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Some fixed point theorems for expansiveness of orthogonal p-contractiveness

Gungor N.B.

Orthogonal set and orthogonal metric spaces are two new notions, which are defined in 2017. In this type metric spaces, a generalization of Banach fixed point theorem is presented. Then in 2019, new fixed point theorems are investigated by using altering distance functions. In this paper, fixed point theorems for expansiveness of orthogonal p-contractiveness via altering distance functions are given inspired by [Rhoades B.E. *Some theorems on weakly contractive maps*. Nonlinear Anal. 2001, **47** (4), 2683–2693] and [Gordji M.E., Rameani M., De La Sen M., Cho Y.J. *On orthogonal sets and Banach fixed point theorem*. Fixed Point Theory 2017, **18** (2), 569–578]. Further, consequences and a restrictive example are offered.

Key words and phrases: fixed point, altering distance function, orthogonal metric space.

Department of Mathematics, Faculty of Science and Arts, Amasya University, 05000, Amasya, Türkiye
E-mail: nurcan.bilgili@amasya.edu.tr, bilgilinurcan@gmail.com

1 Introduction and preliminaries

An important theorem, which is known as Banach contraction principle, is proved by S. Banach in 1922. This principle has been accepted as starting of the fixed point theory in metric spaces.

Some generalizations of Banach contraction principle have been studied on complete metric (see [5, 6, 13, 14, 20]). These studies were developed using two techniques. The first technique is to change the contraction conditions of the mappings and the second technique is to replace the studied metric space with another one.

Considering the first technique, the research of the metric fixed point theory to a new category by presenting a control function is given by M.S. Khan et. al. [15] in 1984.

Definition 1 ([15])**.** *If $\varsigma : [0, \infty) \to [0, \infty)$ is a function, which satisfies the following conditions*

(i) $\varsigma(s)$ is nondecreasing and continuous,

(ii) $s = 0 \iff \varsigma(s) = 0$,

then ς is named altering distance function.

The whole family of altering distance functions will be denoted by Δ.

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Theorem 1 ([15]). *Let (W,τ) be a complete metric space, ς be an altering distance function and $\triangleleft : W \to W$ be a self mapping satisfying the inequality*

$$\varsigma(\tau(\triangleleft\mu, \triangleleft\eta)) \leq \beta\varsigma(\tau(\mu,\eta))$$

for all $\mu, \eta \in W$ and for some $\beta \in (0,1)$. Then $\triangleleft$ has a unique fixed point.

After then, this type functions have been used in a lot of papers in metric fixed point theory (see [4,17,22,23]).

In 1997, the subject of weak contractions is presented by Y.I. Alber and S. Guerre-Delabriere [3], which is an another extension of the contraction principle. Also, B.E. Rhoades [21] enlarged this notion to metric spaces in 2001.

Definition 2 ([21]). *Let (W,τ) be a metric space, ς be an altering distance function and $\triangleleft : W \to W$ be a self mapping satisfying*

$$\tau(\triangleleft\mu, \triangleleft\eta) \leq \tau(\mu,\eta) - \varsigma(\tau(\mu,\eta)),$$

where $\mu, \eta \in W$. Then $\triangleleft$ is said to be weakly contractive mapping.

Theorem 2 ([21]). *Let (W,τ) be a complete metric space and $\triangleleft : W \to W$ be a weakly contractive mapping. Then $\triangleleft$ has a unique fixed point.*

As one of the results in the second technique, M.E. Gordji et. al. [8] defined the subject of an orthogonal set and orthogonal metric spaces. After that, M.E. Gordji and H. Habibi [7] noticed a new subject of generalized orthogonal metric space and they applied the obtained results to show presence and uniqueness of solution of Cauchy problem for the first order differential equation.

Very recently, on orthogonal metric space, N.B. Gungor and D. Turkoglu [12] noticed some fixed point theorems via altering distance functions inspired by [15] and [8]. In 2022, presence and uniqueness of fixed points of the generalizations of contraction principle via auxiliary functions are proved and the homotopy application of the one of the corollaries is given by N.B. Gungor [10].

In recent years, many research articles have presented fixed point theorems and their applications in orthogonal metric spaces (see [1,2,9,11,16,18,19,24,26]).

In this research paper, some fixed points theorems for the generalizations of contraction principle via auxiliary functions are proved. Also, consequences and an illustrative example are presented.

Let $\mathbb{R}, \mathbb{R}^+, \mathbb{Z}$ denote real numbers, positive real numbers and integers, respectively.

Definition 3 ([8]). *Let W be a non-empty set, $\perp$ be a binary relation defined on W. If binary relation $\perp$ fulfils the following condition*

$$\exists\, \mu_0 \in W : \quad (\forall \eta \in W \;\; \eta \perp \mu_0) \quad or \quad (\forall \eta \in W \;\; \mu_0 \perp \eta), \tag{1}$$

then $(W,\perp)$ known as an orthogonal set (an O-set, for short). And μ_0 is named an orthogonal element.

Example 1 ([7]). *Let $W = \mathbb{Z}$. Define $x \perp y$ if there exists $a \in \mathbb{Z}$ such that $x = ay$. One can see that $0 \perp y$ for all $y \in \mathbb{Z}$. Thus, $(W,\perp)$ is an O-set.*

Definition 4 ([8]). *If the following criteria*

$$(\forall n \in \mathbb{N} \ \ \eta_n \perp \eta_{n+1}) \quad or \quad (\forall n \in \mathbb{N} \ \ \eta_{n+1} \perp \eta_n) \tag{2}$$

is satisfied, then the sequence $\{\eta_n\}$ is called orthogonal sequence.

Similarly, if the criteria (2) is satisfied, then a Cauchy sequence $\{\eta_n\}$ is called to be an orthogonally Cauchy sequence.

