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On a generalization of some Shah equation

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A Dirichlet series $F(s)=e^{hs}+\sum_{k=2}^{\infty}f_ke^{s\lambda_k}$ with the exponents $0<h<\lambda_k\uparrow+\infty$ and the abscissa of absolute convergence $\sigma_a[F]\ge 0$ is said to be pseudostarlike of order $\alpha\in[0,\,h)$ and type $\beta\in(0,\,1]$ in $\Pi_0=\{s:\ \mathrm{Re}\,s<0\}$ if $\left|\frac{F'(s)}{F(s)}-h\right|<\beta\left|\frac{F'(s)}{F(s)}-(2\alpha-h)\right|$ for all $s\in\Pi_0$. Similarly, the function F is said to be pseudoconvex of order $\alpha\in[0,\,h)$ and type $\beta\in(0,\,1]$ if $\left|\frac{F''(s)}{F'(s)}-h\right|<\beta\left|\frac{F''(s)}{F'(s)}-(2\alpha-h)\right|$ for all $s\in\Pi_0$, and F is said to be close-to-pseudoconvex if there exists a pseudoconvex (with $\alpha=0$ and $\beta=1$) function Ψ such that $\mathrm{Re}\{F'(s)/\Psi'(s)\}>0$ in Π_0.

Conditions on parameters $a_1,a_2,b_1,b_2,c_1,c_2,$ under which the differential equation $\dfrac{d^nw}{ds^n}+(a_1e^{hs}+a_2)\dfrac{dw}{ds}+(b_1e^{hs}+b_2)w=c_1e^{hs}+c_2$, $n\ge 2$, has an entire solution pseudostarlike or pseudoconvex of order $\alpha\in[0,\,h)$ and type $\beta\in(0,\,1]$, or close-to-pseudoconvex in Π_0 are found. It is proved that for such solution $\ln M(\sigma,F)=(1+o(1))\dfrac{n\sqrt[n]{|b_1|}}{h}e^{h\sigma/n}$ as $\sigma\to+\infty$, where $M(\sigma,F)=\sup\{|F(\sigma+it)|:\ t\in\mathbb{R}\}$.

Key words and phrases: differential equation, Dirichlet series, pseudostarlikeness, pseudoconvexity, close-to-pseudoconvexity.

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Introduction

Let S be a class of functions

$$f(z)=z+\sum_{n=2}^{\infty}f_nz^n \tag{1}$$

analytic univalent in $\mathbb{D}=\{z:\ |z|<1\}$. The function $f\in S$ is said to be convex if $f(\mathbb{D})$ is a convex domain and is said to be starlike if $f(\mathbb{D})$ is starlike domain regarding the origin. It is well known [2] that the condition $\mathrm{Re}\,\{1+zf''(z)/f'(z)\}>0$, $z\in\mathbb{D}$, is necessary and sufficient for the convexity of f, and that the condition $\mathrm{Re}\,\{zf'(z)/f(z)\}>0$, $z\in\mathbb{D}$, is necessary and sufficient for the starlikeness of $f\in S$. By W. Kaplan [7] the function $f\in S$ is said to be close-to-convex (see also [2, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,\big(f'(z)/\Phi'(z)\big)>0$, $z\in\mathbb{D}$.

The concept of the starlikeness of a function $f\in S$ got the series of generalizations. I.S. Jack [6] studied starlike functions of order $\alpha\in[0,\,1)$, i.e. such functions $f\in S$, for which $\mathrm{Re}\,\{zf'(z)/f(z)\}>\alpha$, $z\in\mathbb{D}$. It is proved (see [6], [11, p. 13]) that if $\sum_{n=2}^{\infty}(n-\alpha)|f_n|\le 1-\alpha$, then function $f\in S$ is starlike of order α. V.P. Gupta [4] introduced the concept of starlike function of order $\alpha\in[0,\,1)$ and type $\beta\in(0,\,1]$. A function $f\in S$ is said to be starlike of order $\alpha\in[0,\,1)$ and type $\beta\in(0,\,1]$ if $|zf'(z)/f(z)-1|<\beta|zf'(z)/f(z)+1-2\alpha|$ for all $z\in\mathbb{D}$.

