ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2022, **14** (2), 485–492
doi:10.15330/cmp.14.2.485-492

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2022, Т.14, №2, С.485–492

# Generalized integral type mappings on orthogonal metric spaces

**Acar Ö., Erdoğan E., Özkapu A.S.**

This study is devoted to investigate the problem whether the existence and uniqueness of integral type contraction mappings on orthogonal metric spaces. At the end, we give an example to illustrative our main result.

*Key words and phrases:* fixed point, integral type mapping, orthogonal metric space.

Department of Mathematics, Faculty of Science, Selçuk University, 42003, Konya, Türkiye
E-mail: acarozlem@ymail.com (Acar Ö.), esreferdogan@yahoo.com (Erdoğan E.),
aybalasevdeozkapu.007@yahoo.com (Özkapu A.S.)

## 1 Introduction

In 1922, S. Banach [6] proved that on a complete metric space every contraction mapping has a unique fixed point and also he established the existence of solutions for integral equations. Based on usefulness, applications and simplicity of Banach fixed point theorem, it has become a very popular tool in many branches of mathematical analysis. So, this has led researchers to expand and generalize the principle in many ways. Especially, A. Brianciari [7] extended the Banach fixed point theorem as follows.

**Theorem 1.** *Let $T$ be a mapping from a complete metric space $(M,\rho)$ into itself. Let $c \in ]0,1[$ and $T : M \to M$ be a mapping, such that for each $x, y \in M$ the inequality*

$$\int_0^{\rho(Tx,Ty)} \gamma(s)\, ds \le c \int_0^{\rho(x,y)} \gamma(s)\, ds$$

*holds, where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesque integrable mapping, which is summable on each compact subset of $\mathbb{R}^+$, nonnegative and $\int_0^{\epsilon} \gamma(s)ds > 0$ for each $\varepsilon > 0$.*

*Then $T$ has a unique fixed point $a \in M$ such that $\lim_{n\to+\infty} T^n x = a$ for each $x \in M$.*

Since then, many authors have established fixed point theorems for several classes of contractive mappings of integral type (see [1, 5, 11, 12]). Especially Z. Liu et al. [13] extended the result of A. Brianciari in many different ways.

Recently, M.E. Gordji et al. [8] presented a new generalization of the Banach fixed point theorem by defining the notion of orthogonal sets. The orthogonal set is a non-empty set equipped with a binary relation having a special structure. The metric defined on the orthogonal set is called orthogonal metric space. The orthogonal metric space contains partially ordered metric space and graphical metric space.

УДК 515.126.4
2020 *Mathematics Subject Classification:* 54H25, 47H10.

After defining orthogonality concepts and giving generalization of Banach fixed point theorem, many generalizations were made. For example, M.E. Gordji et al. [10] studied fixed point problems in generalized orthogonal metric spaces. K. Sawangsup et al. [14] showed that for the first time the existence of fixed point in orthogonal metric spaces using the $F$-contraction mapping. Moreover, K. Sawangsup and W. Sintunavarat [15] define the transitive orthogonal set, giving a different perspective to the representation of the uniqueness of the fixed point in orthogonal metric spaces. Also, the reader can see [2,3,9].

In this paper, we investigate whether the fixed point exists and is unique by establishing an integral type contraction mapping in orthogonal metric spaces.

In order to do this we first recall some basic definitions and notations of corresponding mappings and spaces.