Definition 5 ([8]). *Let $(W, \perp)$ be an orthogonal set, τ be a usual metric on W. In this case $(W, \perp, \tau)$ is called an orthogonal metric space.*

Definition 6 ([8]). *Let $(W, \perp, \tau)$ be an orthogonal metric space. If every orthogonally Cauchy sequence converges in W, then $(W, \perp, \tau)$ is called to be a complete orthogonal metric space.*

Definition 7 ([8]). *Let $(W, \perp, \tau)$ be an orthogonal metric space and $\lhd : W \to W$ be a function. If for each orthogonal sequence $\{\eta_n\}$ converging to η we have $\lhd\eta_n \to \lhd\eta$ as $n \to \infty$, then $\lhd$ is named to be orthogonally continuous at η.*

If $\lhd$ is orthogonally continuous in each $\eta \in W$, then $\lhd$ is orthogonally continuous on W.

Definition 8 ([8]). *Let $(W, \perp, \tau)$ be an orthogonal metric space and $\lambda \in \mathbb{R}$, $0 < \lambda < 1$. A function $\lhd : W \to W$ is called to be orthogonal contraction with Lipschitz constant λ if*

$$\tau(\lhd\mu, \lhd\eta) \leq \lambda\tau(\mu, \eta)$$

for all $\mu, \eta \in W$ whenever $\mu \perp \eta$.

Definition 9 ([8]). *Let $(W, \perp, \tau)$ be an orthogonal metric space. A function $\lhd : W \to W$ is named orthogonal preserving if $\lhd\mu \perp \lhd\eta$ whenever $\mu \perp \eta$.*

Remark 1. *In [7], it is shown that the orthogonal continuity and orthogonal contraction notations are weaker than the classical continuity and contraction notations in classical metric spaces.*

Theorem 3 ([8]). *Let $(W, \perp, \tau)$ be an orthogonal complete metric space and $\lhd : W \to W$ be orthogonal continuous, orthogonal contraction (with Lipschitz constant λ, $0 < \lambda < 1$) and orthogonal preserving. Afterwards $\lhd$ has a unique fixed point $\eta^* \in W$ and $\lim_{n\to\infty} \lhd^n(\eta) = \eta^*$ for all $\eta \in W$.*

In 2022, presence and uniqueness of fixed points of the generalizations of contraction principle via auxiliary functions are proved and the homotopy application of the one of the corollaries is given by N.B. Gungor.

In [12], N.B. Gungor and D. Turkoglu presented the remarkable fixed point theorems on orthogonal metric spaces via altering distance functions. And then, in [10], N.B. Gungor proved presence and uniqueness of fixed points of the generalizations of contraction principle via auxiliary functions and gived the homotopy application of the one of the corollaries.

In 2018, T. Senapati et. al. [25] defined the orthogonal lower semi continuity and introduced the concept w-distance in orthogonal metric space. Also they proved a fixed point theorem which is the version of Banach fixed point theorem in orthogonal metric spaces owing to the concept of w-distance.

Definition 10 ([25]). *Let $(W, \perp, \tau)$ be an orthogonal metric space. A function $\lhd : W \to [0, \infty]$ is said to be orthogonal lower semi continuous at η if for every orthogonal sequence $\{\eta_n\}$ converging to η, we have*

$$\liminf_{n\to\infty} \lhd(\eta_n) \geq \lhd(\eta).$$

Remark 2. *The authors of [25] gave an examples, which show orthogonal lower semi continuity is weaker than orthogonal continuity and lower semi continuity.*

Definition 11 ([25]). *Let $(W, \perp, \tau)$ be an orthogonal metric space. A function $p : W \times W \to [0, \infty)$ is said to be w-distance function on W if*

$(p1)$ $p(\mu, \eta) \leq p(\mu, \theta) + p(\theta, \eta)$ *for any* $\mu, \theta, \eta \in W$,

$(p2)$ $p(\mu, \cdot) : W \times W \to [0, \infty)$ *is orthogonal lower semi-continuous for any* $\mu \in W$,

$(p3)$ *for any* $\epsilon > 0$ *there exists* $\gamma > 0$ *such that* $p(\mu, \eta) \leq \gamma$ *and* $p(\theta, \eta) \leq \gamma$ *imply* $d(\mu, \theta) \leq \epsilon$.

Lemma 1 ([25]). *Let $(W, \perp, \tau)$ be an orthogonal metric space and $p : W \times W \to [0, \infty)$ be a w-distance. Suppose $\{\mu_n\}$ and $\{\eta_n\}$ are two orthogonal sequences in W and $\mu, \eta, \theta \in W$. Let $\{u_n\}$ and $\{v_n\}$ be sequences of positive real numbers converging to 0. Then we have the followings.*

(i) *If* $p(\mu_n, \eta) \leq u_n$ *and* $p(\mu_n, \theta) \leq v_n$ *then* $\eta = \theta$. *Moreover, if* $p(\mu, \eta) = 0$ *and* $p(\mu, \theta) = 0$, *then* $\eta = \theta$.

(ii) *If* $p(\mu_n, \eta_n) \leq u_n$ *and* $p(\mu_n, \theta) \leq v_n$, *then* $\eta_n \to \theta$ *as* $n \to \infty$.

(iii) *If* $p(\mu_n, \mu_m) \leq u_n$ *for all* $m > n$, *then* $\{\mu_n\}$ *is an orthogonal Cauchy sequence in* W.

(iv) *If* $p(\mu_n, \eta) \leq u_n$, *then* $\{\mu_n\}$ *is an orthogonal Cauchy sequence in* W.

Definition 12 ([25]). *Let $(W, \perp, \tau)$ be an orthogonal metric space and $p : W \times W \to [0, \infty)$ be a w-distance. A mapping $\lhd : W \to W$ is said to be an orthogonal p-contraction if there exists a $\lambda \in [0, 1)$ such that*

$$p(\lhd\mu, \lhd\eta) \leq \lambda p(\mu, \eta)$$

for all $\mu, \eta \in W$ with $\mu \perp \eta$.

Remark 3. *In [25], a remarkable example is given, which shows that orthogonal p-contraction need not to be an orthogonal contraction.*

Theorem 4 ([25]). *Let $(W, \perp, \tau)$ be an orthogonal complete metric space with a w-distance p. If $\lhd$ is an orthogonal p-contractive, orthogonal preserving and orthogonal continuous self mapping, then*

(a) $\lhd$ *has a unique fixed point* $\mu^* \in W$,

(b) *the Picard sequence* $\lhd^n(\mu)$ *converges to* $\mu^* \in W$ *for every* $\mu \in W$.