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It is proved [4] that if $\sum_{n=2}^{\infty}\{(1+\beta)n-\beta(2\alpha-1)-1\}|f_n|\le 2\beta(1-\alpha)$, then function $f\in S$ is starlike of order α and type β. According to the Alexander criterion [1,3], a function (1) is close-to-convex in $\mathbb{D}$ if $1\ge 2f_2\ge\dots\ge kf_k\ge(k+1)f_{k+1}\ge\dots$.

Studying the properties of the solutions of the differential equation

$$z^2\frac{d^2w}{dz^2}+\left(\beta_1z^2+\beta_2z\right)\frac{dw}{dz}+(\gamma_1z+\gamma_2)\,w=0, \tag{2}$$

S.M. Shah proved [8] that if $\beta_2>0$, $-1\le\beta_1<0$ and either $\gamma_2=0$, $-\beta_2\le\gamma_1<0$ or $\beta_2+\gamma_2=0$, $-\beta_2\le 2\gamma_1<0$, then differential equation (2) has an entire solution such that all derivatives $f^{(j)}$, $j\in\mathbb{Z}_+$, are close-to-convex functions in $\mathbb{D}$ and $\ln M_f(r)=\big(1+o(1)\big)|\beta_1|r$ as $r\to+\infty$, where $M_f(r):=\max\big\{|f(z)|:|z|=r\big\}$. The convexity of solutions of the Shah equation has been studied in [14,15].

Dirichlet series are a direct generalization of power series. For $h>0$, let (λ_k) be an increasing to $+\infty$ sequence of positive numbers such that $\lambda_2>h$. Let Dirichlet series

$$F(s)=e^{sh}+\sum_{k=2}^{\infty}f_k\exp\{s\lambda_k\},\quad s=\sigma+it, \tag{3}$$

be absolutely convergent in a half-plane $\Pi_0=\{s:\operatorname{Re}s<0\}$. It is known (see [5] and [11, p. 135]) that each function F of such type is non-univalent in Π_0, but if $\sum_{k=2}^{\infty}\lambda_k|f_k|\le h$, then function F is conformal in Π_0.

A conformal in Π_0 function F is said to be pseudostarlike if $\operatorname{Re}\big\{F'(s)/F(s)\big\}>0$, $s\in\Pi_0$. If in the definition of the pseudostarlikeness instead of F'/F we put F''/F', then we get (see [5] and [11, p. 139]) the definition of the pseudoconvexity of F. In [5] (see also [11, p. 139]) it is proved that if $\sum_{k=2}^{\infty}\lambda_k|f_k|\le h$, then function (3) is pseudostarlike.

A conformal function (3) is said to be [13] pseudostarlike of the order $\alpha\in[0,h)$ if $\operatorname{Re}\big\{F'(s)/F(s)\big\}>\alpha$ for all $s\in\Pi_0$. As in [13], we call a conformal function (3) in Π_0 pseudostarlike of the order $\alpha\in[0,h)$ and the type $\beta\in(0,1]$ if

$$\big|F'(s)/F(s)-h\big|<\beta\big|F'(s)/F(s)-(2\alpha-h)\big|$$

for $s\in\Pi_0$. Finally, the function (3) is said [11, p. 140] to be close-to-pseudoconvex if there exists a pseudoconvex function Ψ such that $\operatorname{Re}\big\{F'(s)/\Psi'(s)\big\}>0$ in Π_0.

