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Characteristics of linear and nonlinear approximation of isotropic classes of periodic multivariate functions

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Exact order estimates for some characteristics of linear and nonlinear approximation of the isotropic Nikol'skii-Besov classes $\mathbf{B}^r_{p,\theta}$ of periodic multivariate functions in the spaces $B_{q,1}$, $1 \le q \le \infty$, are obtained. Among them are the best orthogonal trigonometric approximations, best m-term trigonometric approximations, Kolmogorov, linear and trigonometric widths.

For all considered characteristics, their estimates coincide in order with the corresponding estimates in the spaces L_q. Moreover, the obtained exact in order estimates (except the case $1 < p < 2 \le q < \frac{p}{p-1}$) are realized by the approximation of functions from the classes $\mathbf{B}^r_{p,\theta}$ by trigonometric polynomials with the spectrum in cubic regions. In any case, they do not depend on the smoothness parameter θ.

Key words and phrases: Nikol'skii-Besov class, best orthogonal trigonometric approximation, best approximation, width.

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Introduction

In the paper, we investigate linear and nonlinear methods of approximation of the isotropic Nikol'skii-Besov classes $\mathbf{B}^r_{p,\theta}$ of periodic multivariate functions in the spaces $B_{q,1}$, $1 \le q \le \infty$. These spaces, as linear subspaces in L_q, $1 \le q \le \infty$, have a peculiarity in that their norm is stronger than the L_q-norm. The motivation to study approximative characteristics in such spaces is the following.

In the number of papers [2, 7, 8, 11, 24–27, 31], the questions concerning approximation of classes of periodic multivariate functions with mixed smoothness (the classes of Nikol'skii-Besov-type $\mathbf{B}^r_{p,\theta}$, Sobolev classes $\mathbf{W}^r_{p,\alpha}$ and some their analogs) in the normed spaces with slightly modified norms comparing to the norm of $B_{q,1}$, $q \in \{1, \infty\}$, were investigated. As a result of the investigations, it was revealed that in many situations the obtained estimates of considered approximative characteristics differ in order from the estimates of corresponding characteristics in the spaces L_q, $q \in \{1, \infty\}$. Besides, they depend on the smoothness parameter θ. It is worth mentioning that optimal (from the point of view of order values) aggregates

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for approximation of the mentioned classes of functions are trigonometric polynomials with "numbers" of harmonics from the so-called step hyperbolic crosses.

A completely different situation is when investigating the approximative characteristics of the isotropic Nikol'skii-Besov classes $\mathbf{B}^r_{p,\theta}$ in the spaces $B_{q,1}$, $1 \le q \le \infty$. The difference is in following.

For all of the considered in the paper characteristics of linear and nonlinear approximation of the classes $\mathbf{B}^r_{p,\theta}$, their estimates in the spaces $B_{q,1}$, $1 \le q \le \infty$, coincide in order with the corresponding estimates in the spaces L_q. Moreover, the obtained exact in order estimates (except the case $1 < p < 2 \le q < \frac{p}{p-1}$) are realized by the approximation of functions from the classes $\mathbf{B}^r_{p,\theta}$ by trigonometric polynomials with the spectrum in cubic regions. In any case, they do not depend on the smoothness parameter θ.

The obtained results generalize and complement the corresponding statements from the papers [4,17,18,21]. We will further comment on this more.

Let us introduce now the needed notations and definitions.

Let $\mathbb{R}^d$, $d \ge 1$, be the d-dimensional Euclidean space of points $x = (x_1, \dots, x_d)$, and $L_p(T^d)$, $1 \le p \le \infty$, $T^d = \prod_{j=1}^d [0; 2\pi)$, be the space of functions $f(x) = f(x_1, \dots, x_d)$ that are 2π-periodic in each variable and their norm

$$\|f\|_p := \left((2\pi)^{-d} \int_{T^d} \Big| f(x) \Big|^p dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in T^d} \Big| f(x) \Big|,$$

is finite.

Furthermore, let $k \in \mathbb{N}$ and $h \in \mathbb{R}^d$. For $f \in L_p(T^d)$, we denote $\Delta_h f(x) = f(x+h) - f(x)$, and define the difference of order k with a step h by the following formula

$$\Delta_h^k f(x) = \Delta_h \Delta_h^{k-1} f(x), \quad \Delta_h^0 f(x) = f(x).$$

The modulus of smoothness of order k for a function $f \in L_p(T^d)$ is defined by the formula

$$w_k(f,t)_p := \sup_{|h| \le t} \|\Delta_h^k f\|_p, \quad \text{where} \quad |h| = \sqrt{h_1^2 + \dots + h_d^2}.$$

We say that a function $f \in L_p(T^d)$ belongs to the space $B^r_{p,\theta}$, $1 \le p, \theta \le \infty$, $r > 0$, if

$$\Big(\int_0^\infty \big(t^{-r} w_k(f,t)_p \big)^\theta \frac{dt}{t} \Big)^{\frac{1}{\theta}} < \infty, \quad 1 \le \theta < \infty,$$

and

$$\sup_{t>0} t^{-r} w_k(f,t)_p < \infty, \quad \theta = \infty.$$

The norm of the space $B^r_{p,\theta}$ is defined by formulas

$$\|f\|_{B^r_{p,\theta}} := \|f\|_p + \Big(\int_0^\infty \big(t^{-r} w_k(f,t)_p \big)^\theta \frac{dt}{t} \Big)^{\frac{1}{\theta}}, \quad 1 \le \theta < \infty,$$

and

$$\|f\|_{B^r_{p,\infty}} := \|f\|_p + \sup_{t>0} t^{-r} w_k(f,t)_p, \quad \theta = \infty, \quad \text{for some} \quad k > r.$$

The spaces $H^r_p \equiv B^r_{p,\infty}$ and $B^r_{p,\theta}$, $1 \le \theta < \infty$, were introduced by S.M. Nikol'skii [14] and O.V. Besov [3], respectively.

The Nikol'skii-Besov class is defined as the unit ball in the space $B^r_{p,\theta}$. We will use for it the same notation, i.e. put

$$\mathbf{B}^r_{p,\theta} := \{f \in B^r_{p,\theta} : \ \|f\|_{B^r_{p,\theta}} \le 1\}.$$

Note that the classes $\mathbf{B}^r_{p,\theta}$ and $\mathbf{H}^r_p$ were studied from the approximation viewpoint in [4, 6, 10, 11, 17–19, 28, 33], where additional relevant references can be found. In what follows, it will be convenient for us to use the equivalent (up to absolute constants) definition for the norm in the space $B^r_{p,\theta}$.

Let $V_l(t)$, $l \in \mathbb{N}$, $t \in \mathbb{R}$, denote the de la Vallee-Poussin kernel of the form

$$V_l(t) := l^{-1} \sum_{k=l}^{2l-1} D_k(t),$$

where

$$D_k(t) := \sum_{m=-k}^{k} e^{imt}$$

is the Dirichlet kernel.

The multidimensional kernel $V_l(x)$, $l \in \mathbb{N}$, $x \in \mathbb{R}^d$, is defined by the formula

$$V_l(x) := \prod_{j=1}^{d} V_l(x_j).$$

Let $\mathbf{V}_l$ be an operator that is defined by the convolution of a function $f \in L_1(T^d)$ with the multidimensional kernel $V_l(x)$, i.e.

$$\mathbf{V}_l f(x) := (f * V_l)(x) = V_l(f, x) := V_l(f).$$

For $f \in L_1(T^d)$, we set

$$\begin{aligned} \sigma_0(f) &:= \sigma_0(f, x) = V_1(f, x), \\ \sigma_s(f) &:= \sigma_s(f, x) = V_{2^s}(f, x) - V_{2^{s-1}}(f, x), \quad s \in \mathbb{N}. \end{aligned}$$

Then for function in $B^r_{p,\theta}$, $1 \le p, \theta \le \infty$, $r > 0$, the following equivalences hold [13]:

$$\begin{aligned} \|f\|_{B^r_{p,\theta}} &\asymp \Big(\sum_{s \in \mathbb{Z}_+} 2^{sr\theta} \|\sigma_s(f)\|_p^\theta \Big)^{\frac{1}{\theta}}, \quad 1 \le \theta < \infty, \\ \|f\|_{B^r_{p,\infty}} &\asymp \sup_{s \in \mathbb{Z}_+} 2^{sr} \|\sigma_s(f)\|_p, \quad \theta = \infty, \end{aligned} \tag{1}$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Here and below, for positive sequences $a(n)$ and $b(n)$, $n \in \mathbb{N}$, we will use the notation $a(n) \asymp b(n)$, that means that there exist the constants $0 < C_1 \le C_2$ such that $C_1 a(n) \le b(n) \le C_2 a(n)$. If only $b(n) \le C_2 a(n)$ $\big($or $b(n) \ge C_1 a(n)\big)$ holds, then we will write $b(n) \ll a(n)$ $\big(b(n) \gg a(n)\big)$.

