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On certain subclasses of univalent functions associated with generalized Poisson distribution series

Porwal S.[1], Ghai S.K.[2]

The purpose of the present paper is to obtain necessary and sufficient condition for generalized Poisson distribution series to be in certain classes of univalent functions. We also consider an integral operator associated with generalized Poisson distribution series.

Key words and phrases: analytic function, univalent function, starlike and convex function, Poisson distribution series.

[1] Department of Mathematics, Ram Sahai Government Degree College, Shivrajpur, Kanpur 209205, Uttar Pradesh, India
[2] Department of Mathematics, Government Degree College, Kailaras, Morena 476001, Madhya Pradesh, India
E-mail: saurabhjcb@rediffmail.com (Porwal S.), sanjaykumarghai@yahoo.com (Ghai S.K.)

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f, defined in the open unit disk

$$\mathbb{U} = \{z : |z| < 1\},$$

normalized by the condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$. We can express the functions of the class $\mathcal{S}$ by the power series expansion about the origin of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We further suppose that $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{2}$$

A function f of the form (2) is said to be in the class $\mathcal{T}(\lambda, \alpha)$ if it satisfies the following condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \alpha \quad \text{for every } \ z \in \mathbb{U}, \tag{3}$$

where α and λ are non-negative real numbers with $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$. Similarly, a function f of the form (2) is said to be in the class $\mathcal{C}(\lambda, \alpha)$ if it satisfies the following analytic criteria

$$\operatorname{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)}\right\} > \alpha \quad \text{for every } \ z \in \mathbb{U}. \tag{4}$$

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From (3) and (4), it is easy to verify that

$$f(z) \in \mathcal{C}(\lambda, \alpha) \quad \Longleftrightarrow \quad zf'(z) \in \mathcal{T}(\lambda, \alpha).$$

The classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ were initially introduced and studied by O. Altintas and S. Owa [3]. Further, some necessary and sufficient conditions for the generalized Bessel functions, Poisson distribution series and generalized distribution series for these classes were investigated by S. Porwal and K.K. Dixit [8–10].

It is worthy to note that for $\lambda = 0$ the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ reduce to the classes $\mathcal{T}(\alpha)$ (the class of starlike functions of order α, $0 \le \alpha < 1$) and $\mathcal{C}(\alpha)$ (the class of convex functions of order α, $0 \le \alpha < 1$), studied by M.S. Robertson [13] and H. Silverman [14].

In 2014, S. Porwal [8] (see also [2, 6]) introduced the Poisson distribution series and gave a nice its application on certain classes of univalent functions. This paper established a link between probability distribution and geometric function theory and opens up a new and interesting direction of research. After the appearance of that paper, several researchers introduced hypergeometric distribution series [1], hypergeometric type distribution series [12], confluent hypergeometric distribution series [11], Borel distribution series [15], binomial distribution series [7], Pascal distribution series [5], generalized distribution series [10], Mittag-Leffler type Poisson distribution series [4] and obtained some necessary and sufficient conditions on certain classes of univalent functions.

In the present paper, motivated by the above mentioned work, we obtain some necessary and sufficient condition for the generalized Poisson distribution series to belong to the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$. For this purpose we introduce a generalized Poisson distribution series. Now, we recall the definition of Poisson distribution series as

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n. \tag{5}$$

Now we define

$$K(\mu, m, z) = (1-\mu)K(m, z) + \mu z K'(m, z).$$

Hence,

$$K(\mu, m, z) = z + \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} z^n. \tag{6}$$

One can see that for $\mu = 0$, the series (6) reduces to the series (5).

Next, we define $\mathcal{T}K(\mu, m, z) = 2z - K(\mu, m, z)$. Thus

$$TK(\mu, m, z) = z - \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} z^n. \tag{7}$$

2 Main Results

To establish our main results we require the following lemmas.

