Rm * A.$$

In this section, C is denotes a constant and it may changes line to line. Therefore, under the GRBF system (4), we get

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla Rm &= \nabla \frac{\partial}{\partial t} Rm + Rm * \nabla\left(Rm + \rho R g + H^{*2}\right) \\
&= \nabla\left(\Delta Rm + Rm^{*2} + \rho\nabla^2 R * g + \rho R Rm + H^{*2} * Rm + \sum_{i=0}^{2} \nabla^i H * \nabla^{2-i} H\right) \\
&\quad + Rm * \nabla\left(Rm + \rho R g + H^{*2}\right) = \Delta\nabla Rm + \nabla Rm * Rm + \rho\nabla^3 R * g \\
&\quad + \rho\sum_{\alpha+\beta=1} \nabla^\alpha R * \nabla^\beta Rm + \sum_{\alpha+\beta=3} \nabla^\alpha H * \nabla^\beta H + \sum_{\alpha+\beta+\gamma=1} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm.
\end{aligned} \tag{31}$$

More general result is as follow.

Proposition 6. *Under the GRBF system* (4) *for any nonnegative integer* k *we have*

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\nabla^k Rm\right) = \Delta\left(\nabla^k Rm\right) &+ \sum_{\alpha+\beta=k} \nabla^\alpha Rm * \nabla^\beta Rm + \rho\nabla^{k+2} R * g + \rho\sum_{\alpha+\beta=k} \nabla^\alpha R * \nabla^\beta Rm \\
&+ \sum_{\alpha+\beta=2+k} \nabla^\alpha H * \nabla^\beta H + \sum_{\alpha+\beta+\gamma=k} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm.
\end{aligned} \tag{32}$$

Proof. From (23) and (31), we see that (32) holds for $k = 0, 1$. For the induction step, assume that (33) holds for all $0 \le j < k$. We have

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\nabla^k Rm\right) &= \frac{\partial}{\partial t}\nabla\left(\nabla^{k-1} Rm\right) = \nabla\frac{\partial}{\partial t}\left(\nabla^{k-1} Rm\right) + \left(\nabla^{k-1} Rm\right) * \nabla\left(Rm + \rho R g + H^{*2}\right) \\
&= \nabla\Bigg[\Delta\left(\nabla^{k-1} Rm\right) + \sum_{\alpha+\beta=k-1} \nabla^\alpha Rm * \nabla^\beta Rm + \rho\nabla^{k+1} R * g \\
&\quad + \rho\sum_{\alpha+\beta=k-1} \nabla^\alpha R * \nabla^\beta Rm + \sum_{\alpha+\beta=1+k} \nabla^\alpha H * \nabla^\beta H \\
&\quad + \sum_{\alpha+\beta+\gamma=k-1} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm\Bigg] + \left(\nabla^{k-1} Rm\right) * \nabla\left(Rm + \rho R g + H^{*2}\right) \\
&= \Delta\left(\nabla^k Rm\right) + \sum_{\alpha+\beta=k} \nabla^\alpha Rm * \nabla^\beta Rm + \rho\nabla^{k+2} R * g + \rho\sum_{\alpha+\beta=k} \nabla^\alpha R * \nabla^\beta Rm \\
&\quad + \sum_{\alpha+\beta=2+k} \nabla^\alpha H * \nabla^\beta H + \sum_{\alpha+\beta+\gamma=k} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm.
\end{aligned} \tag{33}$$

This completes the inductive step and we obtain the required result. □

As an immediate consequence, we get the following assertion.

Corollary 6. *Under the GRBF system* (4) *the evolution of the length of derivative of Riemannian curvature tensor satisfies*

$$\begin{aligned}\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 \\ + |\rho|C|Rm||\nabla^2 R| + |\rho|C|R||Rm|^2 \\ + C|H|^2|Rm|^2 + C\sum_{\alpha+\beta=2}|\nabla^\alpha H|\,|\nabla^\beta H||Rm|\end{aligned} \tag{34}$$

and for any positive integer k we have

$$\begin{aligned}\frac{\partial}{\partial t}\left|\nabla^k Rm\right|^2 \leq \Delta\left|\nabla^k Rm\right|^2 - 2\left|\nabla^{k+1} Rm\right|^2 \\ + C\left|\nabla^k Rm\right|\sum_{\alpha+\beta=k}|\nabla^\alpha Rm|\left|\nabla^\beta Rm\right| \\ + \rho C\left|\nabla^k Rm\right|\left|\nabla^{k+2} R\right| \\ + \rho C\left|\nabla^k Rm\right|\sum_{\alpha+\beta=k}|\nabla^\alpha R|\left|\nabla^\beta Rm\right| \\ + C\left|\nabla^k Rm\right|\sum_{\alpha+\beta=2+k}|\nabla^\alpha H|\left|\nabla^\beta H\right| \\ + C\left|\nabla^k Rm\right|\sum_{\alpha+\beta+\gamma=k}|\nabla^\alpha H|\left|\nabla^\beta H\right||\nabla^\gamma Rm|,\end{aligned} \tag{35}$$

where C represents universal constants depending only on the dimension of M.

Proof. By the evolution equation (23), we get

$$\begin{aligned}\frac{\partial}{\partial t}|Rm|^2 &= 2Rm * \left(\frac{\partial}{\partial t}Rm\right) + (Rm)^{*2} * \left(Rm + \rho Rg + H^{*2}\right) \\ &= 2Rm * \left(\Delta Rm + Rm * Rm + \rho\nabla^2 R * g + \rho RRm + H^{*2} * Rm + \sum_{i=0}^{2}\nabla^i H * \nabla^{2-i} H\right) \\ &\quad + (Rm)^{*2} * \left(Rm + \rho Rg + H^{*2}\right) \\ &\leq \Delta|Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 + |\rho|C|Rm|\left|\nabla^2 R\right| + |\rho|C|R||Rm|^2 + C|H|^2|Rm|^2 \\ &\quad + C\sum_{\alpha+\beta=2}|\nabla^\alpha H|\left|\nabla^\beta H\right||Rm|.\end{aligned}$$

Hence (34) follows. To prove (35), we note

$$\frac{\partial}{\partial t}\left|\nabla^k Rm\right|^2 = 2\left(\nabla^k Rm\right) * \left(\frac{\partial}{\partial t}\left(\nabla^k Rm\right)\right) + \left(\nabla^k Rm\right)^{*2} * \left(Rm + \rho Rg + H^{*2}\right).$$

Combining it with (32), we obtain the required result. □

Now we derive the evolution equations for the covariant derivative of H.

