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New models for some free algebras of small ranks

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Dimonoids, generalized digroups and doppelsemigroups are algebras defined on a set with two binary associative operations. The notion of a dimonoid was introduced by J.-L. Loday during constructing the universal enveloping algebra for a Leibniz algebra. One of the important motivations for studying doppelsemigroups comes from their connections to interassociative semigroups. Generalized digroups are dimonoids with some additional conditions while commutative dimonoids provide the class of examples of doppelsemigroups.

Let V be a variety of universal algebras. One of the main problems is to describe free objects in V. The purpose of this paper is to construct new more convenient free objects in some varieties of dimonoids, generalized digroups and doppelsemigroups. We first construct a new class of abelian dimonoids, give a new model of the free abelian dimonoid of rank 2 and extend it to the case of an arbitrary rank. Then we show that the semigroups of the free generalized digroup are anti-isomorphic, present a new model of the free monogenic generalized digroup and characterize the least group congruence on it. We also prove that there do not exist commutative generalized digroups with different operations. Finally, we construct a new model of the free monogenic commutative doppelsemigroup, characterize the least semigroup congruence on it and establish that every monogenic abelian doppelsemigroup is the homomorphic image of the free monogenic commutative doppelsemigroup.

Key words and phrases: dimonoid, generalized digroup, doppelsemigroup, free abelian dimonoid of rank 2, free monogenic generalized digroup, free monogenic commutative doppelsemigroup.

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1 Introduction

Lately, Loday-type algebras such as dialgebras, trialgebras, dimonoids, trioids, g-dimonoids, (generalized) digroups and other are actively studied. The idea of doppelsemigroups is based on the study of dimonoids. At the current paper, we turn our attention to investigate dimonoids, generalized digroups and doppelsemigroups. The dimonoid structure is introduced by J.-L. Loday [16] as a non-trivial extension of the concept of a semigroup. These algebras have applications in dialgebra theory (see, e.g., [5, 16, 17]). Digroups and generalized digroups are dimonoids with some additional conditions. They were considered by R. Felipe [8],

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M.K. Kinyon [12], K. Liu [13], O.P. Salazar-Díaz, R. Velásquez and L.A. Wills-Toro [23] and other authors. The first examples of digroups were given by J.-L. Loday [16]. Digroups play an important role in Leibniz algebra theory. This theory was actively studied (see, e.g., [1–4, 12, 16]). The study of doppelsemigroups was initiated by the first author in [29]. Doppelalgebras introduced by B. Richter [20] are linear analogs of doppelsemigroups. It is well-known that there exist natural relationships between doppelsemigroups and interassociative semigroups (see, e.g., [27, 29, 34]). Finally, note that the following relations between dimonoids, digroups, generalized digroups and doppelsemigroups take place: every digroup is a generalized digroup, every commutative dimonoid is a doppelsemigroup [29] and three axioms of a doppelsemigroup appear in axiomatics of a dimonoid and a (generalized) digroup.

Let V be a variety of universal algebras. One of the main problems is to describe free objects in V. The motivation for this research comes from the fact that free objects in any variety of algebras are important in the study of that variety. The above mentioned main problem has been solved in many cases. For example, free objects are known for the varieties of dimonoids, g-dimonoids, digroups, trioids, duplexes, (strong) doppelsemigroups and n-tuple semigroups (see [16, 18, 19, 24, 28, 29, 31, 33]). Abelianity in generalized digroups, doppelsemigroups, digroups and dimonoids was considered in [21, 35, 37, 38], respectively. This property has different meanings in universal algebra, and in the latter four papers, the term "abelian" is not equivalent to the term "commutative". A construction for the free abelian dimonoid was given in [38]. The free generalized digroup was presented in [22]. The paper [29] establishes the structure of the free commutative doppelsemigroup. In the present paper, we suggest new models for some free algebras in the varieties of abelian dimonoids, generalized digroups and commutative doppelsemigroups. The obtained models look more convenient than the corresponding original constructions.

The paper is organized as follows. In the next section we construct a new class of abelian dimonoids, give a new model of the free abelian dimonoid of rank 2 and extend it to the case of an arbitrary rank. Then we show that the semigroups of the free generalized digroup are anti-isomorphic, present a new model of the free monogenic generalized digroup and characterize the least group congruence on it (Section 3). We also prove that there do not exist commutative generalized digroups with different operations. In the final section, we construct a new model of the free monogenic commutative doppelsemigroup and characterize the least semigroup congruence on it. In addition, we establish that every monogenic abelian doppelsemigroup is the homomorphic image of the free monogenic commutative doppelsemigroup.