Note, that in the case $1 < p < \infty$ one can write the equivalent to (1) norm definitions, using some unions of "blocks" of the Fourier series of the function f instead of $\sigma_s(f)$. For $f \in L_p(T^d)$, $1 < p < \infty$, we define

$$f_{(0)} := f_{(0)}(x) = \widehat{f}(0), \quad f_{(s)} := f_{(s)}(x) = \sum_{2^{s-1} \le \max\limits_{j=\overline{1,d}} |k_j| < 2^s} \widehat{f}(k) e^{i(k,x)}, \quad s \in \mathbb{N},$$

where $(k,x) = k_1x_1 + \cdots + k_dx_d$ and $\widehat{f}(k) = (2\pi)^{-d}\int_{T^d} f(t)e^{-i(k,t)}dt$ are the Fourier coefficients of f.

Then if $1 < p < \infty,\ 1 \le \theta \le \infty,\ r > 0$, for $f \in B^r_{p,\theta}$ we have

$$\|f\|_{B^r_{p,\theta}} \asymp \left(\sum_{s\in\mathbb{Z}_+} 2^{sr\theta}\|f_{(s)}\|_p^\theta\right)^{\frac{1}{\theta}}, \quad 1 \le \theta < \infty,$$
$$\|f\|_{B^r_{p,\infty}} \asymp \sup_{s\in\mathbb{Z}_+} 2^{sr}\|f_{(s)}\|_p, \quad \theta = \infty.$$

Note, that for the spaces $B^r_{p,\theta}$ the following embeddings hold

$$B^r_{p,1} \subset B^r_{p,\theta_1} \subset B^r_{p,\theta_2} \subset B^r_{p,\infty} \equiv H^r_p, \quad 1 \le \theta_1 \le \theta_2 \le \infty.$$

Now we define a norm $\|\cdot\|_{B_{q,1}}$ in the subspaces $B_{q,1}$, $1 \le q \le \infty$, of functions $f \in L_q(T^d)$. Such norm is similar to the decomposition norm of functions from the Besov spaces $B^r_{q,\theta}$ (see (1)). Hence, for trigonometric polynomials t with respect to the multiple trigonometric system $\{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$, the norm $\|t\|_{B_{q,1}}$ is defined by the formula

$$\|t\|_{B_{q,1}} := \sum_s \|\sigma_s(t)\|_q.$$

Analogically we define the norm $\|f\|_{B_{q,1}}$ for any function $f \in L_q(T^d)$ such that the series $\sum_{s\in\mathbb{Z}_+}\|\sigma_s(f)\|_q$ is convergent. Note, that for $f \in B_{q,1}$, $1 \le q \le \infty$, the following relations hold:

$$\|f\|_q \ll \|f\|_{B_{q,1}}; \quad \|f\|_{B_{1,1}} \le \|f\|_{B_{q,1}} \le \|f\|_{B_{\infty,1}}.$$

1 Best orthogonal trigonometric approximations

Let us define the approximation characteristic that will be investigated in this part of the paper.

Let X be a normed space with the norm $\|\cdot\|_X$ and Θ_m be a set of m arbitrary d-dimensional vectors with integer coordinates. For $f \in X$ we denote

$$S_{\Theta_m}(f) := S_{\Theta_m}(f,x) = \sum_{k\in\Theta_m}\widehat{f}(k)e^{i(k,x)}$$

and consider the quantity

$$e^\perp_m(f)_X := \inf_{\Theta_m}\|f - S_{\Theta_m}(f)\|_X.$$

If $F \subset X$ is a functional class, then we put

$$e^\perp_m(F)_X := \sup_{f\in F} e^\perp_m(f)_X.$$

The quantity $e^\perp_m(F)_X$ is called the best orthogonal trigonometric approximation of the class F in the space X. The quantities $e^\perp_m(F)_X$ for different functional classes F in the Lebesque spaces $L_q(T^d)$ as well as in some of their subspaces were investigated in many papers (see, e.g., [1,8,15,16,18,21,25,27,30]), where one can find a more detailed bibliography.

Now we formulate two further needed statements.

Theorem A ([21]). *Let* $1 \le p,q,\theta \le \infty$, $(p,q) \notin \{(1,1),(\infty,\infty)\}$, $r > d\left(\frac{1}{p}-\frac{1}{q}\right)_+$. *Then*

$$e_m^{\perp}(\mathbf{B}^r_{p,\theta})_q := e_m^{\perp}(\mathbf{B}^r_{p,\theta})_{L_q(T^d)} \asymp m^{-\frac{r}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+}, \quad \text{where} \quad a_+ = \max\{a;0\}.$$

Theorem B. *Let*

$$t(x) = \sum_{|k_j|\le n_j} c_k e^{i(k,x)},$$

where $n_j \in \mathbb{N}$, $j=\overline{1,d}$. *Then for* $1 \le p < q \le \infty$ *the following inequality holds*

$$\|t\|_q \le 2^d \prod_{j=1}^{d} n_j^{\frac{1}{p}-\frac{1}{q}} \|t\|_p. \tag{2}$$

Inequality (2) was obtained by S.M. Nikol'skii [14] and referred to as the inequality for different metrics. Now we formulate and prove the results obtained for the approximation characteristics of the classes $\mathbf{B}^r_{p,\theta}$ in the spaces $B_{q,1}$.

Theorem 1. *Let* $1 \le p,q,\theta \le \infty$, $(p,q) \notin \{(1,1),(\infty,\infty)\}$. *If* $r > d\left(\frac{1}{p}-\frac{1}{q}\right)_+$, *then*

$$e_m^{\perp}(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp m^{-\frac{r}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+}, \quad \text{where} \quad a_+ = \max\{a;0\}. \tag{3}$$

Proof. First we prove the upper estimate in (3). We also note, that due to the embedding $\mathbf{B}^r_{p,\theta} \subset \mathbf{H}^r_p$, $1 \le \theta < \infty$, it is sufficient to get this estimate for the quantity $e_m^{\perp}(\mathbf{H}^r_p)_{B_{q,1}}$.

We will consider several cases.

Case $1 < p = q < \infty$. For a given $m \in \mathbb{N}$ we take a number $n(m)$ such that $2^{(n-1)d} \le m < 2^{nd}$ holds, i.e. $2^n \asymp m^{1/d}$, and consider the approximation of functions $f \in \mathbf{H}^r_p$ by their cubic Fourier sums $S_n(f)$ of the form

$$S_n(f) := S_n(f,x) = \sum_{s=0}^{n} f_{(s)}(x). \tag{4}$$

By the norm definition of the space $B_{q,1}$ and the convolution property, we can write

$$\begin{aligned} e_m^{\perp}(f)_{B_{q,1}} \ll \|f - S_n(f)\|_{B_{q,1}} &= \Big\| \sum_{s=n+1}^{\infty} f_{(s)} \Big\|_{B_{p,1}} = \sum_{s\in\mathbb{Z}_+} \Big\| \sigma_s * \sum_{s'=n+1}^{\infty} f_{(s')} \Big\|_p \\ &\le \sum_{s=n}^{\infty} \Big\| \sigma_s * \sum_{s'=s}^{s+1} f_{(s')} \Big\|_p \le \sum_{s=n}^{\infty} \|\sigma_s\|_1 \Big\| \sum_{s'=s}^{s+1} f_{(s')} \Big\|_p = I_1. \end{aligned} \tag{5}$$