2 Main results

Theorem 5. *Let $(W, \perp, \tau)$ be an orthogonal complete metric space equipped with a w-distance p, $\lhd : W \to W$ be a self map, $\varsigma, \flat \in \Delta$, $\perp$ is transitive binary relation. Suppose that $\lhd$ is orthogonal preserving self mapping satisfying the inequality*

$$\varsigma(p(\lhd\mu, \lhd\eta)) \leq \varsigma(M(\mu,\eta)) - \flat(M(\mu,\eta)) \tag{3}$$

for all orthogonally related $\mu, \eta \in W$, where

$$M(\mu,\eta) = \max\Big\{p(\mu,\eta), \min\{p(\mu,\lhd\mu), p(\eta,\lhd\eta), p(\lhd\mu,\mu), p(\lhd\eta,\eta)\}\Big\}.$$

In this case, there exists a point $\mu^* \in W$ such that for any orthogonal element $\mu_0 \in W$, the iteration sequence $\{\lhd^n(\mu_0)\}$ converges to this point. Also, if $\lhd$ is orthogonal continuous at $\mu^* \in W$, then $\mu^* \in W$ is a fixed point of $\lhd$.

Proof. Because $(W, \perp)$ is an orthogonal set, condition (1) is fulfilled. Since $\lhd$ is a self mapping on W, for any orthogonal element $\mu_0 \in W$, $\mu_1 \in W$ can be choosen as $\mu_1 = \lhd\mu_0$. Thus,

$$\mu_0 \perp \lhd\mu_0 \vee \lhd\mu_0 \perp \mu_0 \Rightarrow \mu_0 \perp \mu_1 \vee \mu_1 \perp \mu_0.$$

Then, if it is similarly proceeded

$$\mu_1 = \lhd\mu_0, \mu_2 = \lhd\mu_1 = \lhd^2\mu_0, \ldots, \mu_n = \lhd\mu_{n-1} = \lhd^n\mu_0,$$

so $\{\lhd^n\mu_0\}$ is an iteration sequence. □

If for any $n \in \mathbb{N}$ we have $\mu_n = \mu_{n+1}$, then $\mu_n = \lhd\mu_n$ and so $\lhd$ has a fixed point.

Assume that $\mu_n \neq \mu_{n+1}$ for all $n \in \mathbb{N}$. Since $\lhd$ is orthogonal preserving, $\{\lhd^n\mu_0\}$ is an orthogonal sequence. Now, we proceed to show that

$$\lim_{n\to\infty} p(\lhd^n \mu_0, \lhd^{n+1}\mu_0) = 0. \tag{4}$$

By using the inequality (3), we have

$$\begin{aligned}
\varsigma\Big(p(\lhd^n \mu_0, \lhd^{n+1}\mu_0)\Big) &= \varsigma\Big(p(\lhd \mu_{n-1}, \lhd\mu_n)\Big)\\
&\leq \varsigma\Big(\max\Big\{p(\mu_{n-1},\mu_n), \min\{p(\mu_{n-1},\mu_n), p(\mu_n,\mu_{n+1}), p(\mu_n,\mu_{n-1}), p(\mu_{n+1},\mu_n)\}\Big\}\Big)\\
&\quad - \flat\Big(\max\Big\{p(\mu_{n-1},\mu_n), \min\{p(\mu_{n-1},\mu_n), p(\mu_n,\mu_{n+1}), p(\mu_n,\mu_{n-1}), p(\mu_{n+1},\mu_n)\}\Big\}\Big),
\end{aligned}$$

so that

$$\varsigma\Big(p(\lhd^n \mu_0, \lhd^{n+1}\mu_0)\Big) \leq \varsigma\Big(p(\lhd^{n-1} \mu_0, \lhd^n\mu_0)\Big) - \flat\Big(p(\lhd^{n-1} \mu_0, \lhd^n\mu_0)\Big) \tag{5}$$

for any $n \in \mathbb{N}$. Also

$$\varsigma\Big(p(\lhd^n \mu_0, \lhd^{n+1}\mu_0)\Big) \leq \varsigma\Big(p(\lhd^{n-1} \mu_0, \lhd^n\mu_0)\Big) - \flat\Big(p(\lhd^{n-1} \mu_0, \lhd^n\mu_0)\Big) \leq \varsigma\Big(p(\lhd^{n-1} \mu_0, \lhd^n\mu_0)\Big).$$

Therefore, owing to the monocity of ς, the inequality $p(\lhd^n\mu_0, \lhd^{n+1}\mu_0) \leq p(\lhd^{n-1}\mu_0, \lhd^n\mu_0)$ is obtained for all $n \in \mathbb{N}$. Thus, for the nonnegative decreasing sequence $\{p(\lhd^n\mu_0, \lhd^{n+1}\mu_0)\}$ there exists some $r \geq 0$ such that

$$\lim_{n\to\infty} p(\lhd^n \mu_0, \lhd^{n+1}\mu_0) = r. \tag{6}$$

Assume that $r > 0$. On letting $n \to \infty$ in (5) besides using (6), we get

$$\varsigma(r) \leq \varsigma(r) - \flat(r),$$

which amounts to say that $\flat(r) = 0$. As, $\flat$ is an altering distance function, $r = 0$, which is a contradiction to nonzeroness of r yielding thereby

$$\lim_{n\to\infty} p\big(\lhd^n \mu_0, \lhd^{n+1}\mu_0\big) = 0$$

is obtained. Similarly, one can also show that

$$\lim_{n\to\infty} p\big(\lhd^{n+1} \mu_0, \lhd^{n}\mu_0\big) = 0. \tag{7}$$

Now, we continue to show

$$\lim_{n,m\to\infty} p\big(\lhd^n \mu_0, \lhd^{m}\mu_0\big) = 0. \tag{8}$$

Suppose (7) is untrue. Then we can find a $\partial > 0$ with sequences $\{m_s\}$ and $\{n_s\}$ such that

$$p\big(\lhd^{n_s} \mu_0, \lhd^{m_s}\mu_0\big) \geq \partial \quad \text{for all} \quad s \in \{1, 2, \dots\} \tag{9}$$

wherein $m_s > n_s$. By (4) there exists $s_0 \in \mathbb{N}$ such that $n_s > s_0$ implies