Lemma 1 ([3]). *A function f defined by* (2) *is in the class $\mathcal{T}(\lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \left\{ n(1 - \lambda\alpha) - \alpha(1 - \lambda) \right\} |a_n| \le 1 - \alpha.$$

Lemma 2 ([3]). *A function f defined by (2) is in the class $\mathcal{C}(\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n\big\{n(1-\lambda\alpha)-\alpha(1-\lambda)\big\}\,|a_n| \le 1-\alpha.$$

Theorem 1. *If $m>0$, $0\le\alpha<1$, $0\le\lambda\le1$ and $0\le\mu\le1$, then $\mathcal{T}K(\mu,m,z)$ is in the class $\mathcal{T}(\lambda,\alpha)$ if and only if*

$$\mu(1-\lambda\alpha)m^2+\big\{(1-\lambda\alpha)(1+2\mu)-\mu\alpha(1-\lambda)\big\}m \le (1-\alpha)e^{-m}. \tag{8}$$

Proof. Note that

$$\mathcal{T}K(\mu,m,z)=z-\sum_{n=2}^{\infty}(1-\mu+\mu n)\frac{m^{n-1}}{(n-1)!}e^{-m}z^n.$$

In view of Lemma 1, to prove that $\mathcal{T}K(\mu,m,z)\in\mathcal{T}(\lambda,\alpha)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\big\{n(1-\lambda\alpha)-\alpha(1-\lambda)\big\}(1-\mu+\mu n)\frac{m^{n-1}}{(n-1)!}e^{-m}\le 1-\alpha.$$

Note that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\big\{n(1-\lambda\alpha)-\alpha(1-\lambda)\big\}(1-\mu+\mu n)\frac{m^{n-1}}{(n-1)!}e^{-m}\\
&\quad=\sum_{n=2}^{\infty}\Big\{\mu(1-\lambda\alpha)(n-1)(n-2)\\
&\qquad\qquad+\big\{(1-\lambda\alpha)(1+2\mu)-\mu\alpha(1-\lambda)\big\}(n-1)+(1-\alpha)\Big\}\frac{m^{n-1}}{(n-1)!}e^{-m}\\
&\quad=e^{-m}\Big[\mu(1-\lambda\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!}\\
&\qquad\qquad+\big\{(1-\lambda\alpha)(1+2\mu)-\mu\alpha(1-\lambda)\big\}\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\Big]\\
&\quad=e^{-m}\Big[\mu(1-\lambda\alpha)m^2e^m+\big\{(1-\lambda\alpha)(1+2\mu)-\mu\alpha(1-\lambda)\big\}me^m+(1-\alpha)(e^m-1)\Big]\\
&\quad=\mu(1-\lambda\alpha)m^2+\big\{(1-\lambda\alpha)(1+2\mu)-\mu\alpha(1-\lambda)\big\}m+(1-\alpha)(1-e^{-m})\le 1-\alpha,
\end{aligned}$$

by the given hypothesis. This completes the proof. □

Theorem 2. *If $m>0$, $0\le\alpha<1$, $0\le\lambda\le1$ and $0\le\mu\le1$, then $\mathcal{T}K(\mu,m,z)$ is in $\mathcal{C}(\lambda,\alpha)$ if and only if*

$$\begin{aligned}
&e^m\Big[\mu(1-\lambda\alpha)m^3+\big\{(1-\lambda\alpha)(1+5\mu)-\mu\alpha(1-\lambda)\big\}m^2\\
&\qquad+\big\{2(1-\lambda\alpha)(1+2\mu)-2\mu\alpha(1-\lambda)+1-\alpha\big\}m\Big]\le 1-\alpha.
\end{aligned}\tag{9}$$

Proof. In view of Lemma 2, to prove that $\mathcal{T}K(\mu,m,z)\in\mathcal{C}(\lambda,\alpha)$, it is sufficient to show that

$$\sum_{n=2}^{\infty}n\big\{n(1-\lambda\alpha)-\alpha(1-\lambda)\big\}(1-\mu+\mu n)\frac{m^{n-1}}{(n-1)!}e^{-m}\le 1-\alpha.$$

Note that

$$\sum_{n=2}^{\infty} n\left\{n(1-\lambda\alpha)-\alpha(1-\lambda)\right\}(1-\mu+\mu n)\frac{m^{n-1}}{(n-1)!}e^{-m}$$

$$=e^{-m}\Bigg[\sum_{n=2}^{\infty}\Big\{\mu(1-\lambda\alpha)(n-1)(n-2)(n-3)$$
$$+\left\{(1-\lambda\alpha)(1+5\mu)-\mu\alpha(1-\lambda)\right\}(n-1)(n-2)$$
$$+\left\{2(1-\lambda\alpha)(1+2\mu)-2\mu\alpha(1-\lambda)+(1-\alpha)\right\}(n-1)+(1-\alpha)\Big\}\frac{m^{n-1}}{(n-1)!}\Bigg]$$

