ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2023, **15** (1), 286–294
doi:10.15330/cmp.15.1.286-294

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2023, Т.15, №1, С.286–294

Approximate properties of Abel-Poisson integrals on classes of differentiable functions defined by moduli of continuity

Kharkevych Yu.I.[1], Stepaniuk T.A.[2]

The paper deals with the problem of approximation in the uniform metric of $W^1 H_\omega$ classes using one of the classical linear summation methods for Fourier series given by a set of functions of a natural argument, namely, using the Abel-Poisson integral. At the same time, emphasis is placed on the study of the asymptotic behavior of the exact upper limits of the deviations of the Abel-Poisson integrals from the functions of the mentioned class.

Key words and phrases: modulus of continuity, Abel-Poisson integral, uniform metric.

[1] Lesya Ukrainka Volyn National University, 13 Voli avenue, 43025, Lutsk, Ukraine
[2] Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01601, Kyiv, Ukraine
E-mail: `kharkevich.juriy@gmail.com` (Kharkevych Yu.I.), `stepaniuk.tet@gmail.com` (Stepaniuk T.A.)

1 Statement of the problem and some historical information

Let L be a space of 2π-periodic summable functions with norm $\|f\|_L = \|f\|_1 = \int\limits_{-\pi}^{\pi} |f(t)|dt$, C be a subset of the continuous functions from L with the uniform norm $\|f\|_C = \max\limits_t |f(t)|$.

A function of the following form

$$A_\rho(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\frac{1-\rho^2}{1-2\rho\cos t+\rho^2}dt \tag{1}$$

is called the Abel-Poisson integral of the function $f \in L$ [1].

Expanding the kernel of the Abel-Poisson integral $K_\rho(t) = \frac{1-\rho^2}{2(1-2\rho\cos t+\rho^2)}$ in a Fourier series and setting $\rho = e^{-\frac{1}{\delta}}$ (see, e.g., [2]), we write the Abel-Poisson integral as

$$A_\delta(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left\{\frac{1}{2}+\sum_{k=1}^{\infty} e^{-\frac{k}{\delta}}\cos kt\right\}dt, \quad \delta>0. \tag{2}$$

Let further

$$H_\omega = \Big\{\varphi \in C: \ |\varphi(t)-\varphi(t')| \leqslant \omega(|t-t'|) \ \forall\, t,t' \in \mathbb{R}\Big\},$$

where $\omega(t)$ is a fixed majorant of the type of modulus continuity.

УДК 517.5
2020 *Mathematics Subject Classification:* 42A05, 41A60.

For a positive integer r, we denote by $W^r H_\omega$ the class of functions f that have continuous derivatives up to the order $(r-1)$ and whose rth derivatives $f^{(r)}$ belong to the class H_ω, namely

$$W^r H_\omega = \{f \in C: \ f^{(r)} \in H_\omega, \ r \in \mathbb{N}\}.$$

Following A.I. Stepanets [3, p. 9], the problem of finding asymptotic equalities for the quantities

$$\mathcal{E}\left(\mathfrak{M}; A_\delta\right)_C = \sup_{f \in \mathfrak{M}} \left\| f(x) - A_\delta(f;x) \right\|_C,$$

where $\mathfrak{M} \subseteq C$ is the given class of functions and $A_\delta(f;x)$ is the Abel-Poisson integral, is called as the Kolmogorov-Nikol'skii problem.

And if we find in the explcit form the function $\varphi(\delta) = \varphi(A_\delta; \delta)$, such that

$$\mathcal{E}\left(\mathfrak{M}; A_\delta\right)_C = \varphi(\delta) + o\big(\varphi(\delta)\big) \quad \text{as} \ \ \delta \to \infty,$$

then the Kolmogorov-Nikol'skii problem is said to be solved for the Abel-Poisson integral $A_\delta(f;x)$ on the class of the functions $\mathfrak{M}$ in the metric of the space C.

Approximation properties of the Abel-Poisson integrals $A_\delta(f;x)$ were studied by numerous researchers. Let us give the short historical overview of the considered problem.