2 Free abelian dimonoids of rank 2

In this section, we construct a new class of abelian dimonoids, give a new model of the free abelian dimonoid of rank 2 and extend it to the case of an arbitrary rank.

Following J.-L. Loday [16], a dimonoid is a nonempty set D equipped with two binary operations $\dashv$ and $\vdash$ satisfying the axioms

$$
\begin{aligned}
(x \dashv y) \dashv z &= x \dashv (y \dashv z),\\
(x \dashv y) \dashv z &= x \dashv (y \vdash z),\\
(x \vdash y) \dashv z &= x \vdash (y \dashv z),\\
(x \dashv y) \vdash z &= x \vdash (y \vdash z),\\
(x \vdash y) \vdash z &= x \vdash (y \vdash z)
\end{aligned}
$$

for all $x, y, z \in D$. Some examples of dimonoids were given in [16,25]. A collection of relatively free dimonoids can be found in [32]. A dimonoid $(D, \dashv, \vdash)$ is called abelian [38] if

$$x \dashv y = y \vdash x \tag{1}$$

for all $x, y \in D$.

Recall the construction of the free abelian dimonoid suggested in [38].

Let X be an arbitrary nonempty set and let $FCm(X)$ be the free commutative monoid on X with the empty word θ. Put $FAd(X) = X \times FCm(X)$ and define operations $\dashv$ and $\vdash$ on $FAd(X)$ as follows

$$(x, v) \dashv (y, q) = (x, vyq), \quad (x, v) \vdash (y, q) = (y, xvq).$$

Theorem 1 ([38, Theorem 1]). *The algebra $\big(FAd(X), \dashv, \vdash\ \big)$ is the free abelian dimonoid on $X \times \{\theta\}$.*

The dimonoid $(FAd(X), \dashv, \vdash)$ is denoted by $FAd[X]$. The free abelian dimonoid of rank 1 was considered in [38]. In the latter paper it was shown that the operations of the free monogenic abelian dimonoid coincide and it is a variant of the additive semigroup of all non-negative integers. In order to present a new model of the free abelian dimonoid of rank 2 we construct a new class of abelian dimonoids. As usual, we denote by $\mathbb{N}^0$ the set $\mathbb{N}$ of all positive integers with zero. Let A, B be arbitrary nonempty subsets of $\mathbb{N}^0$ and let

$$\varphi : A \to B : x \mapsto \varphi_x, \quad \psi : A \to B : y \mapsto \psi_y$$

be arbitrary maps. Define operations $\dashv$ and $\vdash$ on $\mathbb{N}^0 \times A \times \mathbb{N}^0$ by

$$(n, a, m) \dashv (p, b, s) = (n + \varphi_b + p, a, m + \psi_b + s), \tag{2}$$

$$(n, a, m) \vdash (p, b, s) = (n + \varphi_a + p, b, m + \psi_a + s) \tag{3}$$

for all $(n, a, m), (p, b, s) \in \mathbb{N}^0 \times A \times \mathbb{N}^0$. The algebra $(\mathbb{N}^0 \times A \times \mathbb{N}^0, \dashv, \vdash)$ is denoted by $\mathbb{N}^0(A, B)^{\varphi}_{\psi}$.

Lemma 1. *For any nonempty $A, B \subseteq \mathbb{N}^0$ and any maps φ, ψ from A to B, the algebra $\mathbb{N}^0(A, B)^{\varphi}_{\psi}$ is an abelian dimonoid.*

Proof. It is easy to verify that multiplications in $\mathbb{N}^0(A, B)^{\varphi}_{\psi}$ are associative and it is an abelian dimonoid. $\square$

Now we give a new model of the free abelian dimonoid of rank 2.

Let $A = B = \{0, 1\}$ and

$$\varphi_x = \begin{cases} 1, & x = 0, \\ 0, & x = 1, \end{cases} \qquad \psi_x = \begin{cases} 1, & x = 1, \\ 0, & x = 0. \end{cases} \tag{4}$$

Using (4), define operations $\dashv$ and $\vdash$ on $\mathbb{N}^0 \times \{0, 1\} \times \mathbb{N}^0$ by formulas (2) and (3). Then, by Lemma 1, we get the abelian dimonoid $\mathbb{N}^0\big(\{0, 1\}, \{0, 1\}\big)^{\varphi}_{\psi}$ which denote by FAd_2.