$$p\big(\lhd^{n_s} \mu_0, \lhd^{n_{s+1}}\mu_0\big) < \partial. \tag{10}$$

Notice that in view of (9) and (10), $m_s \neq n_{s+1}$. We can assume that m_s is minimum index such that (9) holds, so that

$$p\big(\lhd^{n_s} \mu_0, \lhd^{r}\mu_0\big) \geq \partial \quad \text{for} \quad r \in \{n_{s+1}, n_{s+2}, \dots, m_s - 1\},$$

which in view of (9) gives rise to

$$\begin{aligned} 0 < \partial &\leq p\big(\lhd^{n_s} \mu_0, \lhd^{m_s}\mu_0\big) \\ &\leq p\big(\lhd^{n_s} \mu_0, \lhd^{m_s-1}\mu_0\big) + p\big(\lhd^{m_s-1} \mu_0, \lhd^{m_s}\mu_0\big) < \partial + p\big(\lhd^{m_s-1} \mu_0, \lhd^{m_s}\mu_0\big), \end{aligned}$$

so that

$$\lim_{s\to\infty} p\big(\lhd^{n_s} \mu_0, \lhd^{m_s}\mu_0\big) = \partial.$$

Next, we show that

$$\limsup_{s\to\infty} p\big(\lhd^{n_s+1} \mu_0, \lhd^{m_s+1}\mu_0\big) = \epsilon < \partial.$$

If $\limsup_{s\to\infty} p\left(\lhd^{n_s+1}\mu_0, \lhd^{m_s+1}\mu_0\right) = \epsilon \geq \partial$, then there exists $\{s_r\}$ such that

$$\limsup_{r\to\infty} p\big(\lhd^{n_{s_r}+1} \mu_0, \lhd^{m_{s_r}+1}\mu_0\big) = \epsilon \geq \partial.$$

Since ς is continuous, nondecreasing mapping and also $\lhd^{n_{s_r}}\mu_0 \perp \lhd^{m_{s_r}}\mu_0$, on using inequality (3), one gets

$$\varsigma\Big(p\big(\lhd^{n_{s_r}+1} \mu_0, \lhd^{m_{s_r}+1}\mu_0\big)\Big) \leq \varsigma\Big(M\big(\lhd^{n_s} \mu_0, \lhd^{m_s}\mu_0\big)\Big) - \flat\Big(M\big(\lhd^{n_s} \mu_0, \lhd^{m_s}\mu_0\big)\Big) \tag{11}$$

with

$$M\big(\lhd^{n_s} k\mu_0, \lhd^{m_s}\mu_0\big) = \max\Big\{p\left(\lhd^{n_s}\mu_0, \lhd^{m_s}\mu_0\right), \min\big\{p\big(\lhd^{n_s}\mu_0, \lhd^{n_s+1}\mu_0\big), p\big(\lhd^{m_s}\mu_0, \lhd^{m_s+1}\mu_0\big), p\big(\lhd^{n_s+1}\mu_0, \lhd^{n_s}\mu_0\big), p\big(\lhd^{m_s+1}\mu_0, \lhd^{m_s}\mu_0\big)\big\}\Big\},$$

implying thereby

$$\lim_{s\to\infty} M\big(\lhd^{n_s}\mu_0, \lhd^{m_s}\mu_0\big) = \max\{0, \partial\} = \partial. \tag{12}$$

Letting $s \to \infty$ in (11) and using (12), we get

$$\varsigma(\partial) \le \varsigma(\epsilon) \le \varsigma(\partial) - \flat(\partial) \le \varsigma(\partial),$$

so that $\flat(\partial) = 0$ implying thereby $\partial = 0$, which is a contradiction. Hence,

$$\limsup_{s\to\infty} p\big(\lhd^{n_s+1}\mu_0, \lhd^{m_s+1}\mu_0\big) < \partial$$

and we have

$$\begin{aligned}0 < \partial &\le p\big(\lhd^{n_s}\mu_0, \lhd^{m_s}\mu_0\big)\\ &\le p\big(\lhd^{n_s}\mu_0, \lhd^{n_s+1}\mu_0\big) + p\big(\lhd^{n_s+1}\mu_0, \lhd^{m_s+1}\mu_0\big) + p\big(\lhd^{m_s+1}\mu_0, \lhd^{m_s}\mu_0\big).\end{aligned}$$

Therefore, owing to (4) and (7), we have

$$\begin{aligned}0 < \partial &\le \lim_{s\to\infty} p\big(\lhd^{n_s}\mu_0, \lhd^{m_s}\mu_0\big)\\ &\le \lim_{s\to\infty} p\big(\lhd^{n_s}\mu_0, \lhd^{n_s+1}\mu_0\big) + \limsup_{s\to\infty} p\big(\lhd^{n_s+1}\mu_0, \lhd^{m_s+1}\mu_0\big) + \lim_{s\to\infty} p\big(\lhd^{m_s+1}\mu_0, \lhd^{m_s}\mu_0\big)\\ &= \limsup_{s\to\infty} p\big(\lhd^{n_s+1}\mu_0, \lhd^{m_s+1}\mu_0\big) < \partial,\end{aligned}$$

which is a contradiction. Hence (8) holds. Owing to Lemma 1, $\{\lhd^n\mu_0\}$ is an orthogonal Cauchy sequence in W. Since Ω is an orthogonal complete metric space, there exists μ^* such that $\lim_{n\to\infty} \lhd^n\mu_0 = \mu^*$. Now, assume that $\lhd$ is an orthogonal continuous at μ^*. In this case, $\lim_{n\to\infty} \lhd^{n+1}\mu_0 = \lhd\mu^*$. By using orthogonal lower semi-continuity of p, we have

$$\begin{aligned}p\big(\lhd^n\mu_0, \mu^*\big) &\le \liminf_{m\to\infty} p\big(\lhd^n\mu_0, \lhd^m\mu_0\big) = \alpha_n,\\ p\big(\lhd^n\mu_0, \lhd\mu^*\big) &\le \liminf_{m\to\infty} p\big(\lhd^n\mu_0, \lhd^{m+1}\mu_0\big) = \beta_n.\end{aligned}$$

On using (8), we have $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$. And then, in view of Lemma 1, μ^* is a fixed point of $\lhd$.