Throughout this paper, we denote $\mathbb{R}^+ := [0,\infty)$ and

$$\Phi_1 := \Big\{\gamma : \mathbb{R}^+ \to \mathbb{R}^+ \mid \gamma \text{ is Lebesque integrable function, summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^{\epsilon} \gamma(s)ds > 0 \text{ for each } \epsilon > 0\Big\},$$

$$\Phi_2 := \Big\{\beta : \mathbb{R}^+ \to \mathbb{R}^+ \mid \liminf_{n\to\infty} \beta(a_n) > 0 \Longleftrightarrow \liminf_{n\to\infty} a_n > 0 \text{ for each } (a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+\Big\},$$

$$\Phi_3 := \Big\{\alpha : \mathbb{R}^+ \to \mathbb{R}^+ \mid \alpha \text{ is nondecreasing continuous function and } \alpha(t) = 0 \Longleftrightarrow t = 0\Big\}.$$

**Lemma 1** ([13]). *Let $\gamma \in \Phi_1$ and $\{a_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} a_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{a_n} \gamma(s)\, ds = \int_0^{a} \gamma(s)\, ds.$$

**Lemma 2** ([13]). *Let $\gamma \in \Phi_1$ and $\{a_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty} \int_0^{a_n} \gamma(s)ds = 0 \Longleftrightarrow \lim_{n\to\infty} a_n = 0.$$

**Lemma 3** ([13]). *Let $\beta \in \Phi_2$. Then $\beta(t) > 0 \Longleftrightarrow t > 0$.*

**Definition 1** ([8]). *Let $M$ be a non-empty set and $\curlywedge$ be a binary relation defined on $M$. If binary relation $\curlywedge$ fulfils the following criteria*

$$\exists\, \varsigma_0\, [(\forall\, \omega \in M,\ \omega \curlywedge \varsigma_0) \text{ or } (\forall\, \omega \in M,\ \varsigma_0 \curlywedge \omega)],$$

*then pair $(M, \curlywedge)$ known as an orthogonal set. The element $\varsigma_0$ is called an orthogonal element. We denote this O-set or orthogonal set by $(M, \curlywedge)$.*

**Definition 2** ([8]). *Let $(M, \curlywedge)$ be an orthogonal set (O-set). Any two elements $\varsigma, \omega \in M$, such that $\varsigma \curlywedge \omega$, are said to be orthogonally related.*

**Definition 3** ([8]). *A sequence $\{\varsigma_n\}$ is called an orthogonal sequence (briefly O-sequence) if*

$$(\forall n \in \mathbb{N},\ \varsigma_n \curlywedge \varsigma_{n+1}) \text{ or } (\forall n \in \mathbb{N},\ \varsigma_{n+1} \curlywedge \varsigma_n).$$

*Similarly, a Cauchy sequence $\{\varsigma_n\}$ is said to be an orthogonally Cauchy sequence if*

$$(\forall n \in \mathbb{N},\ \varsigma_n \curlywedge \varsigma_{n+1}) \text{ or } (\forall n \in \mathbb{N},\ \varsigma_{n+1} \curlywedge \varsigma_n).$$

**Definition 4** ([8]). *Let $(M, \perp)$ be an orthogonal set and $\rho$ be a metric on $M$. Then $(M, \perp, \rho)$ is called an orthogonal metric space (O-metric space for short).*

**Definition 5** ([8]). *Let $(M, \perp, \rho)$ be an orthogonal metric space. Then $M$ is said to be an O-complete if every Cauchy O-sequence converges in $M$.*

**Definition 6** ([8]). *Let $(M, \perp, \rho)$ be an orthogonal metric space. A function $f : M \to M$ is said to be orthogonally continuous ($\perp$-continuous) at $\varsigma$ if for each O-sequence $\{\varsigma_n\}$ converging to $\varsigma$ it follows $f(\varsigma_n) \to f(\varsigma)$ as $n \to \infty$. Also $f$ is $\perp$-continuous on $M$ if $f$ is $\perp$-continuous at every $\varsigma \in M$.*

**Definition 7** ([8]). *Let a pair $(M, \perp)$ be an O-set, where $M(\neq \varnothing)$ is a non-empty set and $\perp$ be a binary relation on $M$. A mapping $f : M \to M$ is said to be $\perp$-preserving if $f(\varsigma) \perp f(\omega)$ whenever $\varsigma \perp \omega$ and weakly $\perp$-preserving if $f(\varsigma) \perp f(\omega)$ or $f(\omega) \perp f(\varsigma)$ whenever $\varsigma \perp \omega$.*