I.P. Natanson [4] solved the Kolmogorov-Nikol'skii problem on the class W^1 for Abel-Poisson integral (1):

$$\mathcal{E}\left(W^1, A_\rho\right)_C = \sup_{f \in W^1} \left\| f(x) - A_\rho(f;x) \right\|_C = \frac{2}{\pi}(1-\rho)\left|\ln(1-\rho)\right| + O(1-\rho), \quad \rho \to 1-.$$

In [5], A.F. Timan obtained exact values of the approximation characteristics $\mathcal{E}\left(W^r, A_\rho\right)_C$ for $r \in \mathbb{N}$.

Definition ([6]). *Formal series $\sum\limits_{n=0}^{\infty} g_n(\rho)$ is called as complete asymptotic expansion or complete asymtotics of the function $f(\rho)$ as $\rho \to 1-$, if for arbitrary natural N, as $\rho \to 1-$*

$$f(\rho) = \sum_{n=0}^{N} g_n(\rho) + o\big(g_N(\rho)\big)$$

and for all $n \in N$

$$\left|g_{n+1}(\rho)\right| = o\Big(\left|g_n(\rho)\right|\Big).$$

Shortly we can write it in the following way

$$f(\rho) \cong \sum_{n=0}^{\infty} g_n(\rho).$$

In É.L. Stark's work [1], the complete asymptotic expansion of the quantity $\mathcal{E}\left(W^1, A_\delta\right)_C$ in powers of $(1-\rho)$, $\rho \to 1-$, was found. This allows to write Kolmogorov-Nikol'skii constants of arbitrary order [3, p. 9], which correspond to asymptotic terms of arbitrary order of smallness, namely

$$\mathcal{E}\left(W^1, A_\rho\right)_C \cong \sum_{k=1}^{\infty} \left\{ \alpha_k (1-\rho)^k \ln\frac{1}{1-\rho} + \beta_k (1-\rho)^k \right\},$$

where
$$\alpha_k = \frac{1}{k}, \;\; \beta_k = \frac{1}{k}\left(\frac{1}{k} + \ln 2 - \sum_{i=1}^{k-1} \frac{2^{-i}}{i}\right).$$

Later the results of É.L. Stark were generalized on the classes W^r, $r \in N$, in [7,8].

At the same time V.A. Baskakov [9] found the complete asymptotic expansions for upper bounds of deviations of the functions from the classes H^α та $W^1 H^\alpha$ from Abel-Poisson integrals $A_\delta(f;x)$ in powers of $\frac{1}{\delta}$, $\delta \to \infty$.

The works [10–12] are devoted to the study of the approximative properties of Abel-Poisson integrals on wider classes of differentiated functions, in particular, on the Weyl-Nagy classes W^r_β and Stepanets classes L^ψ_β.

We should note that the Abel-Poisson method is a saturated method, that is, an arbitrary function $f \in L_p$ cannot be approximated by the operators $A_\rho(f)$ with an accuracy better than $1-\rho$. We can not achieve an order of approximation better than $1-\rho$ by any additional restrictions on the smoothness of the function. Therefore, in addition to the Abel-Poisson integral, the Taylor-Abel-Poisson operator has also been widely studied (see, e.g., [13–16]) on some classes of differentiating functions of many variables. The Taylor-Abel-Poisson operator allows to take into account the smoothness properties of functions and be the best linear operator for the given functional class in a certain sense.

The problem of investigation of approximation properties of Abel-Poisson integral (as a linear method of summation of Fourier series, which is defined by a set of functions of natural argument, depending on a real parameter δ [17–19]) on the classes of differentiable functions, defined with the help of modulus of continuity, remained open up to now. But with regard to approximation properties of triangular methods of linear methods of summation of Fourier series [20–22], the situation is totally different on the same classes of functions.

That is why the main aim of our paper is an investigation of approximation behaviour of the quantity
$$\mathcal{E}\left(W^1 H_\omega; A_\delta\right)_C = \sup_{f \in W^1 H_\omega} \left\| f(x) - A_\delta(f,x) \right\|_C, \quad \delta \to \infty. \tag{3}$$

2 Approximation of the functions from the classes $W^1 H_\omega$ by Abel-Poisson integrals in the uniform metric

With the notations introduced above the following theorem takes place.

Theorem 1. *For arbitrary modulus of continuity $\omega(t)$ the following inequality*
$$\mathcal{E}\left(W^1 H_\omega; A_\delta\right)_C \le \frac{1}{\pi\delta} \int_{\frac{\pi}{\delta}}^{\frac{\pi}{2}} \frac{\omega(2t)}{\sin t}\, dt + O\left(\frac{1}{\delta}\omega\left(\frac{1}{\delta}\right)\right) \tag{4}$$
holds. If $\omega(t)$ is a convex modulus of continuity, then the inequality (4) becomes an equality.