Further we use the relation $\|V_{2^s}\|_p \asymp 2^{sd\left(1-\frac{1}{p}\right)}$, $1 \le p \le \infty$, (see, e.g., [32, Ch. 1, §1]), and get

$$\|\sigma_s\|_1 = \|V_{2^s} - V_{2^{s-1}}\|_1 \le \|V_{2^s}\|_1 + \|V_{2^{s-1}}\|_1 \le C_3. \tag{6}$$

Besides, taking into account that

$$\|f_{(s')}\|_p \ll 2^{-s'r}, \quad f \in \mathbf{H}^r_p, \ \ s' \in \mathbb{N}, \ \ 1 < p < \infty,$$

we have

$$\left\| \sum_{s'=s}^{s+1} f_{(s')} \right\|_p \le \sum_{s'=s}^{s+1} \|f_{(s')}\|_p \ll \sum_{s'=s}^{s+1} 2^{-s'r} \ll 2^{-sr}. \tag{7}$$

Hence, (5), (6) and (7) yield the estimate

$$e_m^{\perp}(f)_{B_{q,1}} \ll I_1 \ll \sum_{s=n}^{\infty} 2^{-sr} \ll 2^{-nr} \asymp m^{-\frac{r}{d}}.$$

Case $1 \le p < q \le \infty$. For $f \in \mathbf{H}_p^r$, according to the convolution property and the relation (6), we get

$$e_m^{\perp}(f)_{B_{q,1}} \ll \sum_{s=n}^{\infty} \left\| \sigma_s * \sum_{s'=s}^{s+1} f_{(s')} \right\|_q \le \sum_{s=n}^{\infty} \|\sigma_s\|_1 \left\| \sum_{s'=s}^{s+1} f_{(s')} \right\|_q \ll \sum_{s=n}^{\infty} \|f_{(s)}\|_q = I_2. \tag{8}$$

To further estimate the quantity I_2, we use the inequality (2). Let q_0 is a number that satisfy the condition $p < q_0 < q$. Then

$$\begin{aligned} I_2 &\ll \sum_{s=n}^{\infty} 2^{sd\left(\frac{1}{q_0}-\frac{1}{q}\right)} \|f_{(s)}\|_{q_0} \ll \sum_{s=n}^{\infty} 2^{sd\left(\frac{1}{q_0}-\frac{1}{q}\right)} \|\sigma_s(f)\|_{q_0} \\ &\ll \sum_{s=n}^{\infty} 2^{sd\left(\frac{1}{q_0}-\frac{1}{q}\right)} 2^{sd\left(\frac{1}{p}-\frac{1}{q_0}\right)} \|\sigma_s(f)\|_p = \sum_{s=n}^{\infty} 2^{sd\left(\frac{1}{p}-\frac{1}{q}\right)} \|\sigma_s(f)\|_p. \end{aligned} \tag{9}$$

Taking into account that

$$\|\sigma_s(f)\|_p \ll 2^{-sr}, \quad f \in \mathbf{H}_p^r, \ \ 1 \le p \le \infty, \ \ s \in \mathbb{N},$$

we obtain

$$I_2 \ll \sum_{s=n}^{\infty} 2^{sd\left(\frac{1}{p}-\frac{1}{q}\right)} 2^{-sr} = \sum_{s=n}^{\infty} 2^{-sd\left(\frac{r}{d}-\frac{1}{p}+\frac{1}{q}\right)} \ll 2^{-nd\left(\frac{r}{d}-\frac{1}{p}+\frac{1}{q}\right)} \asymp m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{q}}. \tag{10}$$

From (8), (9) and (10) we get the upper estimate for the quantity $e_m^{\perp}(\mathbf{H}_p^r)_{B_{q,1}}$, $1 \le p < q \le \infty$.

Case $1 < q < p < \infty$. Here the upper estimate for the quantity $e_m^{\perp}(\mathbf{H}_p^r)_{B_{q,1}}$ follows from the case $1 < p = q < \infty$ due to the embedding $\mathbf{H}_p^r \subset \mathbf{H}_q^r$, i.e.

$$e_m^{\perp}(\mathbf{H}_p^r)_{B_{q,1}} \ll e_m^{\perp}(\mathbf{H}_q^r)_{B_{q,1}} \asymp m^{-\frac{r}{d}}. \tag{11}$$

Case $1 < q < \infty$, $p = \infty$. The upper estimate for the quantity $e_m^{\perp}(\mathbf{H}_\infty^r)_{B_{q,1}}$ is a corollary from (11) due to the embedding $\mathbf{H}_\infty^r \subset \mathbf{H}_p^r$, $1 \le p < \infty$.

Case $1 < p \le \infty$, $q = 1$. The upper estimate for the quantity $e_m^{\perp}(\mathbf{H}_p^r)_{B_{1,1}}$ follows from (11) in view of the inequality $\|\cdot\|_{B_{1,1}} < \|\cdot\|_{B_{q,1}}$, $q > 1$, and the embedding $\mathbf{H}_\infty^r \subset \mathbf{H}_p^r$.

The upper estimates for all of the cases considered in the theorem are proved. As to the lower estimate in (3), we note that it is a corollary from Theorem A and the inequality $\|\cdot\|_{B_{q,1}} \gg \|\cdot\|_q$. Theorem 1 is proved. □

Remark 1. *In the case $d = 1$, the estimate of the quantity $e_m^{\perp}(\mathbf{B}_{p,\theta}^r)_{B_{\infty,1}}$ was obtained in the papers [25] and [27] for $1 < p < \infty$ and $p = 1$, respectively.*

Remark 2. *Analysing the proof of Theorem 1 and comparing the obtained there result with the estimate of corresponding approximation characteristics that was considered in Theorem A, we can make a conclusion that the following relation holds*

$$e^{\perp}_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp e^{\perp}_m(\mathbf{B}^r_{p,\theta})_q, \quad 1 \le p,q,\theta \le \infty,\ (p,q) \notin \{(1,1),(\infty,\infty)\},\ r > d\left(\frac{1}{p}-\frac{1}{q}\right)_+.$$

Now we formulate a corollary from Theorem 1, that corresponds to the approximation of functions $f \in \mathbf{B}^r_{p,\theta}$ in the space $B_{q,1}$ by their Fourier sums $S_n(f)$ of the form (4). For the functional class $\mathbf{B}^r_{p,\theta}$ and $n \in \mathbb{N}$, we consider the quantity

$$\mathcal{E}_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}} := \sup_{f\in\mathbf{B}^r_{p,\theta}} \|f - S_n(f)\|_{B_{q,1}}.$$

Corollary 1. *Let* $1 \le p,q,\theta \le \infty$, $(p,q) \notin \{(1,1),(\infty,\infty)\}$. *If* $r > d\left(\frac{1}{p}-\frac{1}{q}\right)_+$, *then the relation*

$$\mathcal{E}_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp 2^{-nd\left(\frac{r}{d}-\left(\frac{1}{p}-\frac{1}{q}\right)_+\right)} \tag{12}$$

holds.

Proof. The upper estimate in (12) was obtained during proving Theorem 1. The corresponding lower estimate also follows from this theorem under the condition $2^{(n-1)d} \le m < 2^{nd}$, i.e.

$$\mathcal{E}_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \gg e^{\perp}_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp m^{-\frac{r}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+} \asymp 2^{-nd\left(\frac{r}{d}-\left(\frac{1}{p}-\frac{1}{q}\right)_+\right)}.$$

□

Remark 3. *Combining the results of Theorem 1 and Corollary 1, we make a conclusion that for* $m \asymp 2^{nd}$ *the following relation holds*

$$e^{\perp}_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp \mathcal{E}_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}}, \quad 1 \le p,q,\theta \le \infty,\ (p,q) \notin \{(1,1),(\infty,\infty)\},\ r > d\left(\frac{1}{p}-\frac{1}{q}\right)_+.$$

2 Best approximations and widths

In this part of the paper, we first get the exact order estimates for the best approximations of the classes $\mathbf{B}^r_{p,\theta}$ in the space $B_{q,1}$, $1 \le p,q,\theta \le \infty$, by trigonometric polynomials with the spectrum in cubic regions. So, let $T\left(C^d(2^n)\right)$, $n \in \mathbb{Z}_+$, be a set of trigonometric polynomials t of the form

$$t := t(x) = \sum_{k\in C^d(2^n)} c_k e^{i(k,x)},\ c_k \in \mathbb{C},$$

where $C^d(2^n) := \{k\colon\ k = (k_1,\dots,k_d) \in \mathbb{Z}^d\colon\ |k_j| < 2^n,\ j = \overline{1,d}\}$.