Proposition 7. *Under the GRBF system* (4) *we have*

$$\frac{\partial}{\partial t}(\nabla H) = \Delta(\nabla H) + \sum_{\alpha+\beta=1} \nabla^\alpha H * \nabla^\beta Rm + \rho H * \nabla R + \nabla H * H^{*2} \tag{36}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla^k H\right) = \Delta\left(\nabla^k H\right) &+ \sum_{\alpha+\beta=k} \nabla^\alpha H * \nabla^\beta Rm + \rho \sum_{\alpha+\beta=k-1} \nabla^\alpha H * \nabla^{1+\beta} R \\ &+ \sum_{\alpha+\beta+\gamma=k} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma H\end{aligned} \tag{37}$$

for all $k \geq 2$.

Proof. Since $\frac{\partial}{\partial t} H = \Delta H + H * Rm$, we conclude

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla H) &= \nabla \frac{\partial H}{\partial t} + H * \nabla \left(Rm + \rho Rg + H^{*2}\right) \\ &= \nabla(\Delta H + H * Rm) + H * \nabla Rm + \rho H * \nabla R + \nabla H * H^{*2} \\ &= \Delta(\nabla H) + \nabla H * Rm + H * \nabla Rm + \rho H * \nabla R + \nabla H * H^{*2}.\end{aligned}$$

Hence (36) holds. Let us prove (37) by induction. Assume the evolution equation $\nabla^j H$ holds for all $1 \leq j < k$ in (37). We have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla^k H\right) &= \frac{\partial}{\partial t}\nabla\left(\nabla^{k-1} H\right) = \nabla\left(\frac{\partial}{\partial t}\left(\nabla^{k-1} H\right)\right) + \nabla^{k-1} H * \nabla\left(Rm + \rho Rg + H^{*2}\right) \\ &= \nabla\Bigg[\Delta\left(\nabla^{k-1} H\right) + \sum_{\alpha+\beta=k-1} \nabla^\alpha H * \nabla^\beta Rm + \rho \sum_{\alpha+\beta=k-2} \nabla^\alpha H * \nabla^{1+\beta} R \\ &\quad + \sum_{\alpha+\beta+\gamma=k-1} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma H\Bigg] + \nabla^{k-1} H * \nabla\left(Rm + \rho Rg + H^{*2}\right) \\ &= \Delta(\nabla^k H) + \sum_{\alpha+\beta=k} \nabla^\alpha H * \nabla^\beta Rm + \rho \sum_{\alpha+\beta=k-1} \nabla^\alpha H * \nabla^{1+\beta} R \\ &\quad + \sum_{\alpha+\beta+\gamma=k} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma H.\end{aligned}$$

This completes the proof. □

By Proposition 7, we get the following result.

Corollary 7. *Under the GRBF system* (4) *we have*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 &+ C|\nabla H| \sum_{\alpha+\beta=1} |\nabla^\alpha H||\nabla^\beta Rm| \\ &+ |\rho|\, C\, |\nabla H| \sum_{\alpha+\beta=1} |\nabla^\alpha H||\nabla^\beta R| + C|\nabla H|^2|H|^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k H|^2 \leq \Delta|\nabla^k H|^2 - 2|\nabla^k H|^2 &+ C|\nabla^k H| \sum_{\alpha+\beta=k} |\nabla^\alpha H||\nabla^\beta Rm| \\ &+ |\rho|\, C\, |\nabla^k H| \sum_{\alpha+\beta=k} |\nabla^\alpha H||\nabla^\beta R| \\ &+ C|\nabla^k H| \sum_{\alpha+\beta+\gamma=k} |\nabla^\alpha H||\nabla^\beta H||\nabla^\gamma H|\end{aligned}$$

for all $k \geq 2$.

Proposition 8. *Under the GRBF system* (4) *we have*

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla R) = \big(1-2(n-1)\rho\big)\Delta\nabla R &+ \big(1-2(n-1)\rho\big)Rm * \nabla R \\ &+ \big(2-2(n-1)\rho\big)R\,\nabla Rm \\ &+ \nabla Rm * Rm - 2\rho R\nabla R \\ &- \frac{3}{4}\Delta\nabla|H|^2 + Rm * \nabla|H|^2 - \frac{1}{2}|H|^2\nabla Rm \\ &+ \sum_{\alpha+\beta=3}\nabla^\alpha H * \nabla^\beta H \\ &+ \sum_{\alpha+\beta+\gamma=1}\nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm \\ &+ R\,H * H + 2\rho R^2 g\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla^k R\right) = \big(1-2(n-1)\rho\big)\Delta\nabla^k R &+ \sum_{\alpha+\beta=k}\nabla^\alpha Rm * \nabla^\beta R \\ &+ \sum_{\alpha+\beta=k}\nabla^\alpha Rm * \nabla^\beta Rm \\ &+ \sum_{\alpha+\beta\le k}\nabla^\alpha R * \nabla^\beta R - \frac{3}{4}\Delta\nabla^2|H|^2 \\ &+ \sum_{\alpha+\beta=k}\nabla^\alpha Rm * \nabla^\beta|H|^2 \\ &+ \sum_{\alpha+\beta=2+k}\nabla^\alpha H * \nabla^\beta H \\ &+ \sum_{\alpha+\beta+\gamma=k}\nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm \\ &+ \sum_{\alpha+\beta+\gamma=k-1}\nabla^\alpha H * \nabla^\beta H * \nabla^\gamma R\end{aligned}$$

for all $k \geq 2$.