**Definition 8** ([15]). *We say that an O-set is a transitive orthogonal set if $\perp$ is transitive.*

**Definition 9** ([15]). *Let $(M, \perp)$ be an O-set. A path of length $k$ in $\perp$ from $x$ to $y$ is a finite sequence $\{z_0, z_1, \ldots, z_k\} \subset M$ such that*

$$z_0 = x,\ z_k = y,\ z_i \perp z_{i+1} \ \text{ or } \ z_{i+1} \perp z_i$$

*for all $i = 0, 1, \ldots, k-1$.*

*Let $\lambda(x, y, \perp)$ denotes the set of all paths of length $k$ in $\perp$ from $x$ to $y$.*

## 2 Main results

**Definition 10.** *Let $(M, \perp, \rho)$ be an orthogonal metric space. A mapping $T : M \to M$ is called an generalized O-integral type mapping if $\forall x, \mu \in X$ with $x \perp \mu$ the inequality*

$$\alpha\left(\int_0^{\rho(T\varsigma, T\mu)} \gamma(s)ds\right) \leq \alpha\left(\int_0^{\rho(\varsigma,\mu)} \gamma(s)ds\right) - \beta\left(\int_0^{\rho(\varsigma,\mu)} \gamma(s)ds\right) \tag{1}$$

*holds, where $\alpha \in \Phi_3, \beta \in \Phi_2, \gamma \in \Phi_1$.*

**Theorem 2.** *Let $(M, \perp, \rho)$ be an O-complete orthogonal metric space, $a_0$ is an orthogonal element of $M$ and $T$ be a self mapping on $M$ such that:*
*$(i)$ $(M, \perp)$ is a transitive orthogonal set;*
*$(ii)$ $T$ is $\perp$-preserving;*
*$(iii)$ $T$ is a generalized O-integral type mapping;*
*$(iv)$ $T$ is $\perp$-continuous.*
*Then $T$ has a unique fixed point in $M$.*

*Proof.* From the definition of the orthogonality, we have $a_0 \perp T(a_0)$ or $T(a_0) \perp a_0$. Let

$$a_1 := Ta_0,\ a_2 := Ta_1 = T^2 a_0,\ \cdots,\ a_n := Ta_{n-1} = T^n a_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If $a_{n^*} = a_{n^*+1}$, then $a_{n^*}$ is a fixed point of $T$ for some $n^* \in \mathbb{N} \cup \{0\}$. So, we suppose that $a_n \neq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we get $\rho(a_n, a_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Using $\perp$-preserving of $T$, we obtain

$$a_n \perp a_{n+1} \ \text{ or } \ a_{n+1} \perp a_n.$$

Thus $\{a_n\}$ is an 0-sequence. Set $\varsigma_n = \rho(T^n a_0, T^{n+1} a_0)$ and show that

$$\varsigma_n \leq \varsigma_{n-1}, \quad \forall\, n \in \mathbb{N}. \tag{2}$$

Suppose that (2) does not hold. It follows that there exists some $n_0 \in \mathbb{N}$ satisfying

$$\varsigma_{n_0} > \varsigma_{n_0-1}. \tag{3}$$

Noting (3) and using $\gamma \in \Phi_1$, we obtain

$$\int_0^{\varsigma_{n_0}} \gamma(s)\, ds > 0. \tag{4}$$

Using (1), (3) and $\alpha \in \Phi_3, \beta \in \Phi_2, \gamma \in \Phi_1$, we conclude that