For $f \in X$ we define by

$$E_n(f)_X := \inf_{t\in T(C^d(2^n))} \|f - t\|_X$$

the best approximation of the function f by trigonometric polynomials from the set $T\left(C^d(2^n)\right)$ in the space X. If $F \subset X$ is some functional class, then we set

$$E_n(F)_X := \sup_{f\in F} E_n(f)_X.$$

Note, that in the case $X = L_q(T^d)$, $1 \le q \le \infty$, the quantities $E_n(\mathbf{B}^r_{p,\theta})_q := E_n(\mathbf{B}^r_{p,\theta})_{L_q(T^d)}$ were investigated in the papers [17, 21] and, in particular, in the work [21] the following statement was proved.

Theorem C. *Let $1 \le p, q, \theta \le \infty$. If $r > d\left(\frac{1}{p} - \frac{1}{q}\right)_+$, then*

$$E_n(\mathbf{B}^r_{p,\theta})_q \asymp 2^{-nd\left(\frac{r}{d} - \left(\frac{1}{p}-\frac{1}{q}\right)_+\right)}, \quad \textit{where} \quad a_+ = \max\{a; 0\}.$$

Theorem 2. *Let $1 \le p, q, \theta \le \infty$. Then for $r > d\left(\frac{1}{p} - \frac{1}{q}\right)_+$ the relation*

$$E_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp 2^{-nd\left(\frac{r}{d} - \left(\frac{1}{p}-\frac{1}{q}\right)_+\right)} \tag{13}$$

holds.

Proof. Let us prove the upper estimate in (13). Note, that for $(p,q) \notin \{(1,1),(\infty,\infty)\}$ the needed estimate follows from Theorem 1 due to the relation

$$E_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \ll \mathcal{E}_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp 2^{-nd\left(\frac{r}{d} - \left(\frac{1}{p}-\frac{1}{q}\right)_+\right)}.$$

Therefore, it is sufficient to get the upper estimate for the quantity $E_n(\mathbf{B}^r_{p,\theta})_{B_{p,1}}$, $p \in \{1,\infty\}$. At the same time, as it was indicated during proving Theorem 1, it is sufficient to obtain the corresponding estimate for the classes $\mathbf{H}^r_p$.

Let us consider the approximation of a function $f \in \mathbf{H}^r_p$ by polynomials $t_n(f) \in T(C^d(2^n))$ of the form

$$t_n(f) := t_n(f,x) = \sum_{s=0}^{n-1} \sigma_s(f,x).$$

Then for $p \in \{1,\infty\}$ we have

$$\begin{aligned} E_n(f)_{B_{p,1}} \le \|f - t_n(f)\|_{B_{p,1}} &= \left\| \sum_{s=n}^{\infty} \sigma_s(f) \right\|_{B_{p,1}} \\ &= \sum_{s \in \mathbb{Z}_+} \left\| \sigma_s * \sum_{s'=n}^{\infty} \sigma_{s'}(f) \right\|_p \le \sum_{s=n-1}^{\infty} \left\| \sigma_s * \sum_{s'=s-1}^{s+1} \sigma_{s'}(f) \right\|_p = I_3. \end{aligned} \tag{14}$$

To further estimate the quantity I_3, let us consider two cases.

Case $p = 1$. Due to the convolution property and the relations $\|\sigma_s\|_1 \le C_3$ (see (6)), $\|\sigma_s(f)\|_1 \ll 2^{-sr}$, $f \in \mathbf{H}^r_1$, we can write

$$\begin{aligned} I_3 \le \sum_{s=n-1}^{\infty} \|\sigma_s\|_1 \left\| \sum_{s'=s-1}^{s+1} \sigma_{s'}(f) \right\|_1 &\ll \sum_{s=n-1}^{\infty} \sum_{s'=s-1}^{s+1} \left\|\sigma_{s'}(f)\right\|_1 \\ &\ll \sum_{s=n-1}^{\infty} \sum_{s'=s-1}^{s+1} 2^{-s'r} \ll \sum_{s=n-1}^{\infty} 2^{-sr} \ll 2^{-nr}. \end{aligned} \tag{15}$$

Case $p=\infty$. The quantity I_3 can be estimated as follows

$$\begin{aligned} I_3 \leq \sum_{s=n-1}^{\infty} \|\sigma_s\|_1 \left\| \sum_{s'=s-1}^{s+1} \sigma_{s'}(f) \right\|_\infty &\ll \sum_{s=n-1}^{\infty} \sum_{s'=s-1}^{s+1} \left\| \sigma_{s'}(f) \right\|_\infty \\ &\ll \sum_{s=n-1}^{\infty} \sum_{s'=s-1}^{s+1} 2^{-s'r} \ll \sum_{s=n-1}^{\infty} 2^{-sr} \ll 2^{-nr}. \end{aligned} \tag{16}$$

Hence, combining the relations (14), (15) and (16), we get the required upper estimate for the quantity $E_n(\mathbf{B}^r_{p,\theta})_{B_{p,1}}$, $p \in \{1,\infty\}$. The lower estimate in (13) follows from Theorem C and the relation $\|\cdot\|_{B_{q,1}} \gg \|\cdot\|_q$. Theorem 2 is proved. □

We now formulate several corollaries that concern other approximation characteristics. The first of them deals with the approximation of functions from the classes $\mathbf{B}^r_{p,\theta}$ in the space $B_{q,1}$ by the de la Vallee-Poussin sums.

Corollary 2. *Let* $1 \leq p,q,\theta \leq \infty$ *and* $r > d\left(\frac{1}{p}-\frac{1}{q}\right)_+$. *Then the relation*

$$\sup_{f \in \mathbf{B}^r_{p,\theta}} \|f - V_{2^{n-1}}(f)\|_{B_{q,1}} \asymp 2^{-nd\left(\frac{r}{d}-\left(\frac{1}{p}-\frac{1}{q}\right)_+\right)} \tag{17}$$

holds.

Proof. The upper estimate in (17) was obtained during proving Theorems 1 and 2. The corresponding lower estimate follows from the inequality

$$\sup_{f \in \mathbf{B}^r_{p,\theta}} \|f - V_{2^{n-1}}(f)\|_{B_{q,1}} \geq E_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}},$$

and the relation (13). □

To formulate the next corollary, let us define the corresponding approximation characteristics of nonlinear approximation. Let Θ_m be a set of m arbitrary d-dimensional vectors with integer coordinates and $P(\Theta_m) := P(\Theta_m, x) = \sum_{k\in\Theta_m} c_k e^{i(k,x)}$, $(k,x) = k_1x_1 + \cdots + k_dx_d$, be trigonometric polynomials with frequencies in the set Θ_m. For $f \in X$, we consider the quantity

$$e_m(f)_X := \inf_{c_k} \inf_{\Theta_m} \|f - P(\Theta_m)\|_X,$$

which is called the best m-term trigonometric approximation of f. For a class $F \subset X$, we set

$$e_m(F)_X := \sup_{f\in F} e_m(f)_X.$$

The quantity $e_m(f)_2 := e_m(f)_{L_2(T)}$ for functions of a single variable was introduced by S.B. Stechkin [29] in order to formulate a criterion of absolute convergence of orthogonal series in the general case of approximations by polynomials from an arbitrary orthogonal system in a Hilbert space.

Further the quantities $e_m(F)_X$ for certain functional classes and spaces $X = L_q(T^d)$, $d \geq 1$, were studied by many authors. The detailed overview can be found in the monographs [5,20,32,34,36].

Note, that from the definitions of the quantities $e_m(F)_X$ and $e^\perp_m(F)_X$ one gets the following relation

$$e_m(F)_X \leq e^\perp_m(F)_X.$$

Now we formulate the known result that will be used in further speculations.