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla R) &= \nabla\frac{\partial R}{\partial t} + R\,\nabla\left(Rm + H * H + 2\rho R g\right) \\ &= \big(1-2(n-1)\rho\big)\nabla\Delta R + 2\nabla|Ric|^2 - 2\rho R\nabla R - \frac{3}{4}\nabla\Delta|H|^2 \\ &\quad + \nabla\left(\sum_{\alpha+\beta=2}\nabla^\alpha H * \nabla^\beta H + H * H * Rm\right) + R\,\nabla Rm + R\,H * H + 2\rho R^2 g \\ &= \big(1-2(n-1)\rho\big)\Delta\nabla R + \big(1-2(n-1)\rho\big)Rm * \nabla R + (1-2(n-1)\rho)R\,\nabla Rm \\ &\quad + \nabla Rm * Rm - 2\rho R\nabla R - \frac{3}{4}\Delta\nabla|H|^2 + Rm * \nabla|H|^2 - \frac{1}{2}|H|^2\nabla Rm \\ &\quad + \sum_{\alpha+\beta=3}\nabla^\alpha H * \nabla^\beta H + \sum_{\alpha+\beta+\gamma=1}\nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm \\ &\quad + R\,\nabla Rm + R\,H * H + 2\rho R^2 g.\end{aligned}$$

By using the induction we conclude the second result. □

Corollary 8. *Under the GRBF system* (4) *we have*

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla R|^2 = &\big(1-2(n-1)\rho\big)\Delta|\nabla R|^2 - 2\big(1-2(n-1)\rho\big)|\nabla^2 R| + \nabla R * Rm * \nabla R + R\,\nabla R * \nabla Rm\\
&+ \nabla R * \nabla Rm * Rm - 2\rho R \nabla R * \nabla R + \nabla R * \Delta\nabla |H|^2 + \nabla R * Rm * \nabla |H|^2\\
&+ |H|^2 \nabla R * \nabla Rm + \nabla R * \sum_{\alpha+\beta=3} \nabla^\alpha H * \nabla^\beta H + \nabla R * \sum_{\alpha+\beta+\gamma=1} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma Rm\\
&+ R\,\nabla R * H * H + R^2 \nabla R * g + Rm * \nabla R * \nabla R + H * H * \nabla R * \nabla R + R\nabla R * g\\
\le &\big(1-2(n-1)\rho\big)\Delta|\nabla R|^2 - 2\big(1-2(n-1)\rho\big)|\nabla^2 R| + C|\nabla R|^2 |Rm|\\
&+ C|Rm||\nabla R||\nabla Rm| + C|\nabla R||\Delta\nabla|H|^2 + C|\nabla R||Rm||\nabla|H|^2 + C|H|^2|\nabla R||\nabla Rm|\\
&+ C|\nabla R| \sum_{\alpha+\beta=3} |\nabla^\alpha H|\,|\nabla^\beta H| + C|\nabla R| \sum_{\alpha+\beta+\gamma=1} |\nabla^\alpha H||\nabla^\beta H||\nabla^\gamma Rm|\\
&+ C|Rm||\nabla R||H|^2 + C|Rm|^2|\nabla R| + C|H|^2|\nabla R|^2 + |Rm||\nabla R|
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\big|\nabla^k R\big|^2 \le \big(1-2(n-1)\rho\big)\Delta\big|\nabla^k R\big|^2 &- 2\big(1-2(n-1)\rho\big)\big|\nabla^{k+1} R\big|^2\\
&+ C\big|\nabla^k R\big| \sum_{\alpha+\beta=k} |\nabla^\alpha Rm|\,\big|\nabla^\beta R\big|\\
&+ C\big|\nabla^k R\big| \sum_{\alpha+\beta=k} |\nabla^\alpha Rm|\,\big|\nabla^\beta Rm\big|\\
&+ C\big|\nabla^k R\big| \sum_{\alpha+\beta\le k} |\nabla^\alpha R|\,\big|\nabla^\beta R\big|\\
&+ C\big|\nabla^k R\big|^2 \big|\Delta\nabla^2\big|\big|H^2\big|\\
&+ C\big|\nabla^k R\big| \sum_{\alpha+\beta=k} |\nabla^\alpha Rm|\,\big|\nabla^\beta\big|\big|H^2\big|\\
&+ C\big|\nabla^k R\big| \sum_{\alpha+\beta=2+k} |\nabla^\alpha H|\,\big|\nabla^\beta H\big|\\
&+ C\big|\nabla^k R\big| \sum_{\alpha+\beta+\gamma=k} |\nabla^\alpha H|\,\big|\nabla^\beta H\big|\big|\nabla^\gamma Rm\big|\\
&+ C\big|\nabla^k R\big| \sum_{\alpha+\beta+\gamma=k-1} |\nabla^\alpha H|\,\big|\nabla^\beta H\big|\big|\nabla^\gamma R\big|\\
&+ C|Rm|\big|\nabla^k R\big|^2.
\end{aligned}$$

for all $k \ge 2$.

Theorem 6. *Let* $\big(g(x,t), H(x,t)\big)$ *be a solution to GRBF system* (4) *on a closed manifold* M^n *on* $0 \le t \le T$ *and* K_1, K_2 *be arbitrary given nonnegative constants. Then there exists a constant* $C(n)$ *depending only on* n *such that if*

$$\big|Rm(x,t)\big|_{g(x,t)} \le K_1 \quad \text{and} \quad \big|H(x)\big| \le K_2$$

for all $(x,t) \in M^n \times [0,T]$, *then*

$$\big|H(x,t)\big|_{g(x,t)} \le K_2 e^{\frac{1}{2}C(n)K_1 t}$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. Since $\frac{\partial H}{\partial t} = \Delta H + Rm * H$ we have

$$\begin{aligned}\frac{\partial}{\partial t}|H|^2 &= 2H_{ikp}H_j^{kp}\Big(2R^{ij} - 2\rho R g^{ij} - \frac{1}{2}h^{ij}\Big) + 2\left(\Delta H + Rm * H\right) * H \\ &= \Delta|H|^2 - 2|\nabla H|^2 + Rm * H * H + \rho R H * H - |H|^4 \\ &\leq \Delta|H|^2 + C_1(n)|Rm||H|^2 + C_1(n)|\rho||R||H|^2 \\ &\leq \Delta|H|^2 + C(n)K_1|H|^2.\end{aligned}$$

Suppose that $y(t)$ is the solution to the corresponding ordinary differential equation $\frac{dy}{dt} = C(n)K_1 y$ with $y(0) = K_2^2$. Then $y(t) = K_2^2 e^{C(n)K_1 t}$ and by the maximum principle we obtain $\big|H(x,t)\big| \leq K_2 e^{\frac{1}{2}C(n)K_1 t}$. □

6 Compactness theorem for the GRBF system

In [2], the compactness theorem and its various versions for solutions of the Ricci flow are discussed to understand singularity formation. This is most effective when the compactness theorem is combined with monotonicity formulas on other geometric techniques. The compactness for the Ricci flow has applications to study solution $(M^n, g(t))$ to the Ricci flow on $t \in (a,b)$, where $b \leq \infty$ is maximal. It helps in understanding the limiting behavior of the solution $g(t)$ as $t \to b$ and to determining, when there exists a subsequence of pointed solution to the Ricci flow $\big(M, g_k(t), O_k\big)$, that converges to a complete solution $\big(M_\infty, g_\infty(t), O_\infty\big)$. In this section, using the definitions and notations of [2], we prove the compactness theorem for solution of the GRBF system.