$$\begin{aligned}
\alpha\left(\int_0^{\varsigma_{n_0-1}} \gamma(s)\, ds\right) &\leq \alpha\left(\int_0^{\varsigma_{n_0}} \gamma(s)\, ds\right) = \alpha\left(\int_0^{\rho(T^{n_0}a_0, T^{n_0+1}a_0)} \gamma(s)\, ds\right) \\
&\leq \alpha\left(\int_0^{\rho(T^{n_0-1}a_0, T^{n_0}a_0)} \gamma(s)\, ds\right) - \beta\left(\int_0^{\rho(T^{n_0-1}a_0, T^{n_0}a_0)} \gamma(s)\, ds\right) \\
&= \alpha\left(\int_0^{\varsigma_{n_0-1}} \gamma(s)\, ds\right) - \beta\left(\int_0^{\varsigma_{n_0-1}} \gamma(s)\, ds\right) \leq \alpha\left(\int_0^{\varsigma_{n_0-1}} \gamma(s)\, ds\right),
\end{aligned}$$

which yields that

$$\alpha\left(\int_0^{\varsigma_{n_0-1}} \gamma(s) ds\right) = \alpha\left(\int_0^{\varsigma_{n_0}} \gamma(s) ds\right) \tag{5}$$

and

$$\beta\left(\int_0^{\varsigma_{n_0-1}} \gamma(s) ds\right) = 0. \tag{6}$$

Combining (6) and Lemma 3 we get

$$\int_0^{\varsigma_{n_0-1}} \gamma(s) ds = 0,$$

which together with $\alpha \in \Phi_3$ and (5) means that

$$\alpha\left(\int_0^{\varsigma_{n_0-1}} \gamma(s)\, ds\right) = \alpha\left(\int_0^{\varsigma_{n_0}} \gamma(s)\, ds\right) = \alpha(0) = 0,$$

that is

$$\alpha\left(\int_0^{\varsigma_{n_0}} \gamma(s)\, ds\right) = 0,$$

which contradicts (4). Hence, (2) holds. Now we show that,

$$\lim_{n\to\infty} \varsigma_n = 0. \tag{7}$$

From (2), we deduce that the nonnegative sequence $\{\varsigma_n\}$ is nonincreasing, which means that there exists a constant $c$ with $\lim_{n\to\infty} \varsigma_n = c \geq 0$. Suppose that $c > 0$. Then from (1) we get

$$\begin{aligned}
\alpha\left(\int_0^{\varsigma_n} \gamma(s)\, ds\right) &= \alpha\left(\int_0^{\rho(T^n a_0, T^{n+1}a_0)} \gamma(s)\, ds\right) \\
&\leq \alpha\left(\int_0^{\rho(T^{n-1}a_0, T^n a_0)} \gamma(s)\, ds\right) - \beta\left(\int_0^{\rho(T^{n-1}a_0, T^n a_0)} \gamma(s)\, ds\right) \\
&= \alpha\left(\int_0^{\varsigma_{n-1}} \gamma(s)\, ds\right) - \beta\left(\int_0^{\varsigma_{n-1}} \gamma(s)\, ds\right), \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{8}$$

Taking upper limit in (8) and using Lemma 1 and $\alpha \in \Phi_3, \beta \in \Phi_2, \gamma \in \Phi_1$, we get

$$\begin{aligned}\alpha\left(\int_0^c \gamma(s)\,ds\right) &= \limsup_{n\to\infty} \alpha\left(\int_0^{\varsigma_n} \gamma(s)\,ds\right)\\ &\le \limsup_{n\to\infty}\left[\alpha\left(\int_0^{\varsigma_{n-1}} \gamma(s)\,ds\right) - \beta\left(\int_0^{\varsigma_{n-1}} \gamma(s)\,ds\right)\right]\\ &\le \limsup_{n\to\infty} \alpha\left(\int_0^{\varsigma_{n-1}} \gamma(s)\,ds\right) - \liminf_{n\to\infty} \beta\left(\int_0^{\varsigma_{n-1}} \gamma(s)\,ds\right)\\ &= \alpha\left(\int_0^c \gamma(s)\,ds\right) - \liminf_{n\to\infty} \beta\left(\int_0^{\varsigma_{n-1}} \gamma(s)\,ds\right) < \alpha\left(\int_0^c \gamma(s)\,ds\right)\end{aligned}$$

which is a contradiction, so $c = 0$.