$$=e^{-m}\Bigg[\mu(1-\lambda\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-4)!}+\left\{(1-\lambda\alpha)(1+5\mu)-\mu\alpha(1-\lambda)\right\}\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!}$$
$$+\left\{2(1-\lambda\alpha)(1+2\mu)-2\mu\alpha(1-\lambda)+(1-\alpha)\right\}\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\Bigg]$$

$$=e^{-m}\Bigg[\mu(1-\lambda\alpha)m^3e^m+\left\{(1-\lambda\alpha)(1+5\mu)-\mu\alpha(1-\lambda)\right\}m^2e^m$$
$$+\left\{2(1-\lambda\alpha)(1+2\mu)-2\mu\alpha(1-\lambda)+(1-\alpha)\right\}me^m+(1-\alpha)(e^m-1)\Bigg]\leq 1-\alpha,$$

by the given hypothesis. This completes the proof. □

3 Integral Operator

In the following theorems, we obtain analogous results for a particular integral operator $\mathcal{TG}(\mu,m,z)$, defined as

$$\mathcal{TG}(\mu,m,z)=\int_0^z\frac{\mathcal{T}K(\mu,m,t)}{t}\,dt. \tag{10}$$

Theorem 3. *If $m>0$, $0\leq\alpha<1$, $0\leq\lambda\leq1$ and $0\leq\mu\leq1$, then $\mathcal{TG}(\mu,m,z)$ defined by (10) is in the class $\mathcal{C}(\lambda,\alpha)$ if and only if (8) is satisfied.*

Proof. From the representation of (10) we have $\mathcal{TG}(\mu,m,z)=z-\sum_{n=2}^{\infty}(1-\mu+\mu n)\frac{m^{n-1}}{n!}e^{-m}z^n$. In view of Lemma 2, to prove that $\mathcal{TG}(\mu,m,z)\in\mathcal{C}(\lambda,\alpha)$, it is sufficient to show that $\sum_{n=2}^{\infty}n\{n(1-\lambda\alpha)-\alpha(1-\lambda)\}(1-\mu+\mu n)\frac{m^{n-1}}{n!}e^{-m}\leq 1-\alpha$. Note that

$$\sum_{n=2}^{\infty}n\{n(1-\lambda\alpha)-\alpha(1-\lambda)\}(1-\mu+\mu n)\frac{m^{n-1}}{n!}e^{-m}$$

$$=e^{-m}\sum_{n=2}^{\infty}\left\{n(1-\lambda\alpha)-\alpha(1-\lambda)\right\}(1-\mu+\mu n)\frac{m^{n-1}}{(n-1)!}$$

$$=e^{-m}\sum_{n=2}^{\infty}\Big[\mu(1-\lambda\alpha)(n-1)(n-2)$$
$$+\left\{(1-\lambda\alpha)(1+2\mu)-\mu\alpha(1-\lambda)\right\}(n-1)+(1-\alpha)\Big]\frac{m^{n-1}}{(n-1)!}$$

$$=e^{-m}\Big[\mu(1-\lambda\alpha)m^2e^m+\{(1-\lambda\alpha)(1+2\mu)-\mu\alpha(1-\lambda)\}me^m+(1-\alpha)(e^m-1)\Big]\leq 1-\alpha,$$

by the given hypothesis. This completes the proof. □

Theorem 4. *If* $m > 0$, $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$ *and* $0 \leq \mu \leq 1$, *then* $\mathcal{TG}(\mu, m, z)$ *defined by* (10) *is in the class* $\mathcal{T}(\lambda, \alpha)$ *if and only if*

$$\mu(1-\lambda\alpha)m + \left\{(1-\alpha\lambda) - \mu\alpha(1-\lambda)\right\}\left(1-e^{-m}\right) - \frac{\alpha(1-\mu)(1-\lambda)}{m}\left(1-e^{-m}-me^{-m}\right) \leq 1-\alpha.$$

Proof. In view of Lemma 1, to prove that $\mathcal{TG}(\mu, m, z) \in \mathcal{T}(\lambda, \alpha)$, it is sufficient to show that

$$\sum_{n=2}^{\infty}\left\{n(1-\lambda\alpha) - \alpha(1-\lambda)\right\}(1-\mu+\mu n)\frac{m^{n-1}}{n!}e^{-m} \leq 1-\alpha.$$