Theorem D ([4])**.** *Let* $1 \le p, q, \theta \le \infty$. *Define*

$$r(p,q) := \begin{cases} d\left(\frac{1}{p} - \frac{1}{q}\right)_+, & 1 \le p \le q \le 2 \text{ and } 1 \le q \le p \le \infty, \\ \max\left\{\frac{d}{p}; \frac{d}{2}\right\}, & \text{otherwise.} \end{cases} \tag{18}$$

Then for $r > r(p,q)$, *we have*

$$e_m(\mathbf{B}^r_{p,\theta})_q \asymp m^{-\frac{r}{d} + \left(\frac{1}{p} - \max\left\{\frac{1}{q}; \frac{1}{2}\right\}\right)_+}.$$

Corollary 3. *Let* $1 \le p \le q \le 2$ *or* $1 \le q \le p \le \infty$. *Then for* $r > d\left(\frac{1}{p} - \frac{1}{q}\right)_+$, $1 \le \theta \le \infty$, *the relation*

$$e_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp m^{-\frac{r}{d} + \left(\frac{1}{p} - \frac{1}{q}\right)_+} \tag{19}$$

holds.

Proof. The upper estimate in (19) follows from Theorem 2 under the condition $m \asymp 2^{nd}$. The lower estimate is a corollary from Theorem D and the relation $\|\cdot\|_q \ll \|\cdot\|_{B_{q,1}}$. $\square$

Remark 4. *Comparing the results of Theorem 2 and Corollary 3 in the cases* $1 \le p \le q \le 2$ *and* $1 \le q \le p \le \infty$ *as* $m \asymp 2^{nd}$, *we verify that the following relation is true*

$$E_n(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp e_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}}.$$

To conclude, let us formulate the estimates of some widths of the classes $\mathbf{B}^r_{p,\theta}$ in the spaces $B_{q,1}$, $1 \le p, q, \theta \le \infty$. Let us define now the corresponding approximation characteristics. Let Y be a normed space with the norm $\|\cdot\|_Y$, $\mathscr{L}_m(Y)$ be a set of the subspaces in the space Y of dimension at most m and W be a centrally-symmetric set in Y. The quantity

$$d_m(W,Y) := \inf_{L_m \in \mathscr{L}_m(Y)} \sup_{w \in W} \inf_{u \in L_m} \|w - u\|_Y$$

is called the Kolmogorov m-width of the set W in the space Y. The width $d_m(W,Y)$ was introduced in 1936 by A.N. Kolmogorov [12]. Let Y and Z be normed spaces and $L(Y,Z)$ be a set of linear continuous mappings of Y to Z. The quantity

$$\lambda_m(W,Y) := \inf_{\substack{L_m \in \mathscr{L}_m(Y) \\ \Lambda \in L(Y, L_m)}} \sup_{w \in W} \|w - \Lambda w\|_Y,$$

where the lower bound is taken over the all subsets L_m in $\mathscr{L}_m(Y)$ of dimension at most m and all linear continuous operators that map Y into L_m, is called the linear m-width of the set W in the space Y. The width $\lambda_m(W,Y)$ was introduced in 1960 by V.M. Tikhomirov [35]. In what follows, we define the approximation characteristics that was introduced by R.S. Ismagilov [9]. Let $Y = L_q(T^d)$ or $Y = B_{q,1}$, $1 \le q \le \infty$, and $F \subset Y$ be a functional class.

Trigonometric m-width of the class F in the space Y (the notation $d^{\top}_m(F,Y)$) is defined by the formula

$$d^{\top}_m(F,Y) := \inf_{\{k^j\}_{j=1}^m} \sup_{f \in F} \inf_{\{c_j\}_{j=1}^m} \left\| f(\cdot) - \sum_{j=1}^m c_j e^{i(k^j, \cdot)} \right\|_Y,$$

where $\{k^j\}_{j=1}^m$ is a set of vectors $k^j = (k_1^j, \dots, k_d^j)$, $j = \overline{1,m}$, from the integer grid $\mathbb{Z}^d$, c_j are arbitrary complex numbers. According to the introduced definitions of widths, the following relations hold:

$$\begin{aligned} d_m(F,Y) &\le \lambda_m(F,Y), \\ d_m(F,Y) &\le d_m^{\top}(F,Y). \end{aligned} \tag{20}$$

The history of investigation of widths for different functional classes of periodic multivariate functions is described in the monographs [5,20,32,34,36]. Now we formulate a corollary from obtained in the paper and earlier known results.

Corollary 4. *Let* $1 \le \theta \le \infty$, $1 \le p \le q \le 2$ *or* $1 \le q \le p \le \infty$. *Then for* $r > d\left(\frac{1}{p} - \frac{1}{q}\right)_+$ *the relations*

$$d_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \asymp \lambda_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \asymp d_m^{\top}(\mathbf{B}^r_{p,\theta}, B_{q,1}) \asymp m^{-\frac{r}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+} \tag{21}$$

hold.

Proof. The upper estimates for each of the considered in (21) widths follow from Theorem 2 under the condition that the numbers n and m are connected by the relation $2^{nd} \asymp m$. As to the lower estimates in (21), we note that due to (20) it is sufficient to prove the corresponding lower estimate for the Kolmogorov width $d_m(\mathbf{B}^r_{p,\theta}, B_{q,1})$. For this reason we use the known statement.

Theorem E ([19]). *Let* $1 \le p, q, \theta \le \infty$ *and* $r(p,q)$ *is defined by* (18). *Then for* $r > r(p,q)$ *we have*

$$d_m(\mathbf{B}^r_{p,\theta}, L_q) \asymp m^{-\frac{r}{d}+\left(\frac{1}{p}-\max\left\{\frac{1}{q};\frac{1}{2}\right\}\right)_+}. \tag{22}$$

Hence, from (22) and the relation $\|\cdot\|_q \ll \|\cdot\|_{B_{q,1}}$, the estimate

$$d_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \gg d_m(\mathbf{B}^r_{p,\theta}, L_q) \asymp m^{-\frac{r}{d}+\left(\frac{1}{p}-\frac{1}{q}\right)_+}$$

follows. ☐

Remark 5. *Estimates of widths considered in* (21) *are realized by approximation of the classes* $\mathbf{B}^r_{p,\theta}$ *by the subspace of trigonometric polynomials with "numbers" of harmonics from the set* $C^d(2^n)$ *under the condition* $m \asymp 2^{nd}$. *Note, that in the case* $d = 1$ *and* $(p,q) \in \{(1,1),(\infty,\infty)\}$ *the estimates* (21) *were obtained in the papers [24] and [22] for* $p = \infty$ *and* $p = 1$, *respectively.*

In conclusion, we consider relations between the parameters p and q for which the estimates of the approximative characteristics from Corollaries 3 and 4 are not realized by the approximations by trigonometric polynomials from the set $T(C^d(2^n))$. First, we formulate a needed in the following speculations auxiliary statement.

Lemma A ([36, Ch. 10, § 10.2]). *Let* $2 \le q < \infty$. *Then for any trigonometric polynomial*

$$P(\Theta_m) := P(\Theta_m, x) = \sum_{j=1}^{m} e^{i(k^j,x)}$$

and any $n \le m$ *there exists a trigonometric polynomial* $\tilde{P}(\Theta_n) := \tilde{P}(\Theta_n, x)$ *that consists of at most* n *harmonics, and a constant* $C(q) > 0$, *such that the inequality*

$$\|P(\Theta_m) - \tilde{P}(\Theta_n)\|_q \le C(q) m n^{-\frac{1}{2}}$$

holds. Besides, $\Theta_n \subset \Theta_m$ *and all of the coefficients of the polynomial* $\tilde{P}(\Theta_n)$ *coincide and do not exceed* mn^{-1} *in their absolute value.*

Theorem 3. *Let $1 \le p < 2 \le q < \frac{p}{p-1}$, $1 \le \theta \le \infty$. Then for $r > d$ the relations*

$$d_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \asymp \lambda_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \asymp d^{\top}_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \asymp m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}} \tag{23}$$

hold.

Proof. Let us first get the upper estimate in (23) for the trigonometric widths $d^{\top}_m(\mathbf{B}^r_{p,\theta}, B_{q,1})$. As it was already indicated, it is sufficient to obtain the required estimate for the classes $\mathbf{H}^r_p$. Let us consider two cases.