Let $z=e^s$. Then equation (2) will be written in the form

$$\frac{d^2w}{ds^2}+(\beta_1e^s+\beta_2-1)\frac{dw}{ds}+(\gamma_1e^s+\gamma_2)\,w=0,$$

and a generalization of this equation is a differential equation

$$\frac{d^nw}{ds^n}+\left(a_1e^{hs}+a_2\right)\frac{dw}{ds}+\left(b_1e^{hs}+b_2\right)w=c_1e^{hs}+c_2,\quad n\ge2,\quad h>0. \tag{4}$$

The purpose of this article is to study the properties of solutions of the equation (4).

1 Recurrent formulas

We will search the solution of the equation (4) in the form

$$w=F(s)=\sum_{k=0}^{\infty}f_ke^{s\lambda_k},\quad s=\sigma+it,$$

where $0 = \lambda_0 < \lambda_k \uparrow +\infty$ as $k \to \infty$. Then

$$\sum_{k=1}^{\infty} (\lambda_k^n + a_2\lambda_k + b_2)\, f_k e^{s\lambda_k} + b_2 f_0 + \sum_{k=1}^{\infty} (a_1\lambda_k + b_1)\, f_k e^{s(\lambda_k+h)} + b_1 e^{hs} f_0 \equiv c_1 e^{hs} + c_2,$$

whence as $\sigma \to -\infty$ we get $b_2 f_0 + o(1) = c_2 + o(1)$, i.e. $f_0 = c_2/b_2$ and

$$(\lambda_1^n + a_2\lambda_1 + b_2) f_1 e^{s\lambda_1} + \sum_{k=2}^{\infty} (\lambda_k^n + a_2\lambda_k + b_2) f_k e^{s\lambda_k} + \sum_{k=1}^{\infty} (a_1\lambda_k + b_1) f_k e^{s(\lambda_k+h)} \equiv (c_1 - b_1 f_0) e^{hs}.$$

If $\lambda_1^n + a_2\lambda_1 + b_2 \neq 0$ and $c_1 - b_1 f_0 \neq 0$, then we obtain

$$\big(1 + o(1)\big)\, (\lambda_1^n + a_2\lambda_1 + b_2)\, f_1 e^{s\lambda_1} = (c_1 - b_1 f_0)\, e^{hs}$$

as $\sigma \to -\infty$, therefore

$$\lambda_1 = h, \quad f_1 = \frac{c_1 - b_1 f_0}{\lambda_1^n + a_2\lambda_1 + b_2} = \frac{c_1 - b_1 f_0}{h^n + a_2 h + b_2}$$

and

$$\sum_{k=2}^{\infty} (\lambda_k^n + a_2\lambda_k + b_2)\, f_k e^{s\lambda_k} + \sum_{k=1}^{\infty} (a_1\lambda_k + b_1)\, f_k e^{s(\lambda_k+h)} \equiv 0. \tag{5}$$

Writing the identity (5) in the form

$$(\lambda_2^n + a_2\lambda_2 + b_2)\, f_2 e^{s\lambda_2} + \sum_{k=3}^{\infty} (\lambda_k^n + a_2\lambda_k + b_2)\, f_k e^{s\lambda_k} + (a_1\lambda_1 + b_1)\, f_1 e^{2hs}$$
$$+ \sum_{k=2}^{\infty} (a_1\lambda_k + b_1)\, f_k e^{s(\lambda_k+h)} \equiv 0,$$

as above, by condition $\lambda_2^n + a_2\lambda_2 + b_2 \neq 0$ and $a_1\lambda_1 + b_1 \neq 0$, we get

$$\lambda_2 = 2h, \quad f_2 = -\frac{a_1\lambda_1 + b_1}{\lambda_2^n + a_2\lambda_2 + b_2} f_1 = -\frac{a_1 h + b_1}{(2h)^n + 2ha_2 + b_2} f_1$$

and

$$\sum_{k=3}^{\infty} (\lambda_k^n + a_2\lambda_k + b_2)\, f_k e^{s\lambda_k} + \sum_{k=2}^{\infty} (a_1\lambda_k + b_1)\, f_k e^{s(\lambda_k+h)} \equiv 0.$$

Continuing this process, we obtain for $k \geq 3$

$$\lambda_k = kh, \quad f_k = -\frac{(k-1)ha_1 + b_1}{(kh)^n + kha_2 + b_2} f_{k-1},$$

provided $(kh)^n + kha_2 + b_2 \neq 0$ and $(k-1)ha_1 + b_1 \neq 0$.