Let us show that $\{T^n a_0\}$ is an $O$-Cauchy sequence. Suppose that $\{T^n a_0\}$ is not an $O$-Cauchy sequence, which means that there is a constant $\epsilon > 0$, such that for each positive integer $k$ there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) < k$ satisfying

$$\rho(T^{m(k)}a_0, T^{n(k)}a_0) > \epsilon. \tag{9}$$

For each positive integer $k$, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying (9). It follows that

$$\rho(T^{m(k)}a_0, T^{n(k)}a_0) > \epsilon$$

and

$$\rho(T^{m(k)-1}a_0, T^{n(k)}a_0) \le \epsilon, \quad \forall\, k \in \mathbb{N}. \tag{10}$$

Note that

$$\rho(T^{m(k)}a_0, T^{n(k)}a_0) \le \rho(T^{m(k)-1}a_0, T^{n(k)}a_0) + \rho(T^{m(k)-1}a_0, T^{m(k)}a_0), \quad \forall\, k \in \mathbb{N}.$$

Hence, for all $k \in \mathbb{N}$

$$\begin{aligned}\left|\rho(T^{m(k)}a_0, T^{n(k)+1}a_0) - \rho(T^{m(k)}a_0, T^{n(k)}a_0)\right| &\le \varsigma_{n(k)},\\ \left|\rho(T^{m(k)+1}a_0, T^{n(k)+1}a_0) - \rho(T^{m(k)}a_0, T^{n(k)+1}a_0)\right| &\le \varsigma_{m(k)},\\ \left|\rho(T^{m(k)+1}a_0, T^{n(k)+1}a_0) - \rho(T^{m(k)+1}a_0, T^{n(k)+2}a_0)\right| &\le \varsigma_{n(k)+1}.\end{aligned} \tag{11}$$

From (10) and (11), we obtain

$$\begin{aligned}\epsilon = \lim_{k\to\infty} \rho(T^{m(k)}a_0, T^{n(k)}a_0) &= \lim_{k\to\infty} \rho(T^{m(k)}a_0, T^{n(k)+1}a_0)\\ &= \lim_{k\to\infty} \rho(T^{m(k)+1}a_0, T^{n(k)+1}a_0) = \lim_{k\to\infty} \rho(T^{m(k)+1}a_0, T^{n(k)+2}a_0).\end{aligned} \tag{12}$$

From (1), we have for all $k \in \mathbb{N}$,

$$\begin{aligned}&\alpha\Bigg(\int_0^{\rho(T^{m(k)+1}a_0, T^{n(k)+2}a_0)} \gamma(s)ds\Bigg)\\ &\qquad \le \alpha\Bigg(\int_0^{\rho(T^{m(k)}a_0, T^{n(k)+1}a_0)} \gamma(s)ds\Bigg) - \beta\Bigg(\int_0^{\rho(T^{m(k)}a_0, T^{n(k)+1}a_0)} \gamma(s)ds\Bigg).\end{aligned} \tag{13}$$

Taking upper limit in (13) and using (12), $\alpha \in \Phi_3, \beta \in \Phi_2, \gamma \in \Phi_1$ and Lemma 1, we obtain

$$\begin{aligned}\alpha\left(\int_0^{\epsilon}\gamma(s)\,ds\right) &= \limsup_{k\to\infty}\alpha\left(\int_0^{\rho(T^{m(k)+1}a_0,T^{n(k)+2}a_0)}\gamma(s)\,ds\right)\\ &\le \limsup_{k\to\infty}\left[\alpha\left(\int_0^{\rho(T^{m(k)}a_0,T^{n(k)+1}a_0)}\gamma(s)\,ds\right)-\beta\left(\int_0^{\rho(T^{m(k)}a_0,T^{n(k)+1}a_0)}\gamma(s)\,ds\right)\right]\\ &\le \limsup_{k\to\infty}\alpha\left(\int_0^{\rho(T^{m(k)}a_0,T^{n(k)+1}a_0)}\gamma(s)\,ds\right)-\liminf_{k\to\infty}\beta\left(\int_0^{\rho(T^{m(k)}a_0,T^{n(k)+1}a_0)}\gamma(s)\,ds\right)\\ &= \alpha\left(\int_0^{\epsilon}\gamma(s)\,ds\right)-\liminf_{k\to\infty}\beta\left(\int_0^{\rho(T^{m(k)}a_0,T^{n(k)+1}a_0)}\gamma(s)\,ds\right)<\alpha\left(\int_0^{\epsilon}\gamma(s)\,ds\right),\end{aligned}$$