In the following, we find bounds on the metric and its derivatives.

Lemma 3. *Let (M^n, g) be a close manifold, $\rho < \frac{1}{2(n-1)}$, U be a compact subset of M, and $(g_k(t), H_k(t))$ be smooth solutions of the GRBF system in neighborhood of $U \times [\beta, \psi]$, where $\beta < 0 < \psi$. At time $t = 0$ on U suppose that:*

- *(a) the metrics $g_k(x,0)$ are all uniformly equivalent to $g(x)$ on U, i.e. for all $V \in T_xM$, k and $x \in U$, we have $cg(V,V) \leq g_k(x,0)(V,V) \leq Cg(V,V)$, where c and C are constants independent of V, k, and x,*
- *(b) $|\nabla^p g_k|_g \leq C_p$ for all $p \geq 1$, where C_p is a constant independent of k,*
- *(c) $|\nabla^p H_k|_g \leq C'_p$ for all $p \geq 0$, where C'_p is a constant independent of k, and in addition*
- *(d) $\sup\limits_{U\times[\beta,\psi]} \big|\nabla_k^p Rm_k\big|_{g_k} \leq C''_p$ for all $p \geq 0$, where C''_p is a constant independent of k,*
- *(e) $\sup\limits_{U\times[\beta,\psi]} \big|\nabla_k^p H_k\big|_{g_k} \leq C'''_p$ for all $p \geq 0$, where C'''_p is a constant independent of k.*

Then we have

- *(i) $\tilde{c}g(V,V) \leq g_k(t)(V,V) \leq \tilde{C}g(V,V)$,*
- *(ii) $\sup\limits_{U\times[\beta,\psi]} \big|\nabla_k^p g_k\big|_g \leq \tilde{C}_p$ for all $p \geq 1$,*
- *(iii) $\sup\limits_{U\times[\beta,\psi]} \big|\nabla_k^p H_k\big|_g \leq \tilde{C'}_p$ for all $p \geq 0$,*

on $U \times [\beta, \psi]$, where $\tilde{c}$, $\tilde{C}$, $\tilde{C}_p$, and $\tilde{C'}_p$ are constants independent of k for all $t \in [0, \tau]$.

Proof. In the process of proving the Lemma, $\bar{C}_k$, $1 \leq k \leq 31$ are constants. For any $V \in T_xM$ we have

$$\frac{\partial}{\partial t} g_k(x,t)(V,V) = -2Ric_k(x,t)(V,V) + 2\rho R_k g_k(x,t)(V,V) + h_k(x,t)(V,V).$$

Then using *(d)* and *(e)*, we infer

$$|Ric_k(x,t)(V,V)|_{g_k} \leq \bar{C}_1(n) C_0'' g_k(x,t)(V,V),$$

$$|R_k(x,t)| \leq \bar{C}_2(n) C_0'',$$

$$|h_k(x,t)(V,V)|_{g_k} \leq \bar{C}_3(n) \, |H_k(x,t)|^2_{g_k} g_k(x,t)(V,V) \leq \bar{C}_3(n) C_0''' g_k(x,t)(V,V),$$

which give

$$\begin{aligned} \left| \frac{\partial}{\partial t} g_k(x,t)(V,V) \right|_{g_k} &\leq 2 \, |Ric_k(x,t)(V,V)|_{g_k} + 2 \, |\rho| \, |R_k| \, g_{(x,t)}(V,V) + |h_k(x,t)(V,V)|_{g_k} \\ &\leq \left(\bar{C}_1(n) C_0'' + 2|\rho| \bar{C}_2(n) C_0'' + \bar{C}_3(n) C_0''' \right) g_k(x,t)(V,V). \end{aligned}$$

Therefore for $0 \leq t \leq \psi$ we obtain

$$\begin{aligned} \left| \log \frac{g_k(x,t)(V,V)}{g_k(x,0)(V,V)} \right| &= \left| \int_0^t \frac{\frac{\partial}{\partial t} g_k(x,t)(V,V)}{g_k(x,t)(V,V)} dt \right| \\ &\leq \int_0^t \left| \frac{\frac{\partial}{\partial t} g_k(x,t)(V,V)}{g_k(x,t)(V,V)} \right|_{g(t)} dt \\ &\leq \left(\bar{C}_1(n) C_0'' + 2|\rho| \bar{C}_2(n) C_0'' + \bar{C}_3(n) C_0''' \right) \psi. \end{aligned}$$

Hence, this inequality and the assumption condition *(a)* complete the proof of *(i)*.

Let $\nabla, \Gamma, {}^k\nabla$ and ${}^k\Gamma$ be connections and Christoffel symbols of metrics g and g_k, respectively. From the definition we have

$${}^k\Gamma^l_{ij} - \Gamma^l_{ij} = \frac{1}{2} (g_k)^{lr} \left\{ \nabla_i (g_k)_{jr} + \nabla_j (g_k)_{ir} - \nabla_r (g_k)_{ij} \right\},$$

thus $\left| {}^k\Gamma(x,t) - \Gamma(x) \right|_g \leq \bar{C}_4(n) \, |\nabla g_k(x,t)|_{g_k}$. On the other hand, for a tensor T, we have

$$\nabla_i T_{jr} = \frac{\partial}{\partial x_i} T_{jr} - \Gamma^l_{ij} T_{lr} - \Gamma^l_{ir} T_{lj} = {}^k\nabla_i T_{jr} - \left(\Gamma^l_{ij} - {}^k\Gamma^l_{ij} \right) T_{lr} - \left(\Gamma^l_{ir} - {}^k\Gamma^l_{ir} \right) T_{lj},$$

therefore,

$$\nabla_i \left(g_k \right)_{jr} = - \left(\Gamma^l_{ij} - ^k \Gamma^l_{ij} \right) (g_k)_{lr} - \left(\Gamma^l_{ir} - ^k \Gamma^l_{ir} \right) (g_k)_{lj}.$$