Theorem 2. *Let $X = \{x, y\}$. Then $FAd[X]$ is isomorphic to the dimonoid FAd_2.*

Proof. Words of $FCm(X)$ we write as $x^n y^m$, where $n, m \in \mathbb{N}^0$. Define the map

$$\alpha : FAd[X] \to FAd_2 : (z, x^n y^m) \mapsto (z, x^n y^m)\alpha = \begin{cases} (n, 0, m), & z = x, \\ (n, 1, m), & z = y. \end{cases}$$

We show that α is an isomorphism. Let $n_i, m_i \in \mathbb{N}^0$, where $1 \le i \le 4$. We have

$$\begin{aligned}
\Big((x, x^{n_1}y^{m_1}) \dashv (y, x^{n_2}y^{m_2})\Big)\alpha &= \Big(x, x^{n_1+n_2}y^{m_1+m_2+1}\Big)\,\alpha \\
&= (n_1 + n_2, 0, m_1 + m_2 + 1) = (n_1 + \varphi_1 + n_2, 0, m_1 + \psi_1 + m_2) \\
&= (n_1, 0, m_1) \dashv (n_2, 1, m_2) = (x, x^{n_1}y^{m_1})\alpha \dashv (y, x^{n_2}y^{m_2})\alpha,
\end{aligned}$$

$$\begin{aligned}
\Big((y, x^{n_2}y^{m_2}) \dashv (x, x^{n_1}y^{m_1})\Big)\alpha &= \Big(y, x^{n_2+n_1+1}y^{m_2+m_1}\Big)\,\alpha \\
&= (n_2 + n_1 + 1, 1, m_2 + m_1) = (n_2 + \varphi_0 + n_1, 1, m_2 + \psi_0 + m_1) \\
&= (n_2, 1, m_2) \dashv (n_1, 0, m_1) = (y, x^{n_2}y^{m_2})\alpha \dashv (x, x^{n_1}y^{m_1})\alpha,
\end{aligned}$$

$$\begin{aligned}
\Big((x, x^{n_1}y^{m_1}) * (x, x^{n_3}y^{m_3})\Big)\alpha &= \Big(x, x^{n_1+n_3+1}y^{m_1+m_3}\Big)\,\alpha \\
&= (n_1 + n_3 + 1, 0, m_1 + m_3) = (n_1 + \varphi_0 + n_3, 0, m_1 + \psi_0 + m_3) \\
&= (n_1, 0, m_1) * (n_3, 0, m_3) = (x, x^{n_1}y^{m_1})\alpha * (x, x^{n_3}y^{m_3})\alpha,
\end{aligned}$$

$$\begin{aligned}
\Big((y, x^{n_2}y^{m_2}) * (y, x^{n_4}y^{m_4})\Big)\alpha &= \Big(y, x^{n_2+n_4}y^{m_2+m_4+1}\Big)\,\alpha \\
&= (n_2 + n_4, 1, m_2 + m_4 + 1) = (n_2 + \varphi_1 + n_4, 1, m_2 + \psi_1 + m_4) \\
&= (n_2, 1, m_2) * (n_4, 1, m_4) = (y, x^{n_2}y^{m_2})\alpha * (y, x^{n_4}y^{m_4})\alpha,
\end{aligned}$$

where $* \in \{\dashv, \vdash\}$, and

$$\begin{aligned}
\Big((x, x^{n_1}y^{m_1}) \vdash (y, x^{n_2}y^{m_2})\Big)\alpha &= \Big(y, x^{n_1+n_2+1}y^{m_1+m_2}\Big)\,\alpha \\
&= (n_1 + n_2 + 1, 1, m_1 + m_2) = (n_1 + \varphi_0 + n_2, 1, m_1 + \psi_0 + m_2) \\
&= (n_1, 0, m_1) \vdash (n_2, 1, m_2) = (x, x^{n_1}y^{m_1})\alpha \vdash (y, x^{n_2}y^{m_2})\alpha,
\end{aligned}$$

$$\begin{aligned}
\Big((y, x^{n_2}y^{m_2}) \vdash (x, x^{n_1}y^{m_1})\Big)\alpha &= \Big(x, x^{n_2+n_1}y^{m_2+m_1+1}\Big)\,\alpha \\
&= (n_2 + n_1, 0, m_2 + m_1 + 1) = (n_2 + \varphi_1 + n_1, 0, m_2 + \psi_1 + m_1) \\
&= (n_2, 1, m_2) \vdash (n_1, 0, m_1) = (y, x^{n_2}y^{m_2})\alpha \vdash (x, x^{n_1}y^{m_1})\alpha.
\end{aligned}$$

Moreover, it is clear that α is a bijection. So, α is a dimonoid isomorphism. □

It is natural to extend the dimonoid FAd_2 to the case of an arbitrary rank.