Thus, the following statement is true.

Lemma 1. *If $b_2 \neq 0$, $c_1 b_2 - c_2 b_1 \neq 0$, $(kh)^n + kha_2 + b_2 \neq 0$ and $kha_1 + b_1 \neq 0$ for all $k \geq 1$, then differential equation* (4) *has a solution*

$$F(s) = \frac{c_2}{b_2} + \frac{c_1 b_2 - c_2 b_1}{b_2(h^n + ha_2 + b_2)} e^{sh} + \sum_{k=2}^{\infty} f_k e^{skh}, \tag{6}$$

where $f_k = -\dfrac{(k-1)ha_1 + b_1}{(kh)^n + kha_2 + b_2} f_{k-1}$ for $k \geq 2$.

2 Growth

For $k \geq 2$ we have

$$|f_k| \leq \prod_{j=2}^{k} \frac{(j-1)h|a_1|+|b_1|}{|(jh)^n + jha_2 + b_2|} \leq \prod_{j=2}^{k} \frac{B}{j^{n-1}}, \quad B = \max_{j\geq 2} \frac{h|a_1|+|b_1|j^{-1}}{|h^n + j^{1-n}ha_2 + b_2 j^{-n}|},$$

whence we have $\ln|f_k| \leq k\ln B - (n-1)\sum_{j=2}^{k} \ln j = -(1+o(1))(n-1)k\ln k$ as $k \to \infty$, i.e. $\lim\limits_{k\to\infty} \frac{1}{\lambda_k} \ln \frac{1}{|f_k|} = \lim\limits_{k\to\infty} \frac{1}{kh} \ln \frac{1}{|f_k|} = +\infty$ and, as is well known (see, e.g., [9, p. 10]), function (6) is entire.

To study the growth of entire function (6) we use the Wiman-Valiron method. Let $M(\sigma, F) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$, $\mu(\sigma, F) = \max\{|f_k|\exp\{\sigma\lambda_k\} : k \geq 0\}$ be the maximal term and $\nu(\sigma, F) = \max\{k : |f_k|\exp\{\sigma\lambda_k\} = \mu(\sigma, F)\}$ be the central index.

Suppose that the exponents of Dirichlet series (6) satisfy the condition $\int_0^\infty \frac{\ln n(t)}{t^2}dt < +\infty$, where $n(t) = \sum\limits_{0<\lambda_n\leq t} 1$, and put $\eta(x) = \int_x^\infty t^{-2}\ln n(t)dt$, $l(x) = \frac{1}{\eta(x)} \ln^{-2} \frac{1}{\eta(x)}$ and $k(x) = x\sqrt{\frac{1}{l(x)}}$. Then (see [12]) for every $m \in \mathbb{N}$ and all s, $\operatorname{Re} s = \tau$, $|\tau - \sigma| < \frac{1}{30k(\lambda_{\nu(\sigma,F)})}$ the following equality holds

$$F^{(m)}(s) = \lambda^m_{\nu(\sigma,F)}\Big(F(s) + o\big(M(\sigma, F)\big)\Big) \tag{7}$$

as $0 \leq \sigma \to +\infty$ outside of some set $E \subset [0, +\infty)$ of finite measure and E is contained in the union of intervals $[R_\nu + \tau_{\nu-1}, R_\nu + \tau_\nu)$, where $\tau_\nu - \tau_{\nu-1} \to 0$ as $\nu \to \infty$.