Proof. Abel-Poisson integral defined according to [9] by means of relation (2), can be represented in the form
$$A_\delta(f;x) = \frac{1}{\pi\delta} \int_{-\pi}^{\pi} f(x+t) \sum_{k=-\infty}^{\infty} \frac{1}{\frac{1}{\delta^2} + (t+2\pi k)^2}\, dt = \frac{1}{\pi\delta} \int_{-\infty}^{\infty} \frac{\big(f(x+t)\big)_{2\pi}}{\frac{1}{\delta^2} + t^2}\, dt, \tag{5}$$
where $\big(f(x+t)\big)_{2\pi}$ is the even 2π-periodic continuation of the function $f(x+t)$.

From (5) it follows that

$$A_\delta(f;0) = \frac{1}{\pi\delta}\int_{-\infty}^{\infty} \frac{f(t)\,dt}{\frac{1}{\delta^2}+t^2} = \frac{1}{\pi\delta}\int_{-\infty}^{0} \frac{f(t)\,dt}{\frac{1}{\delta^2}+t^2} + \int_{0}^{\infty} \frac{f(t)dt}{\frac{1}{\delta^2}+t^2} = \frac{1}{\pi}\int_{0}^{\infty}\left[f\Big(\frac{t}{\delta}\Big) - f\Big(-\frac{t}{\delta}\Big)\right]\frac{dt}{1+t^2}.$$

For the Abel-Poisson integral (in the same way as in [3, Subsection 1.9, Chapter 4]) the following equality is true

$$\mathcal{E}\left(W^1 H_\omega; A_\delta\right)_C = \sup_{f\in W^1 H_\omega} \left|f(0) - A_\delta(f;0)\right|. \tag{6}$$

Moreover

$$\begin{aligned} f(0) - A_\delta(f;0) &= \frac{1}{\pi\delta}\int_{-\infty}^{\infty} \frac{f(0)dt}{\frac{1}{\delta^2}+t^2} - \frac{1}{\pi}\int_0^\infty \left[f\left(\frac{t}{\delta}\right) - f\left(-\frac{t}{\delta}\right)\right]\frac{dt}{1+t^2} \\ &= \frac{2}{\pi}\int_0^\infty \frac{f(0)dt}{1+t^2} - \frac{1}{\pi}\int_0^\infty \left[f\left(\frac{t}{\delta}\right) - f\left(-\frac{t}{\delta}\right)\right]\frac{dt}{1+t^2} \\ &= -\frac{1}{\pi}\int_0^\infty \left[f\left(\frac{t}{\delta}\right) + f\left(-\frac{t}{\delta}\right) - 2f(0)\right]\frac{dt}{1+t^2}. \end{aligned} \tag{7}$$

Combining (6) and (7), we obtain that

$$\mathcal{E}\left(W^1 H_\omega; A_\delta\right)_C = \sup_{f\in W^1 H_\omega} \left|\frac{1}{\pi}\int_0^\infty F_\delta(t)\frac{dt}{1+t^2}\right|, \tag{8}$$

where

$$F_\delta(t) = f\left(\frac{t}{\delta}\right) - f\left(-\frac{t}{\delta}\right) - 2f(0). \tag{9}$$

Repeating the arguments, given in the paper by A.I. Stepanets [3, p. 224] we show that if $f \in W^1 H_\omega$, then

$$\left|\int_0^\pi F_\delta(t)\frac{dt}{1+t^2}\right| = O\left(\frac{1}{\delta}\omega\left(\frac{1}{\delta}\right)\right). \tag{10}$$

