ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2024, **16** (2), 587–592
doi:10.15330/cmp.16.2.587-592

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2024, Т.16, №2, С.587–592

A note on normal maximal subgroups in Mal′cev-Neumann division rings

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The aim of this paper is to describe normal maximal subgroups of the unit groups of Mal′cev-Neumann division rings. As a corollary, we affirmatively answer the conjecture posed in [Akbari S., Mahdavi-Hezavehi M. *On the existence of normal maximal subgroups in division rings.* J. Pure Appl. Algebra 2002, **171** (2–3), 123–131] regarding Mal′cev-Neumann division rings of noncyclic free groups.

Key words and phrases: division ring, Mal′cev-Neumann division ring, maximal subgroup, normal maximal subgroup.

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1 Introduction

Let G be a group with a total order $\preceq$. If for a, b and c in G, the condition $a \preceq b$ implies $ca \preceq cb$ and $ac \preceq bc$, then G is called an *ordered group*. It is well known that a free group is an ordered group with the Magnus order (see, e.g., [5]). A subset S of an ordered group G is called *well-ordered* (*WO* for short) if every nonempty subset of S has a least element. We denote $\min(S)$ the least element of a WO subset S in case S is nonempty.

Let K be a division ring, G an ordered group, and $\omega : G \to \mathrm{Aut}(K)$, $g \mapsto \omega_g$ a group morphism. Here $\mathrm{Aut}(K)$ is the automorphism group of K. For a (formal) sum $\alpha = \sum\limits_{g\in G} a_g g$ with $a_g \in K$, the *support* of α is defined as $\mathrm{supp}(\alpha) = \{g \in G : a_g \neq 0\}$. Put

$$K\big((G,\omega)\big) = \left\{ \alpha = \sum_{g\in G} a_g g : \mathrm{supp}(\alpha) \text{ is WO} \right\}.$$

For every $\alpha = \sum\limits_{g\in G} a_g g$ and $\beta = \sum\limits_{g\in G} b_g g$ in $K\big((G,\omega)\big)$, we define

$$\alpha + \beta = \sum_{g\in G} \left(a_g + b_g\right) g$$

and

$$\alpha\beta = \sum_{u\in G} \left(\sum_{gh=u} a_g \omega_g\left(b_h\right) \right) u.$$

УДК 512.623
2020 *Mathematics Subject Classification:* 12E15, 16K99.

These operators are well-defined and $K\big((G,\omega)\big)$ is a division ring (see [13, Theorem 14.21]). The division ring $K\big((G,\omega)\big)$ is called the *Mal'cev-Neumann division ring* of G over K with respect to ω.

The Mal'cev-Neumann division rings were first introduced in [14] and up to now, they have many applications. Noncrossed product division rings in [6, 11, 12] are constructed by using special cases of the Mal'cev-Neumann division rings over certain groups. The Mal'cev-Neumann division rings are also recently used to construct some examples on division rings which satisfy certain properties (see [1,7–9] in detail). There are many papers which describe properties of Mal'cev Neumann division rings and their special cases (see, e.g., [10, 15, 17]).

The aim of this paper is to describe normal maximal subgroups of the unit group of Mal'cev-Neumann division rings. Among results, we show that if the ordered group G contains a normal maximal subgroup, then so does the unit group $\big(K\big((G,\omega)\big)\big)^*$. As a corollary, we affirmatively answer the conjecture posed in [3] regarding Mal'cev-Neumann division rings of noncyclic free groups.

2 Main results

We begin this section with the following lemma.

Lemma 1. *Let G be an ordered group, K be a division ring, $\omega : G \to \mathrm{Aut}(K)$ be a group morphism, and $D = K\big((G,\omega)\big)$ be the Mal'cev-Neumann division ring of G over K with respect to ω. Then the map $v : D^* \to G$, $\alpha \mapsto \min\big(\mathrm{supp}(\alpha)\big)$ is a surjective group homomorphism.*

Proof. This lemma is just a corollary of [1, Lemma 2.5]. □

In this paper, the morphism v as in Lemma 1 is fixed and used frequently. Lemma 1 has a corollary as follows.

Corollary 1. *Let the assumptions of Lemma 1 hold and the surjective group homomorphism v as above. Assume that M is a maximal subgroup of D^*. Then either $v(M) = G$ or $v(M)$ is a maximal subgroup of G. Moreover, if $v(M) \neq G$, then $v(M) \subseteq M$.*

Proof. Assume that M is a maximal subgroup of D^* and $v(M) \neq G$. For $H \leq G$ such that $v(M) \leq H$ and $v(M) \neq H$, since M is maximal in D^*, $v^{-1}(H) = D^*$. Then $v(D^*) = H$. Since v is surjective, $v\,(D^*) = G$. Thus, $G = H$, and so $v(M)$ is a maximal subgroup of G.

Now we prove the final assertion. Given the hypothesis $v(M) \neq G$, assume that $v(M) \not\subseteq M$. Let $g \in v(M) \setminus M$. Then, since M is maximal in D^*, $\langle M, g\rangle = D^*$. Thus,

$$v\left(\langle M, g\rangle\right) = v\left(D^*\right) = G.$$

Observe that $v\big(\langle M, g\rangle\big) = \big\langle v(M), v(g)\big\rangle = v(M)$. Consequently, $v(M) = G$, a contradiction. Hence, $v(M) \subseteq M$. □

Let G be a group and H its subgroup. The *core* of H in G is the subgroup

$$\mathrm{Core}_G(H) = \bigcap_{g\in G} gHg^{-1}.$$

The core of H is the largest normal subgroup of G contained in H. Moreover, one has the following property.