Let $\delta(\sigma)$ be an arbitrary positive function on $[0, +\infty)$, which tends to zero as $\sigma \to +\infty$, and $\Delta(\sigma) = \{s : \operatorname{Re} s = \sigma, |F(s)| \geq 1 - \delta(\sigma)M(\sigma, F)\}$. Then choosing $\tau = \sigma$, from (7) we get

$$F^{(m)}(s) = \lambda^m_{\nu(\sigma,F)} F(s)\big(1 + \varepsilon(\sigma)\big), \quad s \in \Delta(\sigma), \tag{8}$$

where $\varepsilon(\sigma) \to 0$ as $\nu \to \infty$. Substituting (8) in (4), by the condition $b_1 \neq 0$ we obtain $\lambda^n_{\nu(\sigma,F)} = |b_1|e^{h\sigma}\big(1 + \varepsilon_1(\sigma)\big)$, where $\varepsilon_1(\sigma) \to 0$ as $\sigma \to +\infty$, $\sigma \notin E$. Therefore,

$$\lambda_{\nu(\sigma,F)} = \big(1 + o(1)\big)\sqrt[n]{|b_1|}e^{h\sigma/n}, \quad \sigma \to +\infty, \sigma \notin E. \tag{9}$$

If $\sigma \in E$, i.e. $\sigma_{\nu-1} := R_\nu + \tau_{\nu-1} \leq \sigma \leq \sigma_\nu := R_\nu + \tau_\nu$ for some ν, then $\sigma_\nu - \sigma_{\nu-1} \to 0$ and, thus, $e^{h\sigma_{\nu-1}} = \big(1+o(1)\big)e^{h\sigma} = \big(1+o(1)\big)e^{h\sigma_\nu}$ as $\nu \to \infty$. Therefore,

$$\begin{aligned}\big(1+o(1)\big)e^{h\sigma} = \big(1+o(1)\big)e^{h\sigma_{\nu-1}} &= \lambda_{\nu(\sigma_{\nu-1},F)} \leq \lambda_{\nu(\sigma,F)} \leq \lambda_{\nu(\sigma_\nu,F)} \\ &= \big(1+o(1)\big)e^{h\sigma_\nu} = \big(1+o(1)\big)e^{h\sigma}, \quad \sigma \to +\infty,\end{aligned}$$

i.e. (9) is valid if $\sigma \to +\infty$. Since $\ln\mu(\sigma, F) = \ln\mu(0, F) + \int_0^\sigma \lambda_{\nu(t,F)}dt$ for all $\sigma \in \mathbb{R}$ (see, e.g., [9, p. 17]), by L'Hôpital's rule we get

$$\ln\mu(\sigma, F) = \big(1+o(1)\big)\frac{n\sqrt[n]{|b_1|}}{h}e^{h\sigma/n}, \quad \sigma \to +\infty.$$

Then, since $\lambda_n = nh$ for all integers $n \geq 0$, we have $\ln M(\sigma, F) = \big(1+o(1)\big)\ln\mu(\sigma, F)$ as $\sigma \to +\infty$ (see, e.g., [9, p. 22] and [10]). Therefore, the following theorem is proved.

Theorem 1. *The solution* (6) *of differential equation* (4) *is an entire function and*

$$\ln M(\sigma, F) = \big(1+o(1)\big)\frac{n\sqrt[n]{|b_1|}}{h}e^{h\sigma/n}, \quad \sigma \to +\infty.$$

3 Pseudostarlikeness

Using Lemma 1, we can find the conditions under which solution (6) of equation (4) will be pseudostarlike. First of all, we note that in order for this solution to look like (3), it is necessary that $c_2 = 0$ and $c_1 = h^n + ha_2 + b_2$. In [13], it is proved that if

$$\sum_{k=2}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\} |f_k| \leq 2\beta(h-\alpha), \tag{10}$$

then function (3) is pseudostarlike of the order $\alpha \in [0, h)$ and the type $\beta \in (0, 1]$.