Indeed, from (9) it follows that $F_\delta(0) = 0$, so, using the method of integration by parts and mean value theorem, we get

$$\begin{aligned} \left|\int_0^\pi F_\delta(t)\frac{dt}{1+t^2}\right| &= \left|\int_0^\pi F_\delta(t)d\int_\pi^t \frac{d\tau}{1+\tau^2}\right| = \left|\int_0^\pi F_\delta'(t)\int_\pi^t \frac{d\tau}{1+\tau^2}dt\right| \\ &\le \left|\int_0^\pi F_\delta'(t)\int_\pi^t d\tau dt\right| = \left|F_\delta'(t_0)\right|\int_0^\pi (\pi - t)dt = \frac{\pi^2}{2}\left|F_\delta'(t_0)\right|, \end{aligned} \tag{11}$$

where t_0 is the some point from the interval $(0,\pi)$. At the same time, it is obviously that

$$\left|F_\delta'(t_0)\right| = \frac{1}{\delta}\left|f'\left(\frac{t_0}{\delta}\right) - f'\left(-\frac{t_0}{\delta}\right)\right| \le \frac{1}{\delta}\omega\left(\frac{2t_0}{\delta}\right). \tag{12}$$

Substituting (12) in the right-hand side of equality (11), we get (10). Hence, taking into account the estimate (10), the relation (8) will get the form

$$\mathcal{E}\left(W^1 H_\omega; A_\delta\right)_C = \sup_{f\in W^1 H_\omega} \left|\frac{1}{\pi}\int_\pi^\infty F_\delta(t)\frac{dt}{1+t^2}\right| + O\left(\frac{1}{\delta}\omega\left(\frac{1}{\delta}\right)\right). \tag{13}$$

Further, for the arbitrary function $f \in W^1 H_\omega$ we obtain

$$\frac{1}{\pi}\int_{\pi}^{\infty} F_\delta(t)\frac{dt}{1+t^2} = \frac{1}{\pi}\int_{\pi}^{\infty} F_\delta(t)\frac{dt}{t^2} + R_\delta(f), \tag{14}$$

where

$$R_\delta(f) = \frac{1}{\pi}\int_{\pi}^{\infty} F_\delta(t)\frac{dt}{1+t^2} - \frac{1}{\pi}\int_{\pi}^{\infty} F_\delta(t)\frac{dt}{t^2} = \frac{-1}{\pi}\int_{\pi}^{\infty} F_\delta(t)\frac{dt}{(1+t^2)t^2}.$$

Let us estimate the quantity $\left|R_\delta(f)\right|$, namely

$$\left|R_\delta(f)\right| = \left|\frac{1}{\pi}\int_{\pi}^{\infty} F_\delta(t)\frac{dt}{(1+t^2)t^2}\right| = \frac{1}{\pi}\left|\int_{\pi}^{\infty} F_\delta(t)d\left(\int_{t}^{\infty}\frac{d\tau}{(1+\tau^2)\tau^2}\right)\right|.$$

Integraiting by parts the last equality, we get

$$\begin{aligned}\left|R_\delta(f)\right| &= \frac{1}{\pi}\left|-F_\delta(\pi)\int_{\pi}^{\infty}\frac{dt}{(1+t^2)t^2} - \int_{\pi}^{\infty} F'_\delta(t)\int_{t}^{\infty}\frac{d\tau}{(1+\tau^2)\tau^2}dt\right| \\ &\le \frac{1}{\pi}\left|F_\delta(\pi)\right|\left|\int_{\pi}^{\infty}\frac{dt}{(1+t^2)t^2}\right| + \frac{1}{\pi}\left|\int_{\pi}^{\infty} F'_\delta(t)\int_{t}^{\infty}\frac{d\tau}{(1+\tau^2)\tau^2}dt\right|.\end{aligned} \tag{15}$$

Using mean value theorem we find that

$$\begin{aligned}\left|F_\delta(\pi)\right| &= \left|f\left(\frac{\pi}{\delta}\right) + f\left(-\frac{\pi}{\delta}\right) - 2f(0)\right| = \left|\left(f\left(\frac{\pi}{\delta}\right) - f(0)\right) - \left(f(0) - f\left(-\frac{\pi}{\delta}\right)\right)\right| \\ &= \left|\frac{\pi}{\delta}f'(\xi_1) - \frac{\pi}{\delta}f'(\xi_2)\right| = \frac{\pi}{\delta}\left|f'(\xi_1) - f'(\xi_2)\right|,\end{aligned}$$

where ξ_1 and ξ_2 are some points from the intervals $\left(0, \frac{\pi}{\delta}\right)$ and $\left(-\frac{\pi}{\delta}, 0\right)$ respectively, and that is why

$$|F_\delta(\pi)| \le \frac{\pi}{\delta}\omega(\xi_1 - \xi_2) \le \frac{\pi}{\delta}\omega\left(\frac{2\pi}{\delta}\right) < \frac{\pi(2\pi+1)}{\delta}\omega\left(\frac{1}{\delta}\right).$$