Remark 1. *Let $X = \{a_i \mid i \in I\}$, $\mathbb{N}_i^0 = \mathbb{N}^0$ for all $i \in I$ and*

$$\varphi_i^{a_k} = \begin{cases} 1, & i = k, \\ 0, & i \neq k \end{cases}$$

for any $a_k \in X$, $k \in I$. Define operations $\dashv$ and $\vdash$ on $X \times \prod_{i \in I} \mathbb{N}_i^0$ by

$$\begin{aligned}
\big(a_j, (x_i)_{i\in I}\big) \dashv \big(a_k, (y_i)_{i\in I}\big) &= \big(a_j, (x_i + \varphi_i^{a_k} + y_i)_{i\in I}\big), \\
\big(a_j, (x_i)_{i\in I}\big) \vdash \big(a_k, (y_i)_{i\in I}\big) &= \big(a_k, (x_i + \varphi_i^{a_j} + y_i)_{i\in I}\big)
\end{aligned}$$

for all $(a_j, (x_i)_{i\in I}), (a_k, (y_i)_{i\in I}) \in X \times \prod_{i\in I} \mathbb{N}_i^0$, *where* $j, k \in I$. *An immediate check shows that the map*

$$\tau : (x, v) \mapsto (x, v)\tau = (x, v^*),$$

where the i*th component of the element* v^* *is equal to the number of occurrences of an element* $a_i \in X$ *in* v*, is an isomorphism of* $FAd[X]$ *onto the algebra* $(X \times \prod_{i\in I} \mathbb{N}_i^0, \dashv, \vdash)$.

3 Free monogenic generalized digroups

In this section, we show that the semigroups of the free generalized digroup are anti-isomorphic, give a new model of the free monogenic generalized digroup and present the least group congruence on it. We also prove that there do not exist commutative generalized digroups with different operations.

The notion of a digroup first appeared in J.-L. Loday's work [16] as a dimonoid (see Section 2) satisfying some additional identities. There exist two different definitions of a digroup and examples of such algebras (see [36]). Bar-units are used in the definitions of digroups. Recall that an element e of a dimonoid $(D, \dashv, \vdash)$ is a bar-unit if $x \dashv e = x = e \vdash x$ for every $x \in D$ [16]. Note that in contrast to monoids a dimonoid may possess not only one bar-unit; and the set of all bar-units of a dimonoid $(D, \dashv, \vdash)$, denoted by $E\big((D, \dashv, \vdash)\big)$, is called the halo. If a dimonoid has a unit then its operations coincide, which follows from the axioms of a dimonoid.

Following [9, 12], a dimonoid $(D, \dashv, \vdash)$ is called a digroup if

(G1) there exists a bar-unit $e \in D$;

(G2) for every $g \in D$ there exists a unique element $g^{-1} \in D$ such that $g \vdash g^{-1} = e = g^{-1} \dashv g$. Such an element g^{-1} is said to be inverse to g.

Following [14, 15, 23], a dimonoid $(D, \dashv, \vdash)$ is called a digroup if **(G1)** holds and

(G3) for every $x \in D$ there exist elements $\overset{\ell}{x_e^{-1}}$ and $\overset{r}{x_e^{-1}}$ of D such that $x \vdash \overset{r}{x_e^{-1}} = e = \overset{\ell}{x_e^{-1}} \dashv x$.

In [23], the latter digroups are called generalized digroups. If in **(G3)** $\overset{\ell}{x_e^{-1}} = \overset{r}{x_e^{-1}}$, then we obtain the first definition of a digroup, that is, the class of all digroups is contained in the class of all generalized digroups. Obviously, the nonempty halo of a dimonoid $(D, \dashv, \vdash)$ is an abelian subdimonoid which is a digroup, that is, the digroup $E\big((D, \dashv, \vdash)\big)$ satisfies (1). If a dimonoid is a (generalized) digroup, then its bar-units and the halo are called, respectively, bar-units and the halo of a (generalized) digroup. The problem of adjoining a set of bar-units to dimonoids was studied in [26, 30]. If operations of a (generalized) digroup coincide, then the (generalized) digroup becomes a group. So, (generalized) digroups are a generalization of groups. Recall the construction of the free generalized digroup from [22].