It follows that $|\nabla_i (g_k)(x,t)|_{g_k} \leq \bar{C}_5(n) \left| {}^k\Gamma(x,t) - \Gamma(x) \right|_{g_k}$ and hence ∇g_k is equivalent to ${}^k\Gamma - \Gamma = {}^k\nabla - \nabla$. Since ∇ is independent of time, the evolution equation for ${}^k\Gamma - \Gamma$ is

$$\begin{aligned} \frac{\partial}{\partial t} \left({}^k\Gamma - \Gamma \right) = &- \left\{ {}^k\nabla_i (Ric_k)_{jr} + {}^k\nabla_j (Ric_k)_{ir} - {}^k\nabla_r (Ric_k)_{ij} \right\} \\ &+ \rho \left\{ {}^k\nabla_i R_k (g_k)_{jr} + {}^k\nabla_j R_k (g_k)_{ir} - {}^k\nabla_r R_k (g_k)_{ij} \right\} \\ &+ \frac{1}{2} \left\{ {}^k\nabla_i (h_k)_{jr} + {}^k\nabla_j (h_k)_{ir} - {}^k\nabla_r (h_k)_{ij} \right\}. \end{aligned}$$

It follows from the assumption that

$$\left|\frac{\partial}{\partial t}\left({}^k\Gamma-\Gamma\right)\right|_{g_k} \leq \bar{C}_7(n,|\rho|)\left[\left|{}^k\nabla(Ric_k)\right|_{g_k}+\left|{}^k\nabla(R_k)\right|_{g_k}+\left|{}^k\nabla(h_k)\right|_{g_k}\right]$$
$$\leq \bar{C}_8(n,|\rho|)\left[C''_1+C'''_1C''_0\right].$$

We conclude

$$\left|\frac{\partial}{\partial t}\nabla g_k\right|_g \leq \bar{C}_9\left|\frac{\partial}{\partial t}\left({}^k\Gamma-\Gamma\right)\right|_{g_k} \leq \bar{C}_9\bar{C}_8(n,|\rho|)\left[C''_1+C'''_1C''_0\right],$$

where constant $\bar{C}_9$ comes from *(i)*. Integrating on both sides, we get

$$\left|\frac{\partial}{\partial t}\nabla g_k(t)\right| = \left|\nabla g_k(0)+\int_0^t \nabla g_k(\tau)d\tau\right| \leq \tilde{C}_1+\bar{C}_9\bar{C}_8(n,|\rho|)\left[C''_1+C'''_1C''_0\right]\psi.$$

By *(e)* and *(i)*, we obtain

$$|H_k|_g \leq \bar{C}_{10}\,|H_k|_{g_k} \leq \bar{C}_{10}C'''_0 := \tilde{C}'_0.$$

Since ∇ is independent of time, we get

$$\begin{aligned}\frac{\partial}{\partial t}\nabla H_k &= \nabla\frac{\partial}{\partial t}H_k = \nabla\left[\Delta_k H_k + Rm_k\star H_k\right]\\
&= \left(\nabla-{}^k\nabla\right)\Delta_k H_k + {}^k\nabla\Delta_k H_k + \left(\nabla-{}^k\nabla\right)Rm_k * H_k + {}^k\nabla Rm_k * H_k\\
&\quad + Rm_k * \left(\nabla-{}^k\nabla\right)H_k + Rm_k * {}^k\nabla H_k = \nabla g_k * \Delta_k H_k + {}^k\nabla\Delta_k H_k\\
&\quad + \nabla g_k * Rm_k * H_k + {}^k\nabla Rm_k * H_k + Rm_k * \nabla g_k * H_k + Rm_k * {}^k\nabla H_k,\end{aligned}$$

where Δ_k is the Laplace operator associated to g_k and we used $\nabla g_k \simeq \nabla - {}^k\nabla$. From the assumptions *(d)*, *(e)*, *(i)* and *(ii)* for $p=1$, the above equation implies that $\left|\frac{\partial}{\partial t}\nabla H_k\right| \leq \bar{C}_{11}$. As above

$$|\nabla H_k(t)| \leq |\nabla H_k(0)| + \int_0^t\left|\frac{\partial}{\partial\tau}\nabla H_k(\tau)\right|d\tau \leq C'_1+\bar{C}_{11}\psi := \tilde{C}'_1.$$

For higher derivative of (g_k, H_k) with respect to g, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2 g_k &= \nabla^2\left(-2Ric+2\rho R_k g+h\right)\\
&= -2\nabla^2 Ric+\nabla^2 R_k * g_k + R1 * \nabla^2 g_k + \nabla R * \nabla g_k + H_k * \nabla^2 H_k + \nabla H_k * \nabla H_k.\end{aligned}$$

We can write

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2 Ric_k &= \left(\nabla-{}^k\nabla\right)\left[\left(\nabla-{}^k\nabla\right)Ric_k+{}^k\nabla Ric_k\right]+{}^k\nabla\left(\nabla-{}^k\nabla\right)Ric_k+{}^k\nabla^2 Ric_k\\
&= \nabla g_k * \left[\nabla g_k * Ric_k + {}^k\nabla Ric_k\right] + \nabla^2 g_k * Ric_k + {}^k\nabla^2 Ric_k.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2 g_k =&\nabla g_k * \left[\nabla g_k * Ric_k + {}^k\nabla Ric_k\right] + \nabla^2 g_k * Ric_k\\
&+ {}^k\nabla^2 Ric_k + \nabla^2 R_k * g_k + R_k 1 * \nabla^2 g_k + \nabla R * \nabla g_k\\
&+ H_k * \left[\nabla g_k * \left[\nabla g_k * H_k + {}^k\nabla Ric_k\right] + \nabla^2 g_k * H_k + {}^k\nabla^2 H_k\right] + \nabla H_k * \nabla H_k.\end{aligned}$$