Note that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{n(1-\lambda\alpha) - \alpha(1-\lambda)\}(1-\mu+\mu n)\frac{m^{n-1}}{n!}e^{-m} \\
&\quad= e^{-m}\sum_{n=2}^{\infty}\Big[\mu(1-\lambda\alpha)n(n-1) + \{(1-\lambda\alpha) - \mu\alpha(1-\lambda)\}n - \alpha(1-\mu)(1-\lambda)\Big]\frac{m^{n-1}}{n!} \\
&\quad= e^{-m}\Bigg[\mu(1-\lambda\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!} \\
&\qquad\quad + \{(1-\lambda\alpha) - \mu\alpha(1-\lambda)\}\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!} - \alpha(1-\mu)(1-\lambda)\sum_{n=2}^{\infty}\frac{m^{n-1}}{n!}\Bigg] \\
&\quad= e^{-m}\Bigg[\mu(1-\lambda\alpha)me^{m} \\
&\qquad\quad + \{(1-\lambda\alpha) - \mu\alpha(1-\lambda)\}(e^{m}-1) - \frac{\alpha(1-\mu)(1-\lambda)}{m}\left(e^{m}-1-m\right)\Bigg] \leq 1-\alpha,
\end{aligned}$$

by the given hypothesis. This completes the proof. □

Remark. *If we put* $\mu = 0$ *in results of the above theorems, then we obtain the corresponding results of S. Porwal [8].*

References

[1] Ahmad M.S., Mehmood Q., Nazeer W., Haq A.U. *An application of a hypergeometric distribution series on certain analytic functions*. Sci.Int. (Lahore) 2015, **27** (4), 2989–2992.

[2] Altinkaya Ş., Yalçin S. *Poisson distribution series for analytic univalent functions*. Complex Anal. Oper. Theory 2018, **12** (5), 1315–1319. doi:10.1007/s11785-018-0764-y

[3] Altintas O., Owa S. *On subclasses of univalent functions with negative coefficients*. Pusan Kyŏngnam Math.J. 1988, **4**, 41–56.

[4] Bajpai D. A study of univalent functions associated with distribution series and q-calculus. M. Phil. Diss. CSJM University, Kanpur, India, 2016.

[5] Bulboaca T., Murugusundaramoorthy G. *Univalent functions with positive coefficients involving Pascal distribution series*. Commun. Korean Math. Soc. 2020, **35** (3), 867–877. doi:10.4134/CKMS.c190413

[6] Frasin B.A. *Two subclasses of analytic functions associated with Poisson distribution*. Turkish J. Ineq. 2020, **4** (1), 25–30.

[7] Nazeer W., Mehmood Q., Kang S.M., Haq A.U. *An application of Binomial distribution series on certain analytic functions*. J. Comput. Anal. Appl. 2019, **26** (1), 11–17.

[8] Porwal S. *An application of a Poisson distribution series on certain analytic functions*. J. Complex Anal. 2014, article ID 984135. doi:10.1155/2014/984135

[9] Porwal S., Dixit K.K. *An application of generalized Bessel functions on certain analytic functions*. Acta Univ. M. Belii Ser. Math. 2013, **21**, 51–56.

[10] Porwal S. *Generalized distribution and its geometric properties associated with univalent functions*. J. Complex Anal. 2018, article ID 8654506. doi:10.1155/2018/8654506

[11] Porwal S. *Confluent hypergeometric distribution and its applications on certain classes of univalent functions of conic regions*. Kyungpook Math. J. 2018, **58** (3), 495–505. doi:10.5666/KMJ.2018.58.3.495

[12] Porwal S., Gupta A. *Some properties of convolution for hypergeometric distribution type series on certain analytic univalent functions*. Acta Univ. Apulensis Math. Inform. 2018, **56**, 69–80.

[13] Robertson M.I.S. *On the theory of univalent functions*. Ann. Math. 1936, **37** (2), 374–408. doi:10.2307/1968451

[14] Silverman H. *Univalent functions with negative coefficients*. Proc. Amer. Math. Soc. 1975, **51**, 109–116. doi: 10.1090/S0002-9939-1975-0369678-0

[15] Wanas A.K., Khuttar J.A. *Applications of Borel distribution series on analytic functions*. Earthline J. Math. Sci. 2020, **4** (1), 71–82. doi:10.34198/ejms.4120.7182

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Ключові слова і фрази: аналітична функція, однолиста функція, зіркоподібна та опукла функція, ряд розподілу Пуассона.