Let X be an arbitrary nonempty set and let $F(X)$ be the free group generated by X. Define operations $\dashv$ and $\vdash$ on $F(X) \times X \times F(X)$ by

$$(u, x, a) \dashv (v, y, b) = (u, x, avyb),$$
$$(u, x, a) \vdash (v, y, b) = (uxav, y, b)$$

for all $(u, x, a), (v, y, b) \in F(X) \times X \times F(X)$. The algebra $\big(F(X) \times X \times F(X), \dashv, \vdash\big)$ is denoted by $FD(X)$. By [22, Proposition 4], $FD(X)$ is the free generalized digroup.

The following statement establishes a relationship between the semigroups of the free generalized digroup.

Lemma 2. *$\big(F(X)\times X\times F(X), \dashv\big)$ and $\big(F(X)\times X\times F(X), \vdash\big)$ are anti-isomorphic semigroups.*

Proof. Define the mapping

$$f:\big(F(X)\times X\times F(X), \dashv\big)\to\big(F(X)\times X\times F(X), \vdash\big)$$

by the formula

$$(x_1\ldots x_i, x_{i+1}, x_{i+2}\ldots x_k)f=(x_k\ldots x_{i+2}, x_{i+1}, x_i\ldots x_1)$$

for all $(x_1\ldots x_i, x_{i+1}, x_{i+2}\ldots x_k)\in F(X)\times X\times F(X)$, where $x_i\in X, 1\le i\le k$.

An immediate verification shows that f is an anti-isomorphism. Indeed, f is a bijection and for all $(x_1\ldots x_i, x_{i+1}, x_{i+2}\ldots x_k), (y_1\ldots y_j, y_{j+1}, y_{j+2}\ldots y_m)\in F(X)\times X\times F(X)$, where $x_i, y_j\in X, 1\le i\le k, 1\le j\le m$, we obtain

$$\begin{aligned}
\big((x_1\ldots x_i, x_{i+1}, x_{i+2}\ldots x_k) &\dashv (y_1\ldots y_j, y_{j+1}, y_{j+2}\ldots y_m)\big)f\\
&=(x_1\ldots x_i, x_{i+1}, x_{i+2}\ldots x_k y_1\ldots y_j\ldots y_m)f\\
&=(y_m\ldots y_j\ldots y_1 x_k\ldots x_{i+2}, x_{i+1}, x_i\ldots x_1)\\
&=(y_m\ldots y_{j+2}, y_{j+1}, y_j\ldots y_1)\vdash(x_k\ldots x_{i+2}, x_{i+1}, x_i\ldots x_1)\\
&=(y_1\ldots y_j, y_{j+1}, y_{j+2}\ldots y_m)f\vdash(x_1\ldots x_i, x_{i+1}, x_{i+2}\ldots x_k)f.
\end{aligned}$$

□

Now we give a new model of the free monogenic generalized digroup. As usual, by Z we denote the set of integers. Define operations $\dashv$ and $\vdash$ on $Z\times Z$ by

$$\begin{aligned}
(n,m)\dashv(p,s)&=(n, m+p+s+1),\\
(n,m)\vdash(p,s)&=(n+m+p+1, s)
\end{aligned}$$

for all $(n,m),(p,s)\in Z\times Z$. The algebra $(Z\times Z, \dashv, \vdash)$ is denoted by FD_1.

The main result of this section is the following theorem.

Theorem 3. *FD_1 is the free monogenic generalized digroup with the halo*

$$E(FD_1)=\big\{(n,m)\mid n+m+1=0\big\}$$

and inverses with respect to the bar unit (n,m)

$$(p,s)^{\overset{\ell}{-1}}_{(n,m)}=(n, m-s-p-1)\quad\text{and}\quad(p,s)^{\overset{r}{-1}}_{(n,m)}=(n-s-p-1, m),\quad\text{where}\quad(p,s)\in FD_1.$$

Proof. Let $X=\{x\}$. Define the map

$$\beta: FD(X)\to FD_1: \big(x^n, x, x^m\big)\mapsto(n,m).$$

Prove that β is an isomorphism. For all $n_1, n_2, m_1, m_2\in Z$ we get

$$\begin{aligned}
\Big(\big(x^{n_1}, x, x^{m_1}\big)\dashv\big(x^{n_2}, x, x^{m_2}\big)\Big)\beta&=\Big(x^{n_1}, x, x^{m_1+n_2+m_2+1}\Big)\beta=(n_1, m_1+n_2+m_2+1)\\
&=(n_1,m_1)\dashv(n_2,m_2)=\big(x^{n_1}, x, x^{m_1}\big)\beta\dashv\big(x^{n_2}, x, x^{m_2}\big)\beta,\\
\Big(\big(x^{n_1}, x, x^{m_1}\big)\vdash\big(x^{n_2}, x, x^{m_2}\big)\Big)\beta&=\Big(x^{n_1+m_1+n_2+1}, x, x^{m_2}\Big)\beta=(n_1+m_1+n_2+1, m_2)\\
&=(n_1,m_1)\vdash(n_2,m_2)=\big(x^{n_1}, x, x^{m_1}\big)\beta\vdash\big(x^{n_2}, x, x^{m_2}\big)\beta.
\end{aligned}$$