Lemma 2. *Let $(W, \perp, d)$ be an orthogonal complete metric space equipped with a w-distance p, $\lhd : W \to W$ be a self map, $\varsigma, \flat \in \Delta$. Suppose that*

$$\varsigma\big(p(\lhd\mu, \lhd\eta)\big) \le \varsigma\big(M(\mu,\eta)\big) - \flat\big(M(\mu,\eta)\big)$$

for all orthogonally related $\mu, \eta \in W$, where

$$M(\mu,\eta) = \max\Big\{p(\mu,\eta), \min\big\{p(\mu, h\mu), p(\eta, \lhd\eta), p(\lhd\mu, \mu), p(\lhd\eta, \eta)\big\}\Big\}.$$

If there exists a point $\mu^* \in W$ that is a fixed point of $\lhd$ and $\mu^* \perp \mu^*$, then $p(\mu^*, \mu^*) = 0$.

Proof. On the contrary suppose that $p(\mu^*,\mu^*)\neq 0$. As $\mu^*\perp\mu^*$ and

$$M(\mu^*,\mu^*)=\max\left\{p(\mu^*,\mu^*),\min\left\{p(\mu^*,\lhd\mu^*),p(\mu^*,\lhd\mu^*),p(\lhd\mu^*,\mu^*),p(\lhd\mu^*,\mu^*)\right\}\right\}$$
$$=p(\mu^*,\mu^*).$$

Therefore,

$$\varsigma(p(\lhd\mu^*,\lhd\mu^*))\leq\varsigma(M(\mu^*,\mu^*))-\flat(M(\mu^*,\mu^*))$$

and so

$$\varsigma(p(\mu^*,\mu^*))\leq\varsigma(p(\mu^*,\mu^*))-\flat(p(\mu^*,\mu^*))\leq\varsigma(p(\mu^*,\mu^*)),$$

which amounts to say that $\flat(p(\mu^*,\mu^*))=0$. Since $\flat$ is an altering distance function, we obtain $p(\mu^*,\mu^*)=0$. □

Theorem 6. *If the following $(*)$ condition is added to hypotheses of the Theorem 5, the fixed point of $\lhd$ turns out to be unique. Moreover, $\lim\limits_{n\to\infty}\lhd^n(\mu)=\mu^*$ for every $\mu\in W$ provided, then μ^* is a fixed point of $\lhd$, i.e. the map $\lhd:W\to W$ is a Picard operator.*

$(*)$ *If there exists a point $\mu^*\in W$ such that for any orthogonal element $\mu_0\in W$, the iteration sequence $\{\lhd^n(\mu_0)\}$ converges to this point, then $\mu^*\perp\mu^*$.*

Proof. Following the proof of Theorem 5, there exists a point $\mu^*\in W$ such that for any orthogonal element $\mu_0\in W$, the iteration sequence $\{\lhd^n(\mu_0)\}$ converges to this point. Also, if $\lhd$ is orthogonal continuous at $\mu^*\in W$, then $\mu^*\in W$ is a fixed point of $\lhd$.

Suppose μ^* and η^* are two fixed points of $\lhd$ in W determined in this shape. Then using the condition $(*)$, one get $\mu^*\perp\mu^*$ and $\eta^*\perp\eta^*$. In this case, using the Lemma 2, $p(\mu^*,\mu^*)=0$ and $p(\eta^*,\eta^*)=0$ are obtained. Now, we have two cases.

Case I. If $\mu^*\perp\eta^*$ or $\eta^*\perp\mu^*$, owing to the condition (3) of Theorem 5, we have

$$\varsigma(p(\lhd\mu^*,\lhd\eta^*))\leq\varsigma(M(\mu^*,\eta^*))-\flat(M(\mu^*,\eta^*)).$$

As

$$M(\mu^*,\eta^*)=\max\left\{p(\mu^*,\eta^*),\min\left\{p(\mu^*,\lhd\mu^*),p(\eta^*,\lhd\eta^*),p(\lhd\mu^*,\mu^*),p(\lhd\eta^*,\eta^*)\right\}\right\}$$
$$=p(\mu^*,\eta^*),$$

therefore

$$\varsigma(p(\lhd\mu^*,\lhd\eta^*))\leq\varsigma(p(\mu^*,\eta^*))-\flat(p(\mu^*,\eta^*)),$$

which amounts to say that $\flat(p(\mu^*,\eta^*))=0$. As $\flat$ is an altering distance function, therefore for every $n\in\mathbb{N}$, we have $p(\mu^*,\eta^*)=0$. Also, in view of Lemma 2, we get $p(\mu^*,\mu^*)=0$ and by using Lemma 1, we have $\mu^*=\eta^*$, i.e. the fixed point of $\lhd$ is unique.

Case II. If μ^* and η^* are not orthogonally related elements, then because of $(W,\perp)$ is an orthogonal set, we have

$$\exists\,\mu_0\in W\quad(\mu^*\perp\mu_0\ \text{ and }\ \eta^*\perp\mu_0)\quad\text{or}\quad(\mu_0\perp\mu^*\ \text{ and }\ \mu_0\perp\eta^*).$$

Since $\lhd$ is orthogonal preserving self mapping, we get

$$(\mu^*\perp\lhd^n(\mu_0)\ \text{ and }\ \eta^*\perp\lhd^n(\mu_0))\quad\text{or}\quad(\lhd^n(\mu_0)\perp\mu^*\ \text{ and }\ \lhd^n(\mu_0)\perp\eta^*)$$

for any $n \in \mathbb{N}$ and henceforth

$$\varsigma(p(\lhd^n \mu_0, \mu^*)) = \varsigma(p(\lhd^n \mu_0, h\mu^*)) \leq \varsigma(M(\lhd^{n-1} \mu_0, \mu^*)) - \flat(M(\lhd^{n-1} \mu_0, \mu^*))$$

with

$$\begin{aligned} M\left(\lhd^{n-1}\mu_0, \mu^*\right) &= \max\Big\{p(\lhd^{n-1} \mu_0, \mu^*), \min\{p(\lhd^{n-1} \mu_0, \lhd^n\mu_0), p(\mu^*, \lhd\mu^*), \\ &\qquad p(\lhd^n \mu_0, \lhd^{n-1}\mu_0), p(\lhd \mu^*, \mu^*)\}\Big\} \\ &= p(\lhd^{n-1} \mu_0, \mu^*). \end{aligned}$$