Case $1 < p < 2$. We choose a number $l \in \mathbb{N}$ such that $2^{(l-1)d} \le m \le 2^{ld}$, i.e. $m \asymp 2^{ld}$, and set

$$\alpha = \left(\frac{r}{d} - \frac{1}{p} + \frac{1}{2}\right) / \left(\frac{r}{d} - \frac{1}{p} + \frac{1}{q}\right).$$

For $s \in \mathbb{Z}_+$ we define

$$m_s := \begin{cases} 2^{sd}, & 0 \le s < l, \\ \left[2^{lr} 2^{sd\left(1-\frac{r}{d}\right)}\right], & l \le s \le [\alpha l] + 1, \\ 0, & s > [\alpha l] + 1, \end{cases} \tag{24}$$

where $[a]$ is an integer part of the number a.

Then, taking into account $r > d$, we get

$$\begin{aligned} \sum_{s \in \mathbb{Z}_+} m_s \le \sum_{s=0}^{l-1} 2^{sd} + \sum_{s=l}^{[\alpha l]+1} 2^{lr} 2^{sd\left(1-\frac{r}{d}\right)} &\ll 2^{ld} + 2^{lr} \sum_{s=l}^{[\alpha l]+1} 2^{sd\left(1-\frac{r}{d}\right)} \\ &\ll 2^{ld} + 2^{lr} 2^{ld\left(1-\frac{r}{d}\right)} = 2^{ld+1} \asymp m. \end{aligned} \tag{25}$$

Further, for $s \in \mathbb{Z}_+$ we define the set

$$\mu(s) := \left\{k = (k_1, \dots, k_d) \in \mathbb{Z}^d : \quad 2^{s-1} \le \max_{j=\overline{1,d}} |k_j| < 2^s\right\}$$

and consider the trigonometric polynomial $t_s := t_s(x) = \sum_{k \in \mu(s)} e^{i(k,x)}$. Let $t(\Theta_{m_s}) := t(\Theta_{m_s}, x)$ be a trigonometric polynomial that approximates t_s due to Lemma A, i.e.

$$\|t_s - t(\Theta_{m_s})\|_q \ll 2^{sd} m_s^{-\frac{1}{2}}, \quad \text{where} \quad \Theta_{m_s} \subset \mu(s).$$

We will approximate the function $f \in \mathbf{H}^r_p$, $1 < p < 2$, by trigonometric polynomials t of the form

$$t := t(x) = \sum_{s=0}^{l-1} f_{(s)}(x) + \sum_{s=l}^{[\alpha l]+1} \left(f_{(s)}(x) * t(\Theta_{m_s}, x)\right).$$

As it was shown in (25), the number of harmonics of the polynomial t does not exceed m in order. Therefore, we can write

$$\|f - t\|_{B_{q,1}} \le \left\| \sum_{s=l}^{[\alpha l]+1} \left(f_{(s)} - \left(f_{(s)} * t(\Theta_{m_s})\right)\right) \right\|_{B_{q,1}} + \left\| \sum_{s=[\alpha l]+2}^{\infty} f_{(s)} \right\|_{B_{q,1}} = I_4 + I_5. \tag{26}$$

Let us first estimate the term I_4. According to the definition of $\|\cdot\|_{B_{q,1}}$, we get

$$I_4 = \sum_{s=0}^{\infty} \left\| \sigma_s * \sum_{s'=l}^{[\alpha l]+1} \Big(f_{(s')} - \big(f_{(s')} * t(\Theta_{m_{s'}}) \big) \Big) \right\|_q \le \sum_{s=l-1}^{[\alpha l]} \left\| \sigma_s * \sum_{s'=s}^{s+1} \Big(f_{(s')} - \big(f_{(s')} * t(\Theta_{m_{s'}}) \big) \Big) \right\|_q$$
$$\le \sum_{s=l-1}^{[\alpha l]} \|\sigma_s\|_1 \left\| \sum_{s'=s}^{s+1} \Big(f_{(s')} - \big(f_{(s')} * t(\Theta_{m_{s'}}) \big) \Big) \right\|_q \ll \sum_{s=l-1}^{[\alpha l]} \left\| \sum_{s'=s}^{s+1} \Big(f_{(s')} - \big(f_{(s')} * t(\Theta_{m_{s'}}) \big) \Big) \right\|_q \quad (27)$$
$$\ll \sum_{s=l-1}^{[\alpha l]+1} \left\| \Big(f_{(s)} - \big(f_{(s)} * t(\Theta_{m_s}) \big) \Big) \right\|_q = I_6.$$

To further estimate the quantity I_6, we will need one more auxiliary statement. Let T_s be an operator that acts on a function f by the formula

$$T_s f := (T_s f)(x) = f(x) * \Big(t_s(x) - t(\Theta_{m_s}, x) \Big).$$

Lemma B ([23]). *Let $1 < p < 2 < q < \frac{p}{p-1}$. Then for the norm of the operator T_s that goes from L_p to L_q ($\|T_s\|_{p\to q}$), the following estimate*

$$\|T_s\|_{p\to q} = \sup_{\|f\|_p \le 1} \|T_s f\|_q \ll 2^{sd} m_s^{-\left(\frac{1}{2}+\frac{1}{p'}\right)} \qquad (28)$$

holds, where $p' = \frac{p}{p-1}$.

Hence, using the relation (28), we obtain

$$I_6 = \sum_{s=l-1}^{[\alpha l]+1} \left\| T_s f_{(s)} \right\|_q \le \sum_{s=l-1}^{[\alpha l]+1} \|T_s\|_{p\to q} \|f_{(s)}\|_p \ll \sum_{s=l-1}^{[\alpha l]+1} 2^{sd} m_s^{-\left(\frac{1}{2}+\frac{1}{p'}\right)} \|f_{(s)}\|_p. \qquad (29)$$

Substituting in (29) the values of m_s from (24) and making the elementary transformations, we get

$$I_6 \ll \sum_{s=l-1}^{[\alpha l]+1} 2^{sd} 2^{-lr\left(\frac{1}{2}+\frac{1}{p'}\right)} 2^{-sd\left(1-\frac{r}{d}\right)\left(\frac{1}{2}+\frac{1}{p'}\right)} \|f_{(s)}\|_p \ll 2^{-lr\left(\frac{1}{2}+\frac{1}{p'}\right)} \sum_{s=l-1}^{[\alpha l]+1} 2^{sd-sd\left(1-\frac{r}{d}\right)\left(\frac{1}{2}+\frac{1}{p'}\right)-sr}$$
$$= 2^{-lr\left(\frac{1}{2}+\frac{1}{p'}\right)} \sum_{s=l-1}^{[\alpha l]+1} 2^{sd\left(\frac{1}{p}-\frac{1}{2}\right)\left(1-\frac{r}{d}\right)}. \qquad (30)$$

Then, taking into account that $\frac{1}{p} - \frac{1}{2} > 0$ and $1 - \frac{r}{d} < 0$, we continue the estimation of (29), namely

$$I_6 \ll 2^{-lr\left(\frac{1}{2}+\frac{1}{p'}\right)} 2^{ld\left(\frac{1}{p}-\frac{1}{2}\right)\left(1-\frac{r}{d}\right)} = 2^{-ld\left(\frac{r}{d}-\frac{1}{p}+\frac{1}{2}\right)} \asymp m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}}. \qquad (31)$$

Combining (27), (29), (30) and (31), we obtain the required estimate for the quantity I_4 in the case $1 < p < 2$, i.e.

$$I_4 \ll m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}}. \qquad (32)$$