Hence the assumptions and *(i)–(iii)* for the case $p = 0, 1$ imply that

$$\left|\frac{\partial}{\partial t}\nabla^2 g_k\right| \leq \bar{C}_{12}\Big(\left|\nabla^2 g_k\right| + \left|\nabla^2 H_k\right| \Big) + \bar{C}_{13}. \tag{38}$$

Similarly we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2 H_k &= \nabla^2\left(\Delta_k H_k + Rm_k * H_k\right)\\ &= \nabla g_k * \left[\nabla g_k * \Delta_k H_k + {}^k\nabla\Delta_k H_k\right] + \nabla^2 g_k * \Delta_k H_k + {}^k\nabla^2\Delta_k H_k\\ &\quad + \nabla^2 Rm_k * H_k + Rm_k * \nabla^2 H_k + \nabla g_k * Rm_k * \nabla H_k + {}^k\nabla Rm_k * \nabla H_k\end{aligned}$$

and

$$\left|\frac{\partial}{\partial t}\nabla^2 H_k\right| \leq \bar{C}_{14}\Big(\left|\nabla^2 g_k\right| + \left|\nabla^2 H_k\right|\Big) + \bar{C}_{15}. \tag{39}$$

Putting (38) and (39) together, we arrive at

$$\left|\frac{\partial}{\partial t}\left(\left|\nabla^2 g_k\right| + \left|\nabla^2 H_k\right|\right)\right| \leq \bar{C}_{16}\Big(\left|\nabla^2 g_k\right| + \left|\nabla^2 H_k\right|\Big) + \bar{C}_{17}.$$

Since $\left|\nabla^2 g_k(0)\right| + \left|\nabla^2 H_k(0)\right| \leq C_2 + C_2'$, integrating on both sides of above inequality, we get

$$\left|\nabla^2 g_k(t)\right| + \left|\nabla^2 H_k(t)\right| \leq \bar{C}_{18}.$$

Suppose that the estimates hold for $p < N$ with $N \geq 2$. Let us show that they also hold for $p = N$. Notice

$$\begin{aligned}\left|\nabla^N Ric_k\right| &= \left|\sum_{i=1}^{N} \nabla^{N-i}\left(\nabla - {}^k\nabla\right){}^k\nabla^{i-1} Ric_k + {}^k\nabla^N Ric_k\right|\\ &\leq \sum_{i=1}^{N}\left|\nabla^{N-i}\left(\nabla - {}^k\nabla\right){}^k\nabla^{i-1} Ric_k\right| + \left|{}^k\nabla^N Ric_k\right|.\end{aligned}$$

By induction on p we show that

$$|\nabla^p Ric_k| \leq A_p\,|\nabla^p g_k| + B_p, \qquad |\nabla^p g_k| + |\nabla^p H_k| \leq D_p$$

for all $p \geq 1$, where A_p, B_p, and D_p are constants independent of k. For $i = 1$, by induction and the assumptions we have

$$\begin{aligned}\left|\nabla^{N-1}\left(\nabla - {}^k\nabla\right) Ric_k\right| &\leq \bar{C}_{19}\left|\nabla^{N-1}\left(\nabla g_k Ric_k\right)\right|\\ &\leq \bar{C}_{19}\left|\sum_{j=0}^{N-1}\binom{N-1}{j}\left(\nabla^{N-j} g_k\right)\left(\nabla^j Ric_k\right)\right|\\ &\leq \bar{C}_{19}\sum_{j=0}^{N-1}\binom{N-1}{j}\left|\nabla^{N-j} g_k\right|\left|\nabla^j Ric_k\right|\\ &\leq \bar{C}_{19}\sum_{j=0}^{N-1}\binom{N-1}{j}\left(A_j\left|\nabla^j g_k\right| + B_j\right)\left|\nabla^{N-j} g_k\right|\\ &\leq \bar{C}_{19}\sum_{j=0}^{N-1}\binom{N-1}{j}\left(A_j D_j + B_j\right)\left|\nabla^{N-j} g_k\right|\\ &\leq \bar{C}_{20}\left|\nabla^N g_k\right| + \bar{C}_{21}.\end{aligned}$$

For $2 \leq i \leq N$ we have

$$\begin{aligned}
&\left|\nabla^{N-i}\left(\nabla - {}^k\nabla\right){}^k\nabla^{i-1}\,Ric_k\right| \\
&\qquad \leq \bar{C}_{22}\left|\nabla^{N-i}\left(\nabla g_k{}^k\nabla^{i-1}Ric_k\right)\right| \\
&\qquad \leq \bar{C}_{22}\sum_{j=0}^{N-1}\binom{N-1}{j}\left|\nabla^{N-i-j+1}g_k\right|\left|\nabla^j\left({}^k\nabla^{i-1}Ric_k\right)\right| \\
&\qquad = \bar{C}_{22}\sum_{j=0}^{N-1}\binom{N-1}{j}\left|\nabla^{N-i-j+1}g_k\right|\left|\left(\left(\nabla - {}^k\nabla\right) + {}^k\nabla\right)^j\left({}^k\nabla^{i-1}Ric_k\right)\right| \\
&\qquad = \bar{C}_{23}\sum_{j=0}^{N-1}\binom{N-1}{j}\left|\nabla^{N-i-j+1}g_k\right|\left(\sum_{l=0}^{j}\binom{j}{l}|\nabla^l g_k||{}^k\nabla^{j-l+i-1}Ric_k|\right) \\
&\qquad \leq \bar{C}_{24}.
\end{aligned}$$

Combining the above two inequalities, we deduce

$$|\nabla^N Ric_k| \leq A_N|\nabla^N g_k| + B_N.$$

Similarly we have

$$|\nabla^N Rm_k| \leq A'_N|\nabla^N g_k| + B'_N, \qquad |\nabla^N \Delta_k H_k| \leq A''_N|\nabla^N g_k| + B''_N,$$

where A'_N, A''_N, B'_N and B''_N are constants independent of k. By induction, we have that $|\nabla^p H_k|$ bounded for all $p < N$. Now, for $p \geq 1$, the equality

$$\frac{\partial}{\partial t}\nabla^p g_k = \nabla^p\left(-2Ric + 2\rho R_k g + h\right) = -2\nabla^p Ric + \sum_{i=0}^{p}\nabla^i R_k * \nabla^{p-i}g_k + \sum_{i=0}^{p}\nabla^i H_k * \nabla^{p-i}H_k$$

implies that

$$\left|\frac{\partial}{\partial t}\nabla^N g_k\right| \leq \bar{C}_{25}\left(|\nabla^N g_k| + |\nabla^N H_k|\right) + \bar{C}_{26}. \tag{40}$$