Therefore

$$\begin{aligned} \varsigma(p(\lhd^n \mu_0, \mu^*)) &= \varsigma(p(\lhd^n \mu_0, \lhd\mu^*)) \\ &\leq \varsigma(p(\lhd^{n-1} \mu_0, \mu^*)) - \flat(p(\lhd^{n-1} \mu_0, \mu^*)) \leq \varsigma(p(\lhd^{n-1} \mu_0, \mu^*)). \end{aligned}$$

Since ς is a nondecreasing function, we get $p\left(\lhd^n\mu_0, \mu^*\right) \leq p\left(\lhd^{n-1}\mu_0, \mu^*\right)$, i.e. the non-negative sequence $\left\{p\left(\lhd^n\mu_0, \mu^*\right)\right\}$ is decreasing. As earlier, we have

$$\lim_{n\to\infty} p\left(\lhd^n\mu_0, \mu^*\right) = 0.$$

Also, since μ_0 and η^* are orthogonally related elements, therefore proceeding as earlier, we can prove that

$$\lim_{n\to\infty} p\left(\lhd^n\mu_0, \eta^*\right) = 0.$$

And so, from Lemma 1, we infer that $\eta^* = \mu^*$, i.e. the fixed point of $\lhd$ is unique.

Now, we proceed to show

$$\lim_{n\to\infty} \lhd^n\mu = \mu^*$$

for every $\mu \in W$ provided μ^* is a fixed point of $\lhd$. We distinguish two cases.

Case (i). Let $\mu \in W$, μ^* and μ are orthogonally related elements. As earlier, we have

$$\lim_{n\to\infty} p\left(\lhd^n\mu^*, \lhd^n\mu\right) = 0.$$

Also, in view of Lemma 2, we have $\lim_{n\to\infty} p\left(\lhd^n\mu^*, \mu^*\right) = 0$ and by using Lemma 1, we get

$$\lim_{n\to\infty} \lhd^n\mu = \mu^*.$$

Case (ii). Let $\mu \in W$, μ^* and μ are not orthogonally related elements. Then because of $(W, \perp)$ is an orthogonal set, we have

$$\exists\, \mu_0 \in W \quad (\mu_0 \perp \mu^* \text{ and } \mu_0 \perp \mu) \quad \text{or} \quad (\mu^* \perp \mu_0 \text{ and } \mu \perp \mu_0).$$

As earlier, we can prove $\lim_{n\to\infty} p\left(\lhd^n\mu_0, \mu^*\right) = 0$ and $\lim_{n\to\infty} p\left(\mu^*, \lhd^n\mu_0\right) = 0$. By the triangular inequality, we obtain

$$p\left(\lhd^n\mu_0, \lhd^n\mu_0\right) \leq p\left(\lhd^n\mu_0, \mu^*\right) + p\left(\mu^*, \lhd^n\mu_0\right).$$

Then one get

$$\lim_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu_0\right) = 0.$$

Since μ and μ_0 are orthogonally related elements, due to orthogonally preserving property of $\lhd$, we can see, that $\lhd\mu$ and $\lhd\mu_0$ are orthogonally related elements. Continuing this process inductively, we get $\lhd^n\mu$ and $\lhd^n\mu_0$ are orthogonally related elements. Now, we proceeed to show that

$$\liminf_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu\right) = 0.$$

Suppose $\liminf_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu\right) = \gamma > 0$. Since $\lim_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu_0\right) = 0$, for arbitrary δ, $0 < \delta < \gamma$, there exists $n_1 \in \mathbb{N}$ such that for every $n > n_1$ we have $p\left(\lhd^n\mu_0, \lhd^n\mu_0\right) < \delta$.

Also, since $\liminf_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu\right) = \gamma > \delta > 0$, there exists $n_2 \in \mathbb{N}$ such that for every $n > n_2$, we have $p\left(\lhd^n\mu_0, \lhd^n\mu\right) > \delta$. Therefore, for every $n > N = \max\{n_1, n_2\}$, we get

$$\begin{aligned} M\big(\lhd^{n-1}\mu_0, \lhd^{n-1}\mu\big) &= \max\Big\{p\big(\lhd^{n-1}\mu_0, \lhd^{n-1}\mu\big), \min\big\{p\big(\lhd^{n-1}\mu_0, \lhd^{n}\mu_0\big), p\big(\lhd^{n-1}\mu, \lhd^{n}\mu\big), \\ &\qquad p\big(\lhd^{n}\mu_0, \lhd^{n-1}\mu_0\big), p\big(\lhd^{n}\mu, \lhd^{n-1}\mu\big)\big\}\Big\} \\ &= p\big(\lhd^{n-1}\mu_0, \lhd^{n-1}\mu\big). \end{aligned}$$

Therefore, as ς is an altering distance function, we get that the nonnegative sequence $p\left(\lhd^n\mu_0, \lhd^n\mu\right)$ is decreasing. As earlier, we can prove $\lim_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu\right) = 0$, which is indeed a contradiction to nonzeroness of γ, implying thereby

$$\liminf_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu\right) = 0.$$

Also, since μ^* and μ_0 are orthogonally related elements, using the arguments of the earlier case, we can prove

$$\lim_{n\to\infty} p\left(\lhd^n\mu_0, \lhd^n\mu\right) = 0,$$

and by orthogonally lower semi-continuity of $p\left(\lhd^n\mu_0, .\right)$, we have

$$p\big(\lhd^n\mu_0, \lim_{m\to\infty}\lhd^m\mu\big) \le \liminf_{m\to\infty} p\big(\lhd^n\mu_0, \lhd^m\mu\big) = \alpha_n,$$

and

$$p\big(\lhd^n\mu_0, \mu^*\big) \le \liminf_{m\to\infty} p\big(\lhd^n\mu_0, \lhd^m\mu\big) = \beta_n.$$