Case $p=1$. Let $q_0 \in (1,2)$ be a number that will be specified later. Then, in view of (2), we proceed the estimation of the quantity I_6 (see (27)) as follows

$$I_6 = \sum_{s=l-1}^{[\alpha l]+1} \left\| T_s f_{(s)} \right\|_q \le \sum_{s=l-1}^{[\alpha l]+1} \|T_s\|_{q_0\to q} \|f_{(s)}\|_{q_0} \ll \sum_{s=l-1}^{[\alpha l]+1} \|T_s\|_{q_0\to q} \|\sigma_s(f)\|_{q_0}$$
$$\ll \sum_{s=l-1}^{[\alpha l]+1} \|T_s\|_{q_0\to q} 2^{sd\left(1-\frac{1}{q_0}\right)} \|\sigma_s(f)\|_1 \ll \sum_{s=l-1}^{[\alpha l]+1} 2^{sd} m_s^{-\left(\frac{1}{2}+\frac{1}{q_0'}\right)} 2^{sd\left(1-\frac{1}{q_0}\right)} 2^{-sr}$$
$$\ll \sum_{s=l-1}^{[\alpha l]+1} 2^{sd} 2^{-lr\left(\frac{1}{2}+\frac{1}{q_0'}\right)} 2^{-sd\left(1-\frac{r}{d}\right)\left(\frac{1}{2}+\frac{1}{q_0'}\right)} 2^{sd\left(1-\frac{1}{q_0}\right)} 2^{-sr}$$
$$= 2^{-lr\left(\frac{1}{2}+\frac{1}{q_0'}\right)} \sum_{s=l-1}^{[\alpha l]+1} 2^{sd - sd\left(1-\frac{r}{d}\right)\left(\frac{1}{2}+\frac{1}{q_0'}\right) + sd\left(1-\frac{1}{q_0}\right) - sr} = 2^{-lr\left(\frac{1}{2}+\frac{1}{q_0'}\right)} \sum_{s=l-1}^{[\alpha l]+1} 2^{\frac{sd}{2} - \frac{sr}{2} + \frac{sr}{q_0'}}.$$

Choosing the number $q_0' > 2$ such that it satisfies the condition $\frac{sd}{2} - \frac{sr}{2} + \frac{sr}{q_0'} < 0$, we finish with estimating the quantity I_6

$$I_6 \ll 2^{-lr\left(\frac{1}{2}+\frac{1}{q_0'}\right)} 2^{\frac{ld}{2} - \frac{lr}{2} + \frac{lr}{q_0'}} = 2^{-ld\left(\frac{r}{d}-\frac{1}{2}\right)} \asymp m^{-\frac{r}{d}+\frac{1}{2}}. \tag{33}$$

Hence, combining (27) with (33), we can write

$$I_4 \ll m^{-\frac{r}{d}+\frac{1}{2}}. \tag{34}$$

Moving to estimation of the term I_5, we note the following. Speculating similarly as it was done during proving the upper estimate in Theorem 1 (see the case $1 \le p < q \le \infty$), we get

$$I_5 = \left\| \sum_{s=[\alpha l]+2}^{\infty} f_{(s)} \right\|_{B_{q,1}} \ll \sum_{s=[\alpha l]+2}^{\infty} 2^{-sd\left(\frac{r}{d}-\frac{1}{p}+\frac{1}{q}\right)} \ll 2^{-\alpha l d\left(\frac{r}{d}-\frac{1}{p}+\frac{1}{q}\right)}. \tag{35}$$

Taking into account the value of α, i.e. putting $\alpha = \left(\frac{r}{d} - \frac{1}{p} + \frac{1}{2}\right) / \left(\frac{r}{d} - \frac{1}{p} + \frac{1}{q}\right)$, from (35) we obtain

$$I_5 \ll 2^{-ld\left(\frac{r}{d}-\frac{1}{p}+\frac{1}{2}\right)} \asymp m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}}. \tag{36}$$

Therefore, combining (26), (32), (34) and (36), we get the upper estimate for the trigonometric widths

$$d_m^{\top}(\mathbf{B}_{p,\theta}^r, B_{q,1}) \ll m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}}. \tag{37}$$

From (37), in view of the inequality (20), we immediately obtain the upper estimate for the Kolmogorov width $d_m(\mathbf{B}_{p,\theta}^r, B_{q,1})$:

$$d_m(\mathbf{B}_{p,\theta}^r, B_{q,1}) \le d_m^{\top}(\mathbf{B}_{p,\theta}^r, B_{q,1}) \asymp m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}}.$$

Concerning the upper estimate for the linear widths $\lambda_m(\mathbf{B}_{p,\theta}^r, B_{q,1})$, we note the following.

When estimating above the upper estimate for the trigonometric width $d_m^{\top}(\mathbf{B}_{p,\theta}^r, B_{q,1})$, we in fact were using the linear operator Λ_m, such that acted on the function $f \in \mathbf{H}_p^r$ by the formula

$$\Lambda_m f := (\Lambda_m f)(x) = \sum_{s=0}^{l-1} f_{(s)}(x) + \sum_{s=l}^{[\alpha l]+1} (t(\Theta_{m_s}, x) * f_{(s)}(x)).$$

Taking this into account, we can state that the upper estimate for the linear widths $\lambda_m(\mathbf{B}^r_{p,\theta}, B_{q,1})$ has the form (37), i.e.

$$\lambda_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \ll m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}}.$$

That is, we proved upper estimates for all of the approximative characteristics considered in Theorem 3. As to the lower estimates in (23), we note that it is sufficient to get the respective estimate for the Kolmogorov width $d_m(\mathbf{B}^r_{p,\theta}, B_{q,1})$. In turn, the required estimate follows from Theorem E and the relation $\|\cdot\|_q \ll \|\cdot\|_{B_{q,1}}$. Theorem 3 is proved. $\square$

Corollary 5. *Let* $1 \leq p < 2 \leq q < \frac{p}{p-1}$, $1 \leq \theta \leq \infty$. *Then for* $r > d$ *the relation*

$$e_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}} \tag{38}$$

holds.

Proof. The upper estimate in (38) follows from (23) and the inequality

$$e_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \leq d^{\top}_m(\mathbf{B}^r_{p,\theta}, B_{q,1}) \asymp m^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{2}}.$$

The corresponding lower estimate is a corollary of the Theorem D and the relation $\|\cdot\|_q \ll \|\cdot\|_{B_{q,1}}$. $\square$

References

[1] Belinsky E.S. *Approximation by a "floating" system of exponentials on classes of periodic functions with bounded mixed derivative*. In: Studies in the Theory of Functions of Several Real Variables, Yaroslav State Univ., Yaroslavl, 1988, 16–33. (in Russian)

[2] Belinsky E.S. *Estimates of entropy numbers and Gaussian measures for classes of functions with bounded mixed derivative*. J. Approx. Theory 1998, **93**, 114–127. doi:10.1006/jath.1997.3157

[3] Besov O.V. *Investigation of a class of function spaces in connection with imbedding and extension theorems*. Tr. Mat. Inst. Steklova 1961, **60**, 42–81. (in Russian)

[4] DeVore R.A., Temlyakov V.N. *Nonlinear approximation by trigonometric sums*. J. Fourier Anal. Appl. 1995, **2** (1), 29–48. doi:10.1007/s00041-001-4021-8

[5] Düng D., Temlyakov V., Ullrich T. Hyperbolic cross approximation. Adv. Courses Math. Birkhauser, CRM Barselona, 2018. doi:10.1007/978-3-319-92240-9

[6] Dung D., Thanh V.Q. *On nonlinear n-widths*. Proc. Amer. Math. Soc. 1996, **124** (9), 2757–2765. doi: 10.1090/S0002-9939-96-03337-0

[7] Fedunyk-Yaremchuk O.V., Hembars'kyi M.V., Hembars'ka S.B. *Approximative characteristics of the Nikol'skii-Besov-type classes of periodic functions in the space* $B_{\infty,1}$. Carpathian Math. Publ. 2020, **12** (2), 376–391. doi:10.15330/cmp.12.2.376-391

[8] Fedunyk-Yaremchuk O.V., Hembars'ka S.B. *Best orthogonal trigonometric approximations of the Nikol'skii-Besov-type classes of periodic functions of one and several variables*. Carpathian Math. Publ. 2022, **14** (1), 171–184. doi:10.15330/cmp.14.1.171-184

[9] Ismagilov R.S. *Widths of sets in normed linear spaces and the approximation of functions by trigonometric polynomials*. Russian Math. Surveys 1974, **29** (3), 169–186. doi:10.1070/RM1974v029n03ABEH001287 (translation of Uspekhi Mat. Nauk 1974, **29** (3(177)), 161–178 (in Russian))

[10] Jiang Y., Yongping L. *Average widths and optimal recovery of multivariate Besov classes in* $L_p(\mathbb{R}^d)$. J. Approx. Theory 2000, **102** (1), 155–170. doi:10.1006/jath.1999.3384