Obviously, β is a bijection. Thus, β is an isomorphism and FD_1 is the free monogenic generalized digroup.

Further, by [22, Proposition 4], $E\big(FD(X)\big) = \Big\{\big(x^n, x, x^m\big) \mid n+m+1=0\Big\}$. It is clear that $E\big(FD(X)\big)\beta = \big\{(n,m) \mid n+m+1=0\big\}$. Hence,

$$E\big(FD_1\big) = \big\{(n,m) \mid n+m+1=0\big\}.$$

Moreover, for all $(p,s) \in FD_1$ and $(m,n) \in E(FD_1)$, we have

$$\begin{aligned}(p,s) \vdash (p,s)^{\overset{r}{-1}}_{(n,m)} &= (p,s) \vdash (n-s-p-1, m)\\ &= (n,m) = (n, m-s-p-1) \dashv (p,s) = (p,s)^{\overset{\ell}{-1}}_{(n,m)} \dashv (p,s).\end{aligned}$$

So, FD_1 is the free monogenic generalized digroup with the halo and inverses as in our theorem. □

Corollary 1. *Let $(p,s) \in FD_1$ and $(n,m) \in E(FD_1)$. The inverses $(p,s)^{\overset{\ell}{-1}}_{(n,m)}$ and $(p,s)^{\overset{r}{-1}}_{(n,m)}$ coincide if and only if $(p,s) \in E\big(FD_1\big)$. In this case,*

$$(p,s)^{\overset{\ell}{-1}}_{(n,m)} = (p,s)^{\overset{r}{-1}}_{(n,m)} = (n,m).$$

If ρ is a congruence on a generalized digroup $(D, \dashv, \vdash)$ such that the operations of $(D, \dashv, \vdash)/\rho$ coincide and it is a group, we say that ρ is a group congruence. If $\mu : D_1 \to D_2$ is a homomorphism of generalized digroups, the kernel of μ is denoted by Δ_μ, that is,

$$\Delta_\mu = \big\{(x,y) \in D_1 \times D_1 \,|\, x\mu = y\mu\big\}. \tag{5}$$

Now we present the least group congruence on the free monogenic generalized digroup.

Lemma 3. *The map*

$$\varpi : FD_1 \to (Z, +) : (n,m) \mapsto (n,m)\varpi = n+m+1$$

is an epimorphism inducing the least group congruence on the free monogenic generalized digroup FD_1.

Proof. For arbitrary elements $(n,m), (p,s) \in FD_1$, we have

$$\begin{aligned}\big((n,m) \dashv (p,s)\big)\varpi &= (n, m+p+s+1)\varpi\\ &= n+m+p+s+2 = (n+m+1)+(p+s+1) = (n,m)\varpi + (p,s)\varpi,\end{aligned}$$

$$\begin{aligned}\big((n,m) \vdash (p,s)\big)\varpi &= (n+m+p+1, s)\varpi\\ &= n+m+p+s+2 = (n+m+1)+(p+s+1) = (n,m)\varpi + (p,s)\varpi.\end{aligned}$$

The map ϖ is surjective since for any $m \in Z$ there exists $(m-1, 0) \in FD_1$ such that $(m-1,0)\varpi = m$. Thus, ϖ is an epimorphism. Since $(Z,+)$ is the free group of rank 1, Δ_ϖ is the least group congruence on FD_1. □

We conclude the section with some additional property of generalized digroups. Commutative dimonoids were introduced and studied in [25] (see also [32]). It is natural to introduce the notion of a commutative generalized digroup. A generalized digroup $(D, \dashv, \vdash)$ will be called commutative if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative.