which is impossible. Thus $\{T^n a_0\}$ is a $O$-Cauchy sequence. Since $M$ is $O$-complete, then there exists $z^* \in M$ such that $a_n \to z^*$. Since orthogonal continuity of $T$ implies that $Ta_n \to Tz^*$, then

$$Tz^* = T(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} Ta_n = \lim_{n\to\infty} a_{n+1} = z^*,$$

so, $z^*$ is a fixed point of $T$. Now, we can show the uniqueness of the fixed point. Suppose that there exists two distinct fixed point $z^*$ and $w^*$. Since $\lambda(\varsigma,\mu,\curlywedge)$ is nonempty for all $\varsigma,\mu \in M$, there exists a path $\{z_0, z_1, \ldots, z_k\}$ of some finite lenght $k$ in $\curlywedge$ from $z^*$ to $w^*$ such that

$$z_0 = z^*,\ z_k = w^*,\ z_i \curlywedge z_{i+1} \text{ or } z_{i+1} \curlywedge z_i.$$

Since $(M,\curlywedge)$ is a transitive orthogonal set, we get $z^* \curlywedge w^*$ or $w^* \curlywedge z^*$. Then from (1) we obtain

$$\begin{aligned}\alpha\left(\int_0^{\rho(z^*,w^*)}\gamma(s)\,ds\right) &= \alpha\left(\int_0^{\rho(Tz^*,Tw^*)}\gamma(s)\,ds\right)\\ &\le \alpha\left(\int_0^{\rho(z^*,w^*)}\gamma(s)\,ds\right)-\beta\left(\int_0^{\rho(z^*,w^*)}\gamma(s)\,ds\right)<\alpha\left(\int_0^{\rho(z^*,w^*)}\gamma(s)\,ds\right),\end{aligned}$$

which is a contradiction. So, $z^*$ is a unique fixed point of $T$. □

**Example 1.** *Let $M = \left[0, \frac{1}{2}\right] \cup \{1\} \cup \{3\}$ be endowed with the standard metric $\rho$. Assume that $T : M \to M$ and $\alpha, \beta, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ are defined as in the paper [13] such as*

$$T(\varsigma)=\begin{cases}\frac{\varsigma}{2}, & \forall\varsigma\in\left[0,\frac{1}{2}\right],\\ 0, & \varsigma=1,\\ 1, & \varsigma=3,\end{cases}\qquad \beta(s)=\begin{cases}\frac{s^2}{4}, & \forall s\in[0,1],\\ \frac{s^2}{8}, & \forall s\in(1,+\infty),\end{cases}$$

$$\gamma(s)=\begin{cases}\frac{s}{2}, & \forall s\in[0,1],\\ 1, & \forall s\in(1,+\infty),\end{cases}\qquad \alpha(s)=\begin{cases}s, & \forall s\in[0,1],\\ \frac{s^2+1}{2}, & \forall s\in(1,+\infty).\end{cases}$$

*Define relation $\curlywedge$ on $M$ by $\varsigma \curlywedge \mu \iff \varsigma\mu \in \{\varsigma, \mu\}$. Clearly $(M,\curlywedge)$ is an $O$-complete orthogonal metric space and $(\gamma, \beta, \alpha) \in \Phi_1\times \Phi_2\times \Phi_3$. In order to verify (1), we have to consider the following four cases.*