As $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$, thus, in view of Lemma 1, we conclude that

$$\lim_{n\to\infty} \lhd^n\mu = \mu^*.$$

This completes the proof. □

Example 2. *Let $W = [0,1)$ be a set. Define $d : W \times W \to \mathbb{R}$ by $d(\mu,\eta) = |\mu - \eta|$ and define $p : W \times W \to [0,\infty)$ by $p(\mu,\eta) = \mu + \eta$. Also, let the binary relation $\perp$ on W be defined by $\mu \perp \eta \iff \mu\eta \le \max\left\{\frac{\mu}{5}, \frac{\eta}{5}\right\}$. Then $(W,\perp)$ is an orthogonal set and d is a metric on W. $(W,\perp,d)$ is an orthogonal metric space with w-distance p. In this space, any orthogonal Cauchy sequence is convergent.*

Indeed, if (μ_n) is an arbitrary orthogonal Cauchy sequence in W, then $\mu_n\mu_{n+1} \le \frac{\mu_n}{5}$ or $\mu_n\mu_{n+1} \le \frac{\mu_{n+1}}{5}$. Therefore

$$\mu_n\left(\mu_{n+1} - \frac{1}{5}\right) \le 0 \quad \text{or} \quad \mu_{n+1}\left(\mu_n - \frac{1}{5}\right) \le 0,$$

which implies

$$\left(\mu_n = 0 \quad or \quad \mu_{n+1} \leq \frac{1}{5}\right) \quad or \quad \left(\mu_{n+1} = 0 \quad or \quad \mu_n \leq \frac{1}{5}\right).$$

Therefore, for any $\delta > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > n_0$, $|\mu_n - \mu_{n+1}| < \delta$ is provided. So, for any $\delta > 0$ and for all $n \in \mathbb{N}$, $n > n_0$, we have $|\mu_n - 0| < \delta$, that is $\{\mu_n\}$ is convergent to $0 \in W$. So $(W, \perp, d)$ is an orthogonal complete metric space with a w-distance p. Remark that (W, d) is not a complete sub-metric space of $(\mathbb{R}, d)$, because W is not a closed subset of $(\mathbb{R}, d)$.

Let $\varsigma : [0, \infty) \to [0, \infty)$ be defined by $\varsigma(t) = \frac{t}{3}$ and let $\flat : [0, \infty) \to [0, \infty)$ be defined by $\flat(t) = \frac{t}{7}$. Also let $\lhd : W \to W$ be defined by

$$\lhd(\mu) = \begin{cases} \frac{\mu}{5}, & 0 \leq \mu \leq \frac{1}{5}, \\ 0, & \frac{1}{5} < \mu < 1. \end{cases}$$

One can see that $\varsigma, \flat \in \Delta$ and $\perp$ is a transitive binary relation on W.

Also $\lhd$ is orthogonal preserving mapping. Indeed,

$$\mu \perp \eta \;\; \Rightarrow \;\; \left(\mu\eta \leq \frac{\mu}{5}\right) \text{ or } \left(\mu\eta \leq \frac{\eta}{5}\right).$$

Without loss of generality, assume that $\mu\eta \leq \frac{\mu}{5}$. In this case, $\mu = 0$ or $\eta \leq \frac{1}{5}$. And so the following cases can be seen:

I: $\mu = 0$ and $\eta \leq \frac{1}{5}$; then $\lhd(\mu) = 0$ and $\lhd(\eta) = \frac{\eta}{5}$,

II: $\mu = 0$ and $\eta > \frac{1}{5}$; then $\lhd(\mu) = \lhd(\eta) = 0$,

III: $\eta \leq \frac{1}{5}$ and $\mu \leq \frac{1}{5}$; then $\lhd(\eta) = \frac{\eta}{5}$ and $\lhd(\mu) = \frac{\mu}{5}$,

IV: $\eta \leq \frac{1}{5}$ and $\mu > \frac{1}{5}$; then $\lhd(\eta) = \frac{\eta}{5}$ and $\lhd(\mu) = 0$.

These cases imply that $\lhd(\mu) \lhd (\eta) \leq \frac{\lhd(\mu)}{5}$.

On the other hand, $\lhd$ is orthogonal continuous at $0 \in W$. Indeed, assume that $\{\mu_n\}$ is an orthogonal sequence and $\mu_n \to 0$. In this case, we have $\mu_n\mu_{n+1} \leq \frac{\mu_n}{5}$ or $\mu_n\mu_{n+1} \leq \frac{\mu_{n+1}}{5}$. From this we obtain

$$\left(\mu_n = 0 \quad or \quad \mu_{n+1} \leq \frac{1}{5}\right) \quad or \quad \left(\mu_{n+1} = 0 \quad or \quad \mu_n \leq \frac{1}{5}\right).$$

Therefore, for any $\delta > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > n_0$, the inequality $|\mu_n - 0| < \delta$ is obtained. So, for all $n \in \mathbb{N}$, $n > n_0$, we have $\mu_n \in [0, \frac{1}{5}]$. Thus, from the definition of $\lhd$, for the same $n_0 \in \mathbb{N}$, $n > n_0$, we get $| \lhd (\mu_n) - \lhd(0)| < \delta$, that is $\lhd(\mu_n) \to \lhd(0) = 0$.

Now, it can be shown that h is a self mapping satisfying the inequality (3) for all orthogonally related $\mu, \eta \in W$, where

$$M(\mu, \eta) = \max\Big\{p(\mu, \eta), \min\{p(\mu, \lhd\mu), p(\eta, \lhd\eta), p(\lhd\mu, \mu), p(\lhd\eta, \eta)\}\Big\}.$$

Assume that $\mu, \eta \in W$ are two orthogonally related elements of W. In this case, we have

$$\mu\eta \leq \frac{\mu}{5} \quad or \quad \mu\eta \leq \frac{\eta}{5}.$$

Without loss of generality, assume that $\mu\eta \leq \frac{\mu}{5}$.

Then there exist the following four cases.