Hence,

$$\left|F_\delta(\pi)\right| = O\left(\frac{1}{\delta}\omega\left(\frac{1}{\delta}\right)\right). \tag{16}$$

Obviously,

$$\int_{\pi}^{\infty}\frac{dt}{(1+t^2)t^2} = O(1). \tag{17}$$

From (16) and (17) it follows that

$$\frac{1}{\pi}\left|F_\delta(\pi)\right|\left|\int_{\pi}^{\infty}\frac{dt}{(1+t^2)t^2}\right| = O\left(\frac{1}{\delta}\omega\left(\frac{1}{\delta}\right)\right). \tag{18}$$

Let us estimate the second integral from (15), namely

$$\frac{1}{\pi}\left|\int_{\pi}^{\infty} F'_\delta(t)\int_{t}^{\infty}\frac{d\tau}{(1+\tau^2)\tau^2}dt\right| = \frac{1}{\pi\delta}\left|\int_{\pi}^{\infty}\left(f'\left(\frac{t}{\delta}\right) - f'\left(-\frac{t}{\delta}\right)\right)\int_{t}^{\infty}\frac{d\tau}{(1+\tau^2)\tau^2}dt\right|.$$

Whereas, $f \in W^1 H_\omega$, then $f' \in H_\omega$, therefore

$$
\begin{aligned}
\frac{1}{\pi\delta}\bigg|\int_\pi^\infty \left(f'\Big(\frac{t}{\delta}\Big)-f'\Big(-\frac{t}{\delta}\Big)\right)\int_t^\infty \frac{d\tau}{(1+\tau^2)\tau^2}dt\bigg| \\
\le & \frac{1}{\pi\delta}\int_\pi^\infty \left|f'\Big(\frac{t}{\delta}\Big)-f'\Big(-\frac{t}{\delta}\Big)\right|\left|\int_t^\infty \frac{d\tau}{(1+\tau^2)\tau^2}\right|dt \\
< & \frac{1}{\pi\delta}\int_\pi^\infty \left|f'\Big(\frac{t}{\delta}\Big)-f'\Big(-\frac{t}{\delta}\Big)\right|\int_t^\infty \frac{d\tau}{\tau^4}dt \le \frac{1}{\pi\delta}\int_\pi^\infty \omega\Big(\frac{2t}{\delta}\Big)\frac{1}{3t^3}dt \\
\le & \frac{1}{\pi\delta}\int_\pi^\infty \omega\Big(\frac{1}{\delta}\Big)\frac{2t+1}{3t^3}dt = \frac{1}{\pi\delta}\omega\left(\frac{1}{\delta}\right)\int_\pi^\infty \frac{2t+1}{3t^3}dt,
\end{aligned}
$$

where we have used the following property $\omega(\lambda t) \le (\lambda+1)\omega(t)$, $\lambda > 0$, of the modulus of continuity. Taking into account that

$$\int_\pi^\infty \frac{2t+1}{3t^3}dt = O(1),$$

we have

$$\frac{1}{\pi}\bigg|\int_\pi^\infty F'_\delta(t)\int_t^\infty \frac{d\tau}{(1+\tau^2)\tau^2}dt\bigg| = O\Big(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\Big). \tag{19}$$

Substituting (19) and (18) in (15), we get

$$\big|R_\delta(f)\big| = O\Big(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\Big).$$

Thereby, according to (13), (14) and (19), we have

$$\mathcal{E}\left(W^1H_\omega; A_\delta\right) = \sup_{f\in W^1H_\omega}\left|\frac{1}{\pi}\int_\pi^\infty F_\delta(t)\frac{dt}{t^2}\right| + O\Big(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\Big). \tag{20}$$

Integrating by parts and taking into account (16), we obtain

$$
\begin{aligned}
\frac{1}{\pi}\int_\pi^\infty F_\delta(t)\frac{dt}{t^2} &= \frac{1}{\pi}\int_\pi^\infty F'_\delta(t)\frac{dt}{t} + O\Big(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\Big) \\
&= \frac{1}{\pi\delta}\int_\pi^\infty \frac{f'\left(\frac{t}{\delta}\right)-f'\left(-\frac{t}{\delta}\right)}{t}dt + O\Big(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\Big).
\end{aligned}
\tag{21}
$$