*Case 1. Let $\varsigma = 0$ and $\mu = 3$. It follows that*

$$\begin{aligned}
\alpha\Big(\int_0^{\rho(T\varsigma,T\mu)}\gamma(s)\,ds\Big) &= \alpha\Big(\int_0^{\rho(0,1)}\gamma(s)\,ds\Big) = \alpha\left(\frac{1}{4}\right) = \frac{1}{4} \le \frac{\left(\frac{9}{4}\right)^2+1}{2} - \frac{\left(\frac{9}{4}\right)^2}{8} = \alpha\left(\frac{9}{4}\right) - \beta\left(\frac{9}{4}\right)\\
&= \alpha\Big(\int_0^1 \gamma(s)\,ds + \int_1^3 \gamma(s)\,ds\Big) - \beta\Big(\int_0^1 \gamma(s)\,ds + \int_1^3 \gamma(s)\,ds\Big)\\
&= \alpha\Big(\int_0^3 \gamma(s)\,ds\Big) - \beta\Big(\int_0^3 \gamma(s)\,ds\Big)\\
&= \alpha\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big) - \beta\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big).
\end{aligned}$$

*Case 2. Let $\varsigma = 1$ and $\mu = 3$. It follows that*

$$\begin{aligned}
\alpha\Big(\int_0^{\rho(T\varsigma,T\mu)}\gamma(s)\,ds\Big) &= \alpha\Big(\int_0^{\rho(0,1)}\gamma(s)\,ds\Big) = \alpha\left(\frac{1}{4}\right) = \frac{1}{4} \le \frac{\left(\frac{5}{4}\right)^2+1}{2} - \frac{\left(\frac{5}{4}\right)^2}{8} = \alpha\left(\frac{5}{4}\right) - \beta\left(\frac{5}{4}\right)\\
&= \alpha\Big(\int_0^1 \gamma(s)\,ds + \int_1^2 \gamma(s)\,ds\Big) - \beta\Big(\int_0^1 \gamma(s)\,ds + \int_1^2 \gamma(s)\,ds\Big)\\
&= \alpha\Big(\int_0^2 \gamma(s)\,ds\Big) - \beta\Big(\int_0^2 \gamma(s)\,ds\Big)\\
&= \alpha\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big) - \beta\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big).
\end{aligned}$$

*Case 3. Let $\varsigma = 1, \mu \in \left[0, \frac{1}{2}\right]$. It follows that*

$$\begin{aligned}
\alpha\Big(\int_0^{\rho(T\varsigma,T\mu)}\gamma(s)\,ds\Big) &= \alpha\Big(\int_0^{\rho(0,\frac{\mu}{2})}\gamma(s)\,ds\Big) = \alpha\left(\frac{\mu^2}{16}\right) = \frac{\mu^2}{16} \le \frac{|1-\mu|^2}{4} - \frac{|1-\mu|^4}{64}\\
&= \alpha\Big(\frac{|1-\mu|^2}{4}\Big) - \beta\Big(\frac{|1-\mu|^2}{4}\Big) = \alpha\Big(\int_0^{\rho(1,\mu)}\gamma(s)\,ds\Big) - \beta\Big(\int_0^{\rho(1,\mu)}\gamma(s)\,ds\Big)\\
&= \alpha\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big) - \beta\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big)
\end{aligned}$$

*Case 4. Let $\varsigma = 0, \mu \in \left[0, \frac{1}{2}\right]$. It follows that*

$$\begin{aligned}
\alpha\Big(\int_0^{\rho(T\varsigma,T\mu)}\gamma(s)\,ds\Big) &= \alpha\Big(\int_0^{\rho(0,\frac{\mu}{2})}\gamma(s)\,ds\Big) = \alpha\left(\frac{\mu^2}{16}\right) = \frac{\mu^2}{16} \le \frac{\mu^2}{4} - \frac{\mu^4}{64} = \alpha\left(\frac{\mu^2}{4}\right) - \beta\left(\frac{\mu^2}{4}\right)\\
&= \alpha\Big(\int_0^{\rho(0,\mu)}\gamma(s)\,ds\Big) - \beta\Big(\int_0^{\rho(0,\mu)}\gamma(s)\,ds\Big)\\
&= \alpha\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big) - \beta\Big(\int_0^{\rho(\varsigma,\mu)}\gamma(s)\,ds\Big).
\end{aligned}$$