Case I: $\mu = 0$ *and* $\eta \leq \frac{1}{5}$. *Then* $\lhd(\mu) = 0$, $\lhd(\eta) = \frac{\eta}{5}$. *Clearly,* $\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{\lhd\mu + \lhd\eta}{3} = \frac{\eta}{15}$ *and*

$$\begin{aligned} M(\mu,\eta) &= \max\Big\{p(\mu,\eta), \min\{p(\mu,\lhd\mu), p(\eta,\lhd\eta), p(\lhd\mu,\mu), p(\lhd\eta,\eta)\}\Big\} \\ &= \max\left\{\eta, \min\left\{0, \frac{6\eta}{5}\right\}\right\} = \eta. \end{aligned}$$

So, $\varsigma\big(M(\mu,\eta)\big) = \frac{\eta}{3}$ *and* $\flat\big(M(\mu,\eta)\big) = \frac{\eta}{7}$. *Therefore,*

$$\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{\eta}{15} \leq \frac{\eta}{3} - \frac{\eta}{7} = \frac{4l}{21} = \varsigma\big(M(\mu,\eta)\big) - \flat\big(M(\mu,\eta)\big).$$

Case II: $\mu = 0$ *and* $\eta > \frac{1}{5}$. *Then* $\lhd(\mu) = \lhd(\eta) = 0$. *Clearly,* $\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{0+0}{3} = 0$. *So,*

$$\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = 0 \leq \varsigma\big(M(\mu,\eta)\big) - \flat\big(M(\mu,\eta)\big).$$

Case III: $\mu \leq \frac{1}{5}$ *and* $\eta \leq \frac{1}{5}$. *Then* $\lhd(\mu) = \frac{\mu}{5}$, $\lhd(\eta) = \frac{\eta}{5}$. *There are two cases:* $0 \leq \eta \leq \mu \leq \frac{1}{5}$ *or* $0 \leq \mu \leq \eta \leq \frac{1}{5}$. *It is sufficient to use only one of these situations. Let us assume that the first condition is satisfied. We have* $\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{\lhd\mu + \lhd\eta}{3} = \frac{\mu+\eta}{15}$, *and*

$$\begin{aligned} M(\mu,\eta) &= \max\Big\{p(\mu,\eta), \min\{p(\mu,\lhd\mu), p(\eta,\lhd\eta), p(\lhd\mu,\mu), p(\lhd\eta,\eta)\}\Big\} \\ &= \max\left\{\mu+\eta, \min\left\{\frac{6\mu}{5}, \frac{6\eta}{5}\right\}\right\} \\ &= \max\left\{\mu+\eta, \frac{6\eta}{5}\right\} = k + l. \end{aligned}$$

So, $\varsigma\big(M(\mu,\eta)\big) = \frac{\mu+\eta}{3}$, $\flat\big(M(\mu,\eta)\big) = \frac{\mu+\eta}{7}$. *Therefore,*

$$\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{\mu+\eta}{15} \leq \frac{\mu+\eta}{3} - \frac{\mu+\eta}{7} = \frac{4(\mu+\eta)}{21} = \varsigma\big(M(\mu,\eta)\big) - \flat\big(M(\mu,\eta)\big).$$

Case IV: $\eta \leq \frac{1}{5}$ *and* $\mu > \frac{1}{5}$. *Then* $\lhd\eta = \frac{l}{5}$, $\lhd\mu = 0$. *Clearly,* $\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{l}{15}$, *and*

$$\begin{aligned} M(\mu,\eta) &= \max\Big\{p(\mu,\eta), \min\{p(\mu,\lhd\mu), p(\eta,\lhd\eta), p(\lhd\mu,\mu), p(\lhd\eta,\eta)\}\Big\} \\ &= \max\left\{\mu+\eta, \min\{\mu, \frac{6\mu}{5}\}\right\}. \end{aligned}$$

There are two following cases.

If $\mu \leq \frac{6\eta}{5}$, *then* $M(\mu,\eta) = \mu+\eta$ *and so* $\varsigma\big(M(\mu,\eta)\big) = \frac{\mu+\eta}{3}$, $\flat\big(M(\mu,\eta)\big) = \frac{\mu+\eta}{7}$. *Thus,*

$$\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{\eta}{15} \leq \frac{\mu+\eta}{3} - \frac{\mu+\eta}{7} = \frac{4(\mu+\eta)}{21} = \varsigma\big(M(\mu,\eta)\big) - \flat\big(M(\mu,\eta)\big).$$

If $\frac{6\eta}{5} \leq \mu$, *then* $M(\mu,\eta) = \mu+\eta$ *and so* $\varsigma\big(M(\mu,\eta)\big) = \frac{\mu+\eta}{3}$, $\flat\big(M(\mu,\eta)\big) = \frac{\mu+\eta}{7}$. *Thus,*

$$\varsigma\big(p(\lhd\mu, \lhd\eta)\big) = \frac{\eta}{15} \leq \frac{\mu+\eta}{3} - \frac{\mu+\eta}{7} = \frac{4(\mu+\eta)}{21} = \varsigma\big(M(\mu,\eta)\big) - \flat\big(M(\mu,\eta)\big).$$

Consequently, h *is a self mapping satisfying the inequality (3) for all orthogonally related* $\mu, \eta \in W$. *Thus, all hypothesis of Theorem 5 satisfy and so it is obvious that* $\lhd$ *has a fixed point* $0 \in W$.

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Ґунґор Н.Б. *Деякі теореми про нерухому точку для експансивності ортогональної p-контрактності* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 617–630.

Ортогональні множини та ортогональні метричні простори — це два нові поняття, які були визначені у 2017 році. У цьому типі метричних просторів представлено узагальнення теореми Банаха про нерухому точку. Потім у 2019 році було досліджено нові теореми про нерухому точку з використанням функцій зміненої відстані. Натхненні роботами [Rhoades B.E. *Some theorems on weakly contractive maps*. Nonlinear Anal. 2001, **47** (4), 2683–2693] та [Gordji M.E., Rameani M., De La Sen M., Cho Y.J. *On orthogonal sets and Banach fixed point theorem*. Fixed Point Theory 2017, **18** (2), 569–578], у цій статті ми запропонували теореми про нерухому точку для розширення ортогональної p-контрактності через функції зміненої відстані. Додатково запропоновано наслідки та обмежувальний приклад.

Ключові слова і фрази: нерухома точка, функція зміненої відстані, ортогональний метричний простір.