[11] Kashin B.S., Temlyakov V.N. *Best m-term approximations and the entropy of sets in the space* L_1. Math. Notes 1994, **56** (5-6), 1137–1157. doi:10.1007/BF02274662 (translation of Mat. Zametki 1994, **56** (5), 57–86. (in Russian))

[12] Kolmogorov A.N. *Über die beste Annaherung von Funktionen einer gegebenen Funktionklasse*. Ann. of Math.(2) 1936, **37** (1), 107–110. doi:10.2307/1968691

[13] Lizorkin P.I. *Generalized Holder spaces* $B^{(r)}_{p,\theta}$ *and their correlations with the Sobolev spaces* $L^{(r)}_p$. Sibirsk. Mat. Zh. 1968, **9** (5), 1127–1152. (in Russian)

[14] Nikol'skii S.M. *Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables*. Tr. Mat. Inst. Steklova 1951, **38**, 244–278. (in Russian)

[15] Romanyuk A.S. *Approximation of classes of periodic functions in several variables*. Math. Notes 2002, **71** (1), 98–109. doi:10.1023/A:1013982425195 (translation of Mat. Zametki 2002, **71** (1), 109–121. doi:10.4213/mzm332 (in Russian))

[16] Romanyuk A.S. *Bilinear and trigonometric approximations of periodic functions of several variables of Besov classes* $B^r_{p,\theta}$. Izv. Math. 2006, **70** (2), 277–306. doi:10.1070/IM2006v070n02ABEH002313 (translation of Izv. Ross. Akad. Nauk Ser. Mat. 2006, **70** (2), 69–98. doi:10.4213/im558 (in Russian))

[17] Romanyuk A.S. *Approximation of the isotropic classes* $\mathbf{B}^r_{p,\theta}$ *of periodic functions of several variables in the space* L_q. Approx. Theory of Functions and Related Problems: Proc. Inst. Math. NAS Ukr. 2008, **5** (1), 263–278. (in Russian)

[18] Romanyuk A.S. *Approximative characteristics of the isotropic classes of periodic functions of many variables*. Ukrainian Math. J. 2009, **61** (4), 613–626. doi:10.1007/s11253-009-0232-y (translation of Ukraïn. Mat. Zh. 2009, **61** (4), 513–523. (in Russian))

[19] Romanyuk A.S. *Bilinear approximations and Kolmogorov widths of periodic Besov classes*. Theory of Operators, Differential Equations, and the Theory of Functions: Proc. Inst. Math. NAS Ukr. 2009, **6** (1), 222–236. (in Russian)

[20] Romanyuk A.S. Approximate characteristics of classes of periodic functions. Proc. of the Institute of Mathematics of the NAS of Ukraine, Kiev, 2012, **93**. (in Russian)

[21] Romanyuk A.S. *Best trigonometric and bilinear approximations of classes of functions of several variables*. Math. Notes 2013, **94** (3), 379–391. doi:10.1134/S0001434613090095 (translation of Mat. Zametki 2013, **94** (3), 401–415. doi:10.4213/mzm8892 (in Russian))

[22] Romanyuk A.S. *Entropy numbers and widths for the classes* $B^r_{p,\theta}$ *of periodic functions of many variables*. Ukrainian Math. J. 2017, **68** (10), 1620–1636. doi:10.1007/s11253-017-1315-9 (translation of Ukraïn. Mat. Zh. 2016, **68** (10), 1403–1417. (in Russian))

[23] Romanyuk A.S., Romanyuk V.S. *Trigonometric and orthoprojection widths of classes of periodic functions of many variables*. Ukrainian Math. J. 2009, **61** (10), 1589–1609. doi:10.1007/s11253-010-0300-3 (translation of Ukraïn. Mat. Zh. 2009, **61** (10), 1348–1366. (in Ukrainian))

[24] Romanyuk A.S., Romanyuk V.S. *Approximating characteristics of the classes of periodic multivariate functions in the space* $B_{\infty,1}$. Ukrainian Math. J. 2019, **71** (2), 308–321. doi:10.1007/s11253-019-01646-3 (translation of Ukraïn. Mat. Zh. 2019, **71** (2), 271–281. (in Ukrainian))

[25] Romanyuk A.S., Romanyuk V.S. *Estimation of some approximating characteristics of the classes of periodic functions of one and many variables*. Ukrainian Math. J. 2020, **71** (8), 1257–1272. doi:10.1007/s11253-019-01711-x (translation of Ukraïn. Mat. Zh. 2019, **71** (8), 1102–1115 (in Ukrainian))

[26] Romanyuk A.S., Romanyuk V.S. *Approximative characteristics and properties of operators of the best approximation of classes of functions from the Sobolev and Nikol'skii-Besov spaces*. J. Math. Sci. (N.Y.) 2021, **252** (4), 508–525. doi:10.1007/s10958-020-05177-2 (translation of Ukr. Mat. Visn. 2020, **17** (3), 372–395 (in Ukrainian))

[27] Romanyuk A.S., Yanchenko S.Ya. *Approximation of the classes of periodic functions of one and many variables from the Nikol'skii-Besov and Sobolev spaces*. Ukrainian Math. J. 2022, **74** (6), 967–980. doi:10.1007/s11253-022-02110-5 (translation of Ukraïn. Mat. Zh. 2022, **74** (6), 844–855. doi:10.37863/umzh.v74i6.7141 (in Ukrainian))

[28] Stasyuk S.A. *Best m-term trigonometric approximation of periodic functions of several variables from Nikol'skii-Besov classes for small smoothness*. J. Approx. Theory 2014, **177**, 1–16. doi:10.1016/j.jat.2013.09.006

[29] Stechkin S.B. *On absolute convergence of orthogonal series*. Dokl. Akad. Nauk SSSR 1955, **102** (2), 37–40. (in Russian)

[30] Stepanyuk T.A. *Order estimates of best orthogonal trigonometric approximations of classes of infinitely differentiable functions*. In: Raigorodskii A., Rassias, M. (Eds.) Trigonometric Sums and Their Applications. Springer, Cham., 2020, 273–287. doi:10.1007/978-3-030-37904-9_13

[31] Temlyakov V.N. *Estimates of the asymptotic characteristics of classes of functions with bounded mixed derivative or difference*. Proc. Steklov Inst. Math. 1990, **189**, 161–197 (translation of Tr. Mat. Inst. Steklova 1989, **189**, 138–168. (in Russian))

[32] Temlyakov V.N. Approximation of periodic functions. Nova Sci. Publ., New York, 1993.

[33] Temlyakov V.N. *Greedy algorithm and m-term trigonometric approximation*. Constr. Approx. 1998, **14** (4), 569–587. doi:10.1007/s003659900090

[34] Temlyakov V.N. Multivariate approximation. Cambridge University Press, 2018.

[35] Tikhomirov V.M. *Widths of sets in function spaces and the theory of best approximations*. Russian Math. Surveys 1960, **15** (3), 75–111. doi:10.1070/RM1960v015n03ABEH004093 (translation of Uspekhi Mat. Nauk 1960, **15** (3(93)), 81–120. (in Russian))

[36] Trigub R.M., Belinsky E.S. Fourier Analysis and Approximation of Functions. Kluwer Academic Publishers, Dordrecht, 2004.

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Отримано точні за порядком оцінки деяких характеристик лінійної та нелінійної апроксимації ізотропних класів Нікольського-Бєсова $\mathbf{B}^r_{p,\theta}$ періодичних функцій багатьох змінних у просторах $B_{q,1}$, $1 \leq q \leq \infty$. Серед них: найкращі ортогональні тригонометричні наближення, найкращі m-членні тригонометричні наближення, колмогоровські, лінійні та тригонометричні поперечники.

Для всіх розглянутих у роботі характеристик, їхні оцінки співпадають за порядком із відповідними оцінками у просторах L_q. Більше того, отримані точні за порядком оцінки (крім випадку $1 < p < 2 \leq q < \frac{p}{p-1}$) реалізуються за допомогою наближення функцій з класів $\mathbf{B}^r_{p,\theta}$ тригонометричними поліномами зі спектром у кубічних областях. У жодному з випадків, вони не залежать від гладкісного параметра θ.

Ключові слова і фрази: клас Нікольського-Бєсова, найкраще ортогональне тригонометричне наближення, найкраще наближення, поперечник.