So, the formula (20) can be written in the form

$$
\begin{aligned}
\mathcal{E}\left(W^1H_\omega; A_\delta\right) &= \sup_{f\in W^1H_\omega}\left|\frac{1}{\pi\delta}\int_\pi^\infty \frac{f'\left(\frac{t}{\delta}\right)-f'\left(-\frac{t}{\delta}\right)}{t}dt\right| + O\Big(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\Big) \\
&= \sup_{f\in H_\omega}\left|\frac{1}{\pi\delta}\int_{\frac{\pi}{\delta}}^\infty \frac{f(t)-f(-t)}{t}dt\right| + O\Big(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\Big).
\end{aligned}
\tag{22}
$$

The function

$$f_1(t) = f(t) - f(-t) \tag{23}$$

is 2π–periodic and non-even. After elementary transformations, we find the equality

$$\begin{aligned}
\frac{1}{\pi\delta}\int_{\frac{\pi}{\delta}}^{\infty}\frac{f_1(t)}{t}dt &= \frac{1}{\pi\delta}\bigg(\int_{\frac{\pi}{\delta}}^{\pi}\frac{f_1(t)}{t}dt+\sum_{k=0}^{\infty}\int_{(2k+1)\pi}^{(2k+3)\pi}\frac{f_1(t)}{t}dt\bigg)\\
&= \frac{1}{\pi\delta}\int_{\frac{\pi}{\delta}}^{\pi}\frac{f_1(t)}{t}dt+\frac{1}{\pi\delta}\sum_{k=0}^{\infty}\int_{(2k+1)\pi}^{2(k+1)\pi}\left(\frac{f_1(t)}{t}+\frac{f_1(-t)}{(4k+4)\pi-t}\right)dt\\
&= \frac{1}{\pi\delta}\int_{\frac{\pi}{\delta}}^{\pi}\frac{f_1(t)}{t}dt+\frac{1}{\pi\delta}\sum_{k=0}^{\infty}\int_{0}^{\pi}\left(\frac{f_1(t+\pi)}{(2k+1)\pi+t}+\frac{f_1(\pi-t)}{(2k+3)\pi-t}\right)dt\\
&= \frac{1}{\pi\delta}\int_{\frac{\pi}{\delta}}^{\pi-\frac{\pi}{\delta}}f_1(t)\bigg(\frac{1}{t}+\sum_{k=0}^{\infty}\left(\frac{1}{2(k+1)\pi+t}-\frac{1}{2(k+1)\pi-t}\right)\bigg)dt+R_\delta(f_1),
\end{aligned} \tag{24}$$

where

$$R_\delta(f_1)=\frac{1}{\pi\delta}\left(\int_0^{\frac{\pi}{\delta}}+\int_{\pi-\frac{\pi}{\delta}}^{\pi}\right)f_1(t)\sum_{k=0}^{\infty}\left(\frac{1}{2(k+1)\pi+t}-\frac{1}{2(k+1)\pi-t}\right)dt. \tag{25}$$

If $f\in H_\omega$, then we use (23) and get

$$\big|f_1(t)\big|=\big|f(t)-f(-t)\big|\le\omega(2t). \tag{26}$$

The series in (25) is uniformly convergent for all $t\in[0,\pi]$, so its sum is bounded on this segment. So,

$$\big|R_\delta(f_1)\big|=O\bigg(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\bigg). \tag{27}$$

Because of (24) and (27), the relation (22) can be rewritten in the form

$$\mathcal{E}\left(W^1H_\omega;A_\delta\right)_C=\sup_{f\in H_\omega}\left|\frac{1}{\pi\delta}\int_{\frac{\pi}{\delta}}^{\pi-\frac{\pi}{\delta}}f_1(t)S(t)dt\right|+O\bigg(\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)\bigg), \tag{28}$$

where

$$S(t)=\frac{1}{t}+\sum_{k=0}^{\infty}\left(\frac{1}{2(k+1)\pi+t}-\frac{1}{2(k+1)\pi-t}\right).$$