On the other hand, equality

$$\frac{\partial}{\partial t}\nabla^p H_k = \nabla^p\left[\Delta_k H_k + Rm_k \star H_k\right] = \nabla^p \Delta_k H_k + \sum_{i=1}^{p}\nabla^i Rm_k * \nabla^{p-i}H_k$$

yields

$$\left|\frac{\partial}{\partial t}\nabla^N \Delta_k H_k\right| \leq \bar{C}_{27}\left(|\nabla^N g_k| + |\nabla^N H_k|\right) + \bar{C}_{28}. \tag{41}$$

Combining (40) and (41), we conclude

$$\left|\frac{\partial}{\partial t}\left(|\nabla^N g_k| + |\nabla^N H_k|\right)\right| \leq \bar{C}_{29}\left(|\nabla^N g_k| + |\nabla^N H_k|\right) + \bar{C}_{30}.$$

Since $|\nabla^N g_k(0)| + |\nabla^N H_k(0)| \leq C_N + C'_N$, integrating on both sides of the above inequality, we get

$$|\nabla^2 g_k(t)| + |\nabla^2 H_k(t)| \leq \bar{C}_{31}.$$

This completes the proof of the lemma. □

Definition 3. *Let E be a vector bundle on a Riemannian manifold M, and let metric g and connection ∇ be given on E and on TM. Let $\Omega \subset M$ be an open set with compact closure $\bar{\Omega}$ in M, and let (η_k) be a sequence of sections of E.*

For any $p \geq 0$ we say that η_k converges in $C^p(M)$ to $\eta_\infty \in \Gamma\left(E|_{\bar{\Omega}}\right)$ if for any $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that

$$\sup_{0\leq\alpha\leq p}\sup_{x\in\bar{\Omega}}\left|\nabla^{\alpha}\left(\eta_k - \eta_\infty\right)\right| < \epsilon,$$

whenever $k > k_0$.

We say η_K converges in C^∞ to η_∞ on $\bar{\Omega}$ if η_K converges in C^p to η_∞ on $\bar{\Omega}$ for any $p \in \mathbb{N}$.

Definition 4. *A pointed Riemannian manifold is a 4-tuple (M, g, H, O), where (M, g) is a Riemannian manifold and $O \in M$ is a choice of point. If the metric g is complete, then the 4-tuple is called a complete pointed Riemannian manifold. We say that $\big(M, g(t), H(t), O\big)$, $t \in (a, b)$, is a pointed solution to the GRBF system if $\big(M, g(t), H(t)\big)$ is a solution to the GRBF system.*

Definition 5. *A sequence $\left\{\left(M_k, g_k, H_k, O_k\right)\right\}$ of complete pointed Riemannian manifolds converges to a complete pointed Riemannian manifold $(M_\infty, g_\infty, H_\infty, O_\infty)$ as Cheeger-Gromov convergence if there exist*

1) *an exhaustion (U_k) of M_∞ with $O_\infty \in U_k$ such that $\bar{U}_k$ is compact and $\bar{U}_k \subset U_{k+1}$ for all k, and $\bigcup_{k\geq 1} U_k = M$,*

2) *a sequence of diffeomorphisms $\phi_k : U_k \to V_k \subset M_k$ with $\phi\left(O_\infty\right) = O_k$ such that $\left(\phi_k^* g_k, \phi_k^* H_k\right)$ converges in C^∞ to (g_∞, H_∞) on compact sets in M_∞.*

Definition 6. *A sequence $\left\{\left(M_k, g_k(t), H_k(t), O_k\right)\right\}$, $t \in (a, b)$, of complete pointed solutions to the GRBF system converges to a complete pointed solution to the GRBF system $(M_\infty, g_\infty(t), H_\infty(t), O_\infty)$, $t \in (a, b)$, if there exist*

1) *an exhaustion (U_k) of M_∞ with $O_\infty \in U_k$,*

2) *a sequence of diffeomorphisms $\phi_k : U_k \to V_k \subset M_k$ with $\phi\left(O_\infty\right) = O_k$ such that $\left(\phi_k^* g_k(t), \phi_k^* H_k(t)\right)$ converges in C^∞ to $(g_\infty(t), H_\infty(t))$ on compact sets in $M_\infty \times (a, b)$.*

In [8], the following theorem about compactness of metrics has been proven.

Theorem 7. *Let $\left\{\left(M_k, g_k, O_k\right)\right\}$ be a sequence of complete pointed Riemannian manifolds that satisfy*

1) *$\left|\nabla_k^p Rm_k\right|_k \leq C_p$ on M_k for each $p \geq 0$ and k, where $C_p < \infty$ is a sequence of constants independent of k, and*

2) *$inj_{g_k}\left(O_k\right) \geq k_0$ for some constant k_0, where $inj_{g_k}\left(O_k\right)$ is the injectivity radius of the metric g at the point O_k.*

Then there exists a subsequence $\{j_k\}$ such that $\left\{\left(M_{j_k}, g_{j_k}, O_{j_k}\right)\right\}$ converges to a complete pointed Riemannian manifold $(M_\infty, g_\infty, O_\infty)$ as $k \to \infty$.

Theorem 8 (compactness for the GRBF system)**.** *Let* $\{(M_k, g_k(t), H_k(t), O_k)\}$*,* $t \in (a,b)$*,* $-\infty \leq a < 0 < b \leq \infty$*, be a sequence of complete pointed solutions to the GRBF system such that*

(i) the curvature is uniformly bounded, i.e.

$$|Rm_k|_{g_k} \leq K_0, \qquad |H_k| \leq K_1 \quad on \quad M_k \times (a,b)$$

for some constants $K_0, K_1 < \infty$ *independent of* k*, and*

(ii) injectivity radius at $t = 0$ *has the following estimate*

$$inj_{g_k(0)}(O_k) \geq k_0$$

for some constant $k_0 > 0$*.*

Then there exists a subsequence $\{j_k\}$ *such that* $\{(M_{j_k}, g_{j_k}(t), H_{j_k}(t), O_{j_k})\}$ *converges to a complete pointed solution to the GRBF system* $(M_\infty, g_\infty(t), H_\infty(t), O_\infty)$ *as* $k \to \infty$ *for* $t \in (a,b)$*.*

Proof. We only prove the case, where each M_k is compact. Consider a sequence of pointed solutions $\{M_k, g_k(t), H_k(t), O_k\}$, $t \in (a,b)$, to the GRBF system, where

$$\sup_{M_k \times (a,b)} |Rm_k| \leq K_0 \quad \text{and} \quad \sup_{M_k \times (a,b)} |H_k)| \leq K_1.$$

The Lemma 3 gives bounds of the form

$$|\nabla^p Rm(x,t)| \leq A$$

for all $x \in M$ and $t \in [a+\epsilon, b)$ for each small $\epsilon > 0$.