*That is,* (1) *holds. Thus Theorem 2 guarantees that $T$ has a unique fixed point $0 \in M$.*

## References

[1] Altun I., Türkoğlu D., Rhoades B.E. *Fixed points of weakly compatible maps satisfying a general contractive condition of integral type*. Fixed Point Theory Appl. 2007, **017301** (2007). doi:10.1155/2007/17301

[2] Acar Ö., Özkapu A.S. *Multivalued rational type F-contraction on orthogonal metric space*. Math. Found. Comput. 2022. doi:10.3934/mfc.2022026

[3] Acar Ö., Erdoğan E. *Some fixed point results for almost contraction on orthogonal metric space*. Creat. Math. Inform. 2022, **31** (2), 147–153. doi:10.37193/CMI.2022.02.01

[4] Acar Ö., Özkapu A.S., Erdoğan E. *Some Fixed Point Results on Orthogonal Metric Space*. Bull. Comput. Appl. Math., accepted.

[5] Aliouche A. *A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type*. J. Math. Anal. Appl. 2006, **322** (2), 796–802.

[6] Banach S. *Sur les fonctions dérivées des fonctions mesurables*. Fund. Math. 1922, **3** (1), 128–132.

[7] Branciari A. *A fixed point theorem for mappings satisfying a general contractive condition of integral type*. Int. J. Math. Math. Sci. 2002, **29**, 531–536. doi:10.1155/S0161171202007524

[8] Gordji M.E., Rameani M., Sen M., Cho Y.J. *On orthogonal sets and Banach fixed point theorem*. Fixed Point Theory 2017, **18**, 569–578. doi:10.24193/FPT-RO.2017.2.45

[9] Gungor N.B. *Extensions of Orthogonal p-Contraction on Orthogonal Metric Spaces*. Symmetry 2022, **14** (4), 746. doi:10.3390/sym14040746

[10] Gordji M.E., Habibi H. *Fixed point theory in generalized orthogonal metric space*. J. Linear Topol. Algebra 2017, **6** (3), 251–260.

[11] Jachymski J. *Remarks on contractive conditions of integral type*. Nonlinear Anal. 2009, **71** (3–4), 1073–1081. doi:10.1016/j.na.2008.11.046

[12] Liu Z., Li X., Kang S.M., Cho S.Y. *Fixed point theorems for mappings satisfying contractive conditions of integral type and applications*. Fixed Point Theory Appl. 2011, 64. doi:10.1186/1687-1812-2011-64

[13] Liu Z., Li J., Kang S.M. *Fixed point theorems of contractive mappings of integral type*. Fixed Point Theory Appl. 2013, **300** (2013). doi:10.1186/1687-1812-2013-300

[14] Sawangsup K., Sintunavarat W., Cho Y.J. *Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces*. J. Fixed Point Theory Appl. 2020. doi:10.1007/s11784-019-0737-4

[15] Sawangsup K., Sintunavarat W. *Fixed Point Results for Orthogonal Z-Contraction Mappings in O-Complete Metric Spaces*. IJAPM 2020, **10** (1). doi:10.17706/ijapm.2020.10.1.33-40

*Received 08.11.2022*

Акар О., Ердоган Е., Озкапу А.С. *Узагальнені відображення інтегрального типу на ортогональних метричних просторах* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 485–492.

Ця стаття присвячена дослідженню проблеми існування та єдиності стискуючих відображень інтегрального типу на ортогональних метричних просторах. Наведено приклад, що ілюструє основний результат статті.

*Ключові слова і фрази:* нерухома точка, відображення інтегрального типу, ортогональний метричний простір.