It is easy to see that on the segment $[0,\pi]$ the function $S(t)$ is nonnegative, that is why, taking into account the relation (26), for the arbitrary function $f\in H_\omega$ we have

$$\begin{aligned}
\frac{1}{\pi\delta}\left|\int_{\frac{\pi}{\delta}}^{\pi-\frac{\pi}{\delta}}f_1(t)S(t)dt\right| &\le \frac{1}{\pi\delta}\left(\int_{\frac{\pi}{\delta}}^{\frac{\pi}{2}}\big|f(t)-f(-t)\big|S(t)dt+\int_{\frac{\pi}{2}}^{\pi-\frac{\pi}{\delta}}\big|f(t)-f(2\pi-t)\big|S(t)dt\right)\\
&\le \frac{1}{\pi\delta}\left(\int_{\frac{\pi}{\delta}}^{\frac{\pi}{2}}\omega(2t)S(t)dt+\int_{\frac{\pi}{2}}^{\pi-\frac{\pi}{\delta}}\omega\big(2(\pi-t)\big)S(t)dt\right)\\
&= \frac{1}{\pi\delta}\int_{\frac{\pi}{\delta}}^{\frac{\pi}{2}}\omega(2t)\big(S(t)+S(\pi-t)\big)dt.
\end{aligned} \tag{29}$$

Since,

$$S(t)+S(\pi-t)=\frac{1}{t}+\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{t-k\pi}+\frac{1}{t+k\pi}\right)=\frac{1}{\sin t}, \tag{30}$$

then combining the formulas (28)–(30), we obtain (4).

The relation (4) has been proved. If $\omega = \omega(t)$ is a convex modulus of continuity, then for non-even 2π-periodic function

$$f_0(t) = \begin{cases} \frac{1}{2}\omega(2t), & t \in \left[0, \frac{\pi}{2}\right], \\ \frac{1}{2}\omega\big(2(\pi - t)\big), & t \in \left[\frac{\pi}{2}, \pi\right], \end{cases} \tag{31}$$

the relation (4) becomes the equality. Thus theorem has been proved. □

We should note that similar result for approximation by Fejer means was obtained in [3, Subsection 3.2, Chapter 4].

References

[1] Stark É.L. *The complete asymptotic expansion for the measure of approximation of Abel-Poisson's singular integral for Lip 1*. Math. Notes of the Academy of Sciences of the USSR 1973, **13** (1), 14–18. doi: 10.1007/BF01093622

[2] Abdullayev F.G., Kharkevych Yu.I. *Approximation of the classes $C^{\psi}_{\beta}H^{\alpha}$ by biharmonic Poisson integrals*. Ukrainian Math. J. 2020, **72** (1), 21–38. doi:10.1007/s11253-020-01761-6

[3] Stepanets A.I. Uniform Approximations by Trigonometric Polynomials. Naukova Dumka, Kiev, 1981. (in Russian)

[4] Natanson I.P. *On the order of approximation of a continuous 2π–periodic function by its Poisson integral*. Dokl. Akad. Nauk SSSR 1950, **72** (1), 11–14. (in Russian)

[5] Timan A.F. *Sharp estimate for a remainder in the approximation of periodic differentiable functions by Poisson integrals*. Dokl. Akad. Nauk SSSR 1950, **74** (1), 17–20. (in Russian)

[6] Kal'chuk I.V., Kharkevych Yu.I., Pozharska K.V. *Asymptotics of Approximation of Functions by Conjugate Poisson Integrals*. Carpathian Math. Publ. 2020, **12** (1), 138–147. doi:10.15330/cmp.12.1.138-147

[7] Zhyhallo K.M., Kharkevych Yu.I. *Complete asymptotics of the deviation of a class of differentiable functions from the set of their harmonic Poisson integrals*. Ukrainian Math. J. 2002, **54** (1), 51–63. doi:10.1023/A:1019789402502

[8] Kharkevych Yu.I. *Asymptotic expansions of upper bounds of deviations of functions of class W^r from their generalized Poisson integrals*. J. Automat. Inf. Scien. 2018, **50** (8), 38–49. doi:10.1615/JAutomatInfScien.v50.i8.40

[9] Baskakov V.A. *Some properties of operators of Abel-Poisson type*. Math. Notes of the Academy of Sciences of the USSR 1975, **17** (2), 101–107. doi:10.1007/BF01161864