For function (6) with $f_0 = 0$ and $f_1 = 1$ by the condition $(kh)^n + kha_2 + b_2 > 0$ we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\} |f_k| \\
&\quad \leq \sum_{k=2}^{\infty} \{(1+\beta)kh - 2\beta\alpha - h(1-\beta)\} \frac{(k-1)h|a_1| + |b_1|}{(kh)^n + kha_2 + b_2} |f_{k-1}| \\
&\quad = \sum_{k=1}^{\infty} \{(1+\beta)(k+1)h - 2\beta\alpha - h(1-\beta)\} \frac{kh\,|a_1| + |b_1|}{\left((k+1)h\right)^n + (k+1)ha_2 + b_2} |f_k| \\
&\quad = \{(1+\beta)2h - 2\beta\alpha - h(1-\beta)\} \frac{h\,|a_1| + |b_1|}{(2h)^n + 2ha_2 + b_2} \\
&\qquad + \sum_{k=2}^{\infty} \{(1+\beta)(k+1)h - 2\beta\alpha - h(1-\beta)\} \frac{kh\,|a_1| + |b_1|}{\left((k+1)h\right)^n + (k+1)ha_2 + b_2} |f_k| \\
&\quad = \sum_{k=2}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\} A_k\,|f_k| + B,
\end{aligned} \tag{11}$$

where

$$A_k = \frac{(1+\beta)(k+1)h - 2\beta\alpha - h(1-\beta)}{(1+\beta)kh - 2\beta\alpha - h(1-\beta)} \frac{kh|a_1| + |b_1|}{\left((k+1)h\right)^n + (k+1)ha_2 + b_2}$$

and

$$B = \{(1+\beta)2h - 2\beta\alpha - h(1-\beta)\} \frac{h\,|a_1| + |b_1|}{(2h)^n + 2ha_2 + b_2}.$$

Suppose that $a_2 \geq 0$ and $b_2 > 0$. Then, since $\alpha < h$, for all $k \geq 2$ we have

$$\frac{(1+\beta)(k+1)h - 2\beta\alpha - h(1-\beta)}{(1+\beta)kh - 2\beta\alpha - h(1-\beta)} = 1 + \frac{(1+\beta)h}{(1+\beta)kh - 2\beta\alpha - h(1-\beta)} \leq 2$$

and for all $k \geq 1$ we have

$$\frac{kh\,|a_1| + |b_1|}{\left((k+1)h\right)^n + (k+1)ha_2 + b_2} \leq \frac{kh|a_1| + |b_1|}{(kh)^n + kha_2 + b_2} \leq \frac{|b_1|}{b_2},$$

provided $|a_1|\, b_2 \leq |b_1|\, a_2$. Therefore, $A_k \leq 2\,|b_1|\,/b_2$.

Also $(1+\beta)2h - 2\beta\alpha - h(1-\beta) \leq (3\beta+1)h$ and $B \leq (3\beta+1)h\,|b_1|\,/b_2$. Therefore, (11) implies

$$\begin{aligned}
&\sum_{k=2}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\} |f_k| \\
&\qquad \leq 2\frac{|b_1|}{b_2} \sum_{k=2}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\} |f_k| + \frac{(3\beta+1)h\,|b_1|}{b_2}.
\end{aligned} \tag{12}$$

Using (12), now we prove the following theorem.

Theorem 2. *Let $a_2 \geq 0$, $b_2 > 0$, $c_2 = 0$, $c_1 = h^n + 2ha_2 + b_2$ and $|a_1|\, b_2 \leq |b_1|\, a_2$. If*

$$(3\beta + 1)h\,|b_1| \leq 2\beta\,(b_2 - 2\,|b_1|)\,(h - \alpha), \tag{13}$$

then differential equation (4) has solution (6) pseudostarlike of the order $\alpha \in [0, h)$ and the type $\beta \in (0,\, 1]$.