Theorem 7 implies that there is a subsequence of $\{(M_k, g_k(0), O_k)\}$, which converges to $(M_\infty, g_\infty, O_\infty)$. We write this subsequence again by $\{(M_k, g_k(0), O_k)\}$ if there is no ambiguous. Let us show that there are metrics $g_\infty(t)$, $t \in (a,b)$, such that $g_\infty(0) = g_\infty$ and $\{(M_k, g_k(t), H_k(t), O_k)\}$ converges to $(M_\infty, g_\infty(t), H_\infty(t), O_\infty)$. Since $\{(M_k, g_k(0), O_k)\}$ converges to $(M_\infty, g_\infty(t), O_\infty)$, there are an exhaustion (U_k) of M_∞ and smooth maps $\phi_k : U_k \to V_k$ taking O_∞ to O_k such that $(\tilde{g}_k(0), \tilde{H}_k(0)) = (\phi_k^*(g_k(0)), \phi_k^*(H_k(0)))$ uniformly converges in C^∞ on compact sets of M_∞ to (g_∞, H_∞). The metrics $\tilde{g}_k(0)$ are uniformly comparable to $\bar{g} = g_\infty$, then by Lemma 3, $(\tilde{g}_k(t), \tilde{H}_k(t)) = (\phi_k^*(g_k(t)), \phi_k^*(H_k(t)))$ remain comparable for other $t \in (a,b)$.

We have $|\bar{\nabla}^p \tilde{g}_k| \leq B$. Now, by the Arzela-Ascoli theorem, there is a subsequence, which converges to $(M_\infty, g_\infty(t), H_\infty, O_\infty)$ in C^∞, where $(g_\infty(t), H_\infty(t))$ is defined to be the limit of $(\phi_k^*(g_k(t)), \phi_k^*(H_k(t)))$. Since all derivatives of the metric converge, the Ricci curvature of $g_k(t)$ converges to the Ricci curvature of $g_\infty(t)$ and hence the limit is a solution of the GRBF system. Since any complete manifold is a σ-compact, locally compact Hausdorff space, then for complete manifolds, it is sufficient to use Arzela-Ascoli theorem. This completes the proof. □

References

[1] Anastasiei M. *A generalization of Myers theorem*. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 2007, 53 (1), 33–40.

[2] Andrews B., Hopper C. The Ricci flow in Riemannian geometry. A complete proof of the differentiable 1/4-pinching sphere theorem. In: Lecture Notes in Mathematics, 2011. Springer, 2011.

[3] Bourguignon J.P. Ricci curvature and Einstein metrics. In: Lecture notes in Math., 838. Springer, Berlin, 1981.

[4] Catino G., Cremaschi L., Djadli Z., Mantegazza C., Mazzieri L. *The Ricci-Bourguignon flow*. Pacific J. Math. 2015, **287** (2), 337–370. doi:10.48550/arXiv.1507.00324

[5] Chow B., Knopf D. The Ricci flow: an introduction. In: Math. Surveys Monogr., 110. Amer. Math. Soc., 2004.

[6] DeTurck D. *Deforming metrics in direction of their Ricci tensors*. J. Differential Geom. 1983, **18** (1), 157–162. doi:10.4310/jdg/1214509286

[7] Hamilton R.S. *Three-manifolds with positive Ricci curvature*. J. Differential Geom. 1982, **17** (2), 255–306. doi:10.4310/jdg/1214436922

[8] Hamilton R.S. *A compactness property for solutions of the Ricci flow*. Amer. J. Math. 1995, **117** (3), 545–572.

[9] Li Y. *Generalized Ricci flow I: higher-derivative estimates for compact manifolds*. Anal. PDE 2012, **5** (4), 747–775. doi:10.2140/apde.2012.5.747

[10] Shi W.-X. *Deforming the metric on complete Riemannian manifolds*. J. Differential Geom. 1989, **30** (1), 223–301. doi:10.4310/JDG/1214443292

[11] Topping P. Lectures on the Ricci flow. In: London Mathematical Society Lecture Note, 325. Cambridge Univ. Press, 2006.

[12] Wu J.Y. *A general Ricci flow system*. J. Korean Math. Soc. 2018, **55** (2), 253–292. doi:10.4134/JKMS.j170037

[13] Wylie W. *Complete shrinking Ricci solitons have finite fundemental group*. Proc. Amer. Math. Soc. 2008, **136** (5), 1803–1806. doi:10.48550/arXiv.0704.031

Received 05.04.2023
Revised 30.11.2024

Азамі Ш. *Узагальнений потік Річчі-Бурґіньйона* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 638–662.

У цій роботі ми розглядаємо узагальнену систему потоку Річчі-Бурґіньйона, яка має схожість із потоком Річчі-Бурґіньйона та володіє градієнтною формою. Ми встановлюємо існування та єдиність розв'язку цього потоку на n-вимірному замкненому рімановому многовиді. Ми вводимо узагальнений солітон системи Річчі-Бурґіньйона та надаємо умову, за якої градієнтний узагальнений солітон системи Річчі-Бурґіньйона є ізометричним до евклідової сфери. Потім ми досліджуємо еволюцію деяких геометричних структур многовида вздовж цього потоку та встановлюємо оцінки для похідних вищих порядків для компактних многовидів, а також теорему компактності для цієї узагальненої системи потоку Річчі-Бурґіньйона на замкнених ріманових многовидах.

Ключові слова і фрази: потік Річчі-Бурґіньйона, градієнтний солітон Річчі-Бурґіньйона, градієнтна оцінка, теорема компактності.