[10] Kal'chuk I., Kharkevych Y. *Approximation Properties of the Generalized Abel-Poisson Integrals on the Weyl-Nagy Classes*. Axioms 2022, **11** (4), 161. doi:10.3390/axioms11040161

[11] Zhyhallo T.V., Kharkevych Yu.I. *On approximation of functions from the class $L^{\psi}_{\beta,1}$ by the Abel-Poisson integrals in the integral metric*. Carpathian Math. Publ. 2022, **14** (1), 223–229. doi:10.15330/cmp.14.1.223-229

[12] Kal'chuk I.V., Kharkevych Yu.I. *Approximation of the classes $W^r_{\beta,\infty}$ by generalized Abel-Poisson integrals*. Ukrainian Math. J. 2022, **74** (4), 575–585. doi:10.1007/s11253-022-02084-4.

[13] Savchuk V.V. *Approximation of holomorphic functions by Taylor-Abel-Poisson means*. Ukrainian Math. J. 2007, **59** (9), 1397–1407. doi:10.1007/s11253-007-0094-0

[14] Savchuk V.V., Shidlich A.L. *Approximation of functions of several variables by linear methods in spaces S^p*. Problems of the theory of approximation of functions and related questions. Proc. Inst. Math. NAS of Ukraine 2007, **4** (1), 302–317. (in Ukrainian)

[15] Prestin J., Savchuk V.V., Shidlich A.L. *Direct and inverse theorems on the approximation of 2π-periodic functions by Taylor-Abel-Poisson operators*. Ukrainian Math. J. 2017, **69** (5), 766–781. doi:10.1007/s11253-017-1394-7

[16] Prestin J., Savchuk V.V., Shidlich A.L. *Approximation theorems for multivariate Taylor-Abel-Poisson means*. Stud. Univ. Babeş-Bolyai Math. 2019, **64** (3), 313–329. doi:10.24193/subbmath.2019.3.03

[17] Hrabova U.Z., Kal'chuk I.V. *Approximation of the classes* $W^r_{\beta,\infty}$ *by three-harmonic Poisson integrals*. Carpathian Math. Publ. 2019, **11** (2), 321–324. doi:10.15330/cmp.11.2.321-334

[18] Kal'chuk I.V., Kravets V.I., Hrabova U.Z. *Approximation of the classes* $W^r_\beta H^\alpha$ *by three-harmonic Poisson integrals*. J. Math. Sci. (N.Y.) 2020, **246** (1), 39–50. doi:10.1007/s10958-020-04721-4

[19] Hrabova U.Z., Kal'chuk I.V., Filozof L.I. *Approximative properties of the three-harmonic Poisson integrals on the classes* $W^r_\beta H^\alpha$. J. Math. Sci. (N.Y.) 2021, **254** (3), 397–405.

[20] Rukasov V.I., Chaichenko S.O. *Approximation of the Classes* $C^{\overline{\psi}} H_\omega$ *by de la Vallée-Poussin Sums*. Ukrainian Math. J. 2002, **54** (5), 839–851. doi:10.1023/A:1021643732395

[21] Serdyuk A.S., Ovsii I.Y. *Uniform approximation of Poisson integrals of functions from the class* H_ω *by de la Vallee Poussin sums*. J. Anal. Math. 2012, **38** (4), 305–325. doi:10.1007/s10476-012-0403-1

[22] Serdyuk A.S., Ovsii E.Y. *Approximation of the classes* $C^\psi_\beta H_\omega$ *by generalized Zygmund sums*. Ukrainian Math. J. 2009, **61** (4), 627–644. doi:10.1007/s11253-009-0229-6

Received 28.09.2022

Харкевич Ю.І., Степанюк Т.А. *Апроксимативні властивості інтегралів Абеля-Пуассона на класах диференційовних функцій, означених за допомогою модуля неперервності* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 286–294.

У роботі розглядається задача наближення в рівномірній метриці класів $W^1 H_\omega$ за допомогою одного з класичних лінійних методів підсумовування рядів Фур'є, що задаються множиною функцій натурального аргументу, а саме за допомогою інтеграла Абеля-Пуассона. При цьому робиться акцент на вивченні асимптотичної поведінки точних верхніх меж відхилень інтегралів Абеля-Пуассона від функцій із згаданого класу.

Ключові слова і фрази: модуль неперервності, інтеграл Абеля-Пуассона, рівномірна метрика.