Proof. Conditions (13) and $|a_1|\, b_2 \leq |b_1|\, a_2$ imply $b_2 - 2\,|b_1| > 0$, i.e. $2\frac{|b_1|}{b_2} < 1$. Therefore, from (12) we obtain

$$\left(1 - 2\frac{|b_1|}{b_2}\right) \sum_{k=2}^{\infty} \big\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\big\}\,|f_k| \leq \frac{(3\beta+1)h|b_1|}{b_2},$$

whence in view of (13) we get

$$\sum_{k=2}^{\infty} \big\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\big\}\,|f_k| \leq \frac{(3\beta+1)h\,|b_1|}{b_2 - 2\,|b_1|} \leq 2\beta(h-\alpha),$$

i.e. the inequality (10) holds, and function (6) is pseudostarlike of the order $\alpha \in [0, h)$ and the type $\beta \in (0,\, 1]$. □

4 Pseudoconvexity

In [13], it is proved that if

$$\sum_{k=2}^{\infty} \lambda_k \big\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\big\}\,|f_k| \leq 2\beta h(h-\alpha), \tag{14}$$

then function (3) is pseudoconvex of the order $\alpha \in [0, h)$ and the type $\beta \in (0,\, 1]$.

Theorem 3. *Let $a_2 \geq 0$, $b_2 > 0$, $c_2 = 0$, $c_1 = h^n + 2ha_2 + b_2$ and $|a_1|\, b_2 \leq |b_1|\, a_2$. If*

$$(3\beta + 1)h\,|b_1| \leq \beta\big(b_2 - 4|b_1|\big)(h - \alpha), \tag{15}$$

then differential equation (4) has solution (6) pseudoconvex of the order $\alpha \in [0, h)$ and the type $\beta \in (0,\, 1]$.

Proof. As above, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \lambda_k \big\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\big\}\,|f_k| \\
&\quad\leq \sum_{k=2}^{\infty} kh\big\{(1+\beta)kh - 2\beta\alpha - h(1-\beta)\big\}\frac{(k-1)h\,|a_1| + |b_1|}{(kh)^n + kha_2 + b_2}\,|f_{k-1}| \\
&\quad= \sum_{k=1}^{\infty} (k+1)h\big\{(1+\beta)(k+1)h - 2\beta\alpha - h(1-\beta)\big\}\frac{kh\,|a_1| + |b_1|}{\big((k+1)h\big)^n + (k+1)ha_2 + b_2}\,|f_k| \\
&\quad= 2h\big\{(1+\beta)2h - 2\beta\alpha - h(1-\beta)\big\}\frac{h\,|a_1| + |b_1|}{(2h)^n + 2ha_2 + b_2} \\
&\qquad + \sum_{k=2}^{\infty} (k+1)h\{(1+\beta)(k+1)h - 2\beta\alpha - h(1-\beta)\}\frac{kh\,|a_1| + |b_1|}{((k+1)h)^n + (k+1)ha_2 + b_2}\,|f_k| \\
&\quad= \sum_{k=2}^{\infty} \lambda_k \big\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\big\} A_k^*\,|f_k| + B^*,
\end{aligned} \tag{16}$$

where $A_k^* = (k+1)A_k/k \le 2A_k$ and $B^* = 2hB$. Using the above estimates for A_k and B, we obtain $A_k^* \le 4|b_1|/b_2$ and $B^* = 2(3\beta+1)h^2|b_1|/b_2$. Condition (15) implies $4|b_1|/b_2 < 1$. Therefore, from (16) as above we get

$$\sum_{k=2}^{\infty} \lambda_k\big\{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\big\}|f_k| \le \frac{2(3\beta+1)h^2|b_1|}{b_2 - 4|b_1|}$$

and, thus, (15) implies (14), i.e. function (3) is pseudoconvex of the order $\alpha \in [0,h)$ and the type $\beta \in (0, 1]$. □

5 Close-to-pseudoconvexity

It is known [11, p. 140] that if $\lambda_k = \lambda_{k-1} + h$ and $f_k > 0$ for all $k \ge 2$ and

$$h \ge \lambda_2 f_2 \ge \dots \ge \lambda_{k-1} f_{k-1} \ge \lambda_k f_k \ge \dots, \tag{17}$$

then function (3) is close-to-pseudoconvex.