Proposition 1. *There do not exist commutative generalized digroups with different operations.*

Proof. Let $(D, \dashv, \vdash)$ be a commutative generalized digroup and let $e \in D$ be a bar-unit. From [25, Lemma 2] it follows that

$$x \vdash (y \dashv z) = x \vdash (y \vdash z)$$

for all $x, y, z \in D$. If $x = e$, then from the latter equality we obtain $y \dashv z = y \vdash z$. $\square$

4 Free monogenic commutative doppelsemigroups

In this section, we construct a new model of the free monogenic commutative doppelsemigroup and characterize the least semigroup congruence on it. Moreover, we establish that every monogenic abelian doppelsemigroup is the homomorphic image of the free monogenic commutative doppelsemigroup.

Recall that a doppelsemigroup [29] is a nonempty set D with two binary operations $\dashv$ and $\vdash$ satisfying the axioms

$$(x \dashv y) \vdash z = x \dashv (y \vdash z),$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(x \dashv y) \dashv z = x \dashv (y \dashv z),$$
$$(x \vdash y) \vdash z = x \vdash (y \vdash z)$$

for all $x, y, z \in D$.

One of the important motivations for studying doppelsemigroups comes from their connections to interassociative semigroups. Recall that two semigroups defined on the same set are interassociative (see, e.g., [6, 7, 10]) provided that they satisfy the first two axioms of a doppelsemigroup. This definition implies that two interassociative semigroups give rise to a doppelsemigroup, and therefore, we can study interassociative semigroups via doppelsemigroups, applying methods of universal algebra. Commutative dimonoids are examples of doppelsemigroups [25, 29]. Some other examples of doppelsemigroups can be found in [29, 31].

A doppelsemigroup $(D, \dashv, \vdash)$ is called commutative [29] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. Let S be a semigroup and $a \in S$. Define a new operation $\circ_a$ on S by $x \circ_a y = xay$ for all $x, y \in S$. Then $\circ_a$ is associative and hence $(S, \circ_a)$ is a semigroup (see, e.g., [11]).

Let T^* be the free commutative monoid on the two-element set $\{a, b\}$.

Lemma 4 ([29, Corollary 4.4]). $(T^*, \circ_a, \circ_b)$ *is the free commutative doppelsemigroup of rank* 1.

As in Section 2, $\mathbb{N}^0$ denotes the set $\mathbb{N}$ of all positive integers with zero. Define operations $\dashv$ and $\vdash$ on $\mathbb{N}^0 \times \mathbb{N}^0$ by

$$(n, m) \dashv (p, s) = (n + p + 1, m + s),$$
$$(n, m) \vdash (p, s) = (n + p, m + s + 1)$$

for all $(n, m), (p, s) \in \mathbb{N}^0 \times \mathbb{N}^0$. The algebra $\left(\mathbb{N}^0 \times \mathbb{N}^0, \dashv, \vdash\right)$ is denoted by FCD_1.

If ρ is a congruence on a doppelsemigroup $(D, \dashv, \vdash)$ such that the operations of $(D, \dashv, \vdash)/\rho$ coincide, we say that ρ is a semigroup congruence. If $\mu : D_1 \to D_2$ is a homomorphism of doppelsemigroups, the kernel Δ_μ of μ is defined by (5).

Define a relation ϵ on FCD_1 by $(n, m)\epsilon(p, s)$ if and only if $n + m = p + s$.

We are ready to formulate the main result of this section.

Theorem 4. *The relation ϵ is the least semigroup congruence on the free monogenic commutative doppelsemigroup FCD_1.*

Proof. First we show that $(T^*, \circ_a, \circ_b)$ is isomorphic to the algebra FCD_1. Words of T^* we write as $a^n b^m$, where $n, m \in \mathbb{N}^0$. Define the map

$$\gamma : (T^*, \circ_a, \circ_b) \to FCD_1 : a^n b^m \mapsto (n, m).$$

For all $n_1, n_2, m_1, m_2 \in \mathbb{N}^0$ we get

$$\begin{aligned}\big((a^{n_1}b^{m_1}) \circ_a (a^{n_2}b^{m_2})\big)\gamma &= \Big(a^{n_1+n_2+1}b^{m_1+m_2}\Big)\gamma = (n_1+n_2+1, m_1+m_2)\\ &= (n_1, m_1) \dashv (n_2, m_2) = \big(a^{n_1}b^{m_1}\big)\gamma \dashv \big(a^{n_2}b^{m_2}\big)\gamma,\\ \big((a^{n_1}b^{m_1}) \circ_b (a^{n_2}b^{m_2})\big)\gamma &= \Big(a^{n_1+n_2}b^{m_1+m_2+1}\Big)\gamma = (n_1+n_2, m_1+m_2+1)\\ &= (n_1, m_1) \vdash (n_2, m_2) = \big(a^{n_1}b^{m_1}\big)\gamma \vdash \big(a^{n_2}b^{m_2}\big)\gamma.\end{aligned}$$

Clearly, γ is a bijection. Thus, γ is an isomorphism and FCD_1 is the free monogenic commutative doppelsemigroup.