Theorem 4. *Let* $c_2 = 0$, $c_1 = h^n + ha_2 + b_2$, $a_2 \ge 0$, $b_2 > 0$, $-b_2/2 \le b_1 < 0$ *and* $-a_2 \le a_1 \le 0$. *Then differential equation* (4) *has close-to-pseudoconvex solution* (6).

Proof. Since $a_2 \ge 0$, $b_2 > 0$, $b_1 < 0$ and $a_1 \le 0$, we have $f_1 = 1$ and

$$f_k = \frac{(k-1)h|a_1| + |b_1|}{(kh)^n + kha_2 + b_2} f_{k-1} > 0 \quad \text{for} \quad k \ge 2.$$

Also

$$\lambda_2 f_2 = 2h\frac{h|a_1| + |b_1|}{(2h)^n + 2ha_2 + b_2} \le h\frac{2h|a_1| + 2|b_1|}{2ha_2 + b_2} \le h$$

and

$$\frac{\lambda_k f_k}{\lambda_{k-1} f_{k-1}} = \frac{k}{k-1}\frac{(k-1)h|a_1| + |b_1|}{(kh)^n + kha_2 + b_2} \le \frac{k(k-1)h|a_1| + k|b_1|}{k(k-1)ha_2 + (k-1)b_2}$$
$$\le \frac{k(k-1)ha_2 + kb_2/2}{k(k-1)ha_2 + (k-1)b_2} \le 1$$

for $k \ge 3$, i.e. (17) holds and function (6) is close-to-pseudoconvex. □

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Ряд Діріхле $F(s) = e^{hs} + \sum_{k=2}^{\infty} f_k e^{s\lambda_k}$ з показниками $0 < h < \lambda_k \uparrow +\infty$ і абсцисою абсолютної збіжності $\sigma_a[F] \geq 0$ називається псевдозірковим порядку $\alpha \in [0, h)$ і типу $\beta \in (0, 1]$ в $\Pi_0 = \{s : \operatorname{Re} s < 0\}$, якщо $\left|\frac{F'(s)}{F(s)} - h\right| < \beta \left|\frac{F'(s)}{F(s)} - (2\alpha - h)\right|$ для всіх $s \in \Pi_0$. Аналогічно, функція F називається псевдоопуклою порядку $\alpha \in [0, h)$ і типу type $\beta \in (0, 1]$, якщо $\left|\frac{F''(s)}{F'(s)} - h\right| < \beta \left|\frac{F''(s)}{F'(s)} - (2\alpha - h)\right|$ для всіх $s \in \Pi_0$, а F називається близькою до псевдоопуклої, якщо існує така псевдоопукла (з $\alpha = 0$ і $\beta = 1$) функція Ψ, що $\operatorname{Re}\{F'(s)/\Psi'(s)\} > 0$ в Π_0.

Знайдено умови на параметри a_1, a_2, b_1, b_2, c_1, c_2, за яких диференціальне рівняння $\dfrac{d^n w}{ds^n} + (a_1 e^{hs} + a_2)\dfrac{dw}{ds} + (b_1 e^{hs} + b_2)w = c_1 e^{hs} + c_2$, $n \geq 2$, має цілий розв'язок, псевдозірковий, або псевдоопуклий порядку $\alpha \in [0, h)$ і типу $\beta \in (0, 1]$, або близький до псевдоопуклого в Π_0. Доведено, що для такого розв'язку $\ln M(\sigma, F) = (1 + o(1))\dfrac{n\sqrt[n]{|b_1|}}{h} e^{h\sigma/n}$ при $\sigma \to +\infty$, де $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$.

Ключові слова і фрази: диференціальне рівняння, ряд Діріхле, псевдозірковість, псевдоопуклість, близькість до псевдоопуклості.