Further consider the map

$$\varpi : FCD_1 \to (\mathbb{N}, +) : (n, m) \mapsto (n, m)\varpi = n + m + 1.$$

A direct verification shows that ϖ is an epimorphism. Indeed, for $(n, m), (p, s) \in FCD_1$, we have

$$\begin{aligned}\big((n, m) \dashv (p, s)\big)\varpi &= (n+p+1, m+s)\varpi = n+p+m+s+2\\ &= (n+m+1) + (p+s+1) = (n, m)\varpi + (p, s)\varpi,\\ \big((n, m) \vdash (p, s)\big)\varpi &= (n+p, m+s+1)\varpi = n+p+m+s+2\\ &= (n+m+1) + (p+s+1) = (n, m)\varpi + (p, s)\varpi,\end{aligned}$$

and $(k-1, 0)\varpi = k$ for any $k \in \mathbb{N}$.

Since $(\mathbb{N}, +)$ is the free semigroup of rank 1, Δ_ϖ is the least semigroup congruence on FCD_1. From the definition of ϖ it follows that $\Delta_\varpi = \epsilon$. □

Recall that a doppelsemigroup $(D, \dashv, \vdash)$ is called abelian [35] if it satisfies the identity (1). Note that the variety of abelian doppelsemigroups does not coincide with the variety of commutative doppelsemigroups.

Corollary 2. *Every monogenic abelian doppelsemigroup is the homomorphic image of the free monogenic commutative doppelsemigroup FCD_1.*

Proof. In [35, Corollary 4.2], it is stated that the operations of a singly generated free abelian doppelsemigroup coincide and it is the additive semigroup $(\mathbb{N}^0, +)$ of positive integers. By the proof of Theorem 4,

$$\varpi_1 : FCD_1 \to (\mathbb{N}^0, +) : (n, m) \mapsto (n, m)\varpi_2 = n + m + 1$$

is a homomorphism. In addition, there exists a homomorphism from $(\mathbb{N}^0, +)$ to an arbitrary monogenic abelian doppelsemigroup E which we denote by ϖ_2. Obviously, the composition $\varpi_1 \circ \varpi_2$ of homomorphisms ϖ_1 and ϖ_2 is a homomorphism from FCD_1 to E. □

Remark 2. *The models FAd_2, FD_1 and FCD_1 look more convenient than the corresponding original constructions since their operations $\dashv$ and $\vdash$ reduce to the addition of non-negative integers.*

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Дімоноїди, узагальнені дігрупи та допельнапівгрупи є алгебрами, визначеними на множині з двома бінарними асоціативними операціями. Поняття дімоноїда було введено Ж.-Л. Лоде під час побудови універсальної обгортуючої алгебри для алгебри Лейбніца. Одна з важливих мотивацій для вивчення допельнапівгруп випливає з їх зв'язків з інтерасоціативними напівгрупами. Узагальнені дігрупи є дімоноїдами з деякими додатковими умовами, в той час як комутативні дімоноїди забезпечують клас прикладів допельнапівгруп.

Нехай V — многовид універсальних алгебр. Однією з основних проблем є опис вільних об'єктів у V. Метою цієї статті є побудова нових більш зручних вільних об'єктів у деяких многовидах дімоноїдів, узагальнених дігруп та допельнапівгруп. Спочатку побудовано новий клас абелевих дімоноїдів, наведено нову модель вільного абелевого дімоноїда рангу 2 та поширено його на випадок довільного рангу. Потім показано, що напівгрупи вільної узагальненої дігрупи є антиізоморфними, представлено нову модель вільної моногенної узагальненої дігрупи та охарактеризовано найменшу групову конгруенцію на ній. Також доведено, що не існує комутативних узагальнених дігруп з різними операціями. Нарешті, побудовано нову модель вільної моногенної комутативної допельнапівгрупи, охарактеризовано найменшу напівгрупову конгруенцію на ній та встановлено, що кожна моногенна абелева допельнапівгрупа є гомоморфним образом вільної моногенної комутативної допельнапівгрупи.

Ключові слова і фрази: дімоноїд, узагальнена дігрупа, допельнапівгрупа, вільний абелевий дімоноїд рангу 2, вільна моногенна узагальнена дігрупа, вільна моногенна комутативна допельнапівгрупа.