Lemma 2 ([16, 3.3.5]). *Let G be a group and H a subgroup of G. If the index of H in G is finite, then $G/\mathrm{Core}_G(H)$ is a finite group.*

Now we show the first main result of this paper.

Theorem 1. *Let G be an ordered group, K be a division ring, $\omega : G \to \mathrm{Aut}(K)$ be a group morphism, and $D = K\big((G,\omega)\big)$ be the Mal'cev-Neumann division ring of G over K with respect to ω.*

1. *If G has a maximal subgroup, which is normal in G, then D^* also has a normal maximal subgroup of prime index.*
2. *If G has a maximal subgroup of finite index n, then D^* also has a maximal subgroup of index n.*

Proof. 1. Assume that M is a maximal subgroup of G which is normal. Then G/M is a simple group and G/M has only two subgroups which are $\langle\overline{1}\rangle$ and G/M. This leads to the fact that $G/M = \langle\overline{g_0}\rangle$ for some $g_0 \in G \setminus M$. Since G/M is simple, it must be a cyclic group of prime degree. Put φ to be the surjective group morphism $\varphi : G \to G/M$, $g \mapsto \overline{g}$. Then the composition $v \circ \varphi : D^* \to G/M$ is a surjective group morphism, where v is the group morphism as defined in Lemma 1. Therefore, the quotient group $D^*/\ker(v \circ \varphi)$ is a cyclic group of prime degree. Hence, $\ker(v \circ \varphi)$ is a normal maximal subgroup of prime index of D^*.

2. Assume that M is a maximal subgroup of G of index n. By Lemma 2, the quotient group $G/\mathrm{Core}_G(M)$ is finite. Using the same group morphism $\varphi : G \to G/\mathrm{Core}_G(M)$, $g \mapsto \overline{g}$, and $v : D^* \to G$ as Case 1. Then the composition $v \circ \varphi : D^* \to G/\mathrm{Core}_G(M)$ is also a surjective group morphism. This follows that

$$D^*/\ker(v \circ \varphi) \overset{v\circ\varphi}{\cong} G/\mathrm{Core}_G(M).$$

Then there exists a subgroup H of D^* such that

$$H/\ker(v \circ \varphi) \overset{v\circ\varphi}{\cong} M/\mathrm{Core}_G(M),$$

that is, $H/\ker(v \circ \varphi)$ is a maximal subgroup of $D^*/\ker(v \circ \varphi)$. Clearly, H is a maximal subgroup of index n of D^*. ☐

The previous result seems to be interesting because the existence of normal maximal subgroups in $K\big((G,\omega)\big)$ does not depend on the base division ring K and the morphism ω. Moreover, by applying the previous theorem, we answer affirmatively a conjecture on the existence of maximal subgroups in division rings. More precisely, the following conjecture posed in [3].

Conjecture 1 ([3, Conjecture]). *Let D be a noncommutative division rings. The unit group D^* contains a maximal subgroup.*

This conjecture holds for some certain classes of division rings (see [2–4]). However, it is still open in general. In this paper, we show this conjecture holds for the Mal'cev-Neumann division rings of free groups. We note that almost all division rings mentioned in [3] for which the conjecture holds are finite dimensional over its center. The following corollary is an infinite-dimensional case.

Corollary 2. *Let G be a noncyclic free subgroup, K be a division ring, $\omega : G \to \mathrm{Aut}(K)$ be a group morphism, and $D = K\big((G,\omega)\big)$ be the Mal'cev-Neumann division ring of G over K with respect to ω. Then D^* contains infinitely many normal maximal subgroups.*

Proof. Assume that the free group G has the rank at least two. Select a generator x of G. Let C_p be the cyclic group of prime order p. Assume that $C_p = \langle c\rangle$. Define a map $\varphi : G \to C_p$ as follows: $x \mapsto c$ and $y \mapsto 1$ for any generator y of G such that $y \neq x$. According to the universal property of the free group, the map φ is a group morphism. Moreover, φ is surjective. Hence, since C_p is simple, the kernel $\ker(\varphi)$ is a normal maximal subgroup of index p of G. According to Theorem 1, the multiplicative group D^* also has a maximal subgroup of index p, and obviously this subgroup is normal in D^*. Thus, D^* has infinitely many normal maximal subgroups. □

Now, we will present a description of a normal maximal subgroup in the special case, when the base division ring K is a field and ω is trivial, that is, $\omega(g) = \mathrm{Id}_K$ for every $g \in G$. In this case, we write shortly $K\big((G)\big)$ for $K\big((G,\omega)\big)$. To show the next main result, we borrow the following lemma.

Lemma 3. *Let G be a noncyclic free group and F be a field. For every $\alpha \in F\big((G)\big)$ with $y = v(\alpha) > 1$, there exists $\beta \in F\big((G)\big)^*$ such that*

$$\beta\alpha\beta^{-1} = \sum_{i=n}^{\infty} a_i y^i,$$

where $n \in \mathbb{Z}$ and $a_i \in F$ for every $i \geq n$.

Proof. It follows from [1, Lemma 4.2]. □

Theorem 2. *Let G be a noncyclic free group, F be a field and $D = F\big((G)\big)$ be the Mal'cev-Neumann division ring of G over F. Assume that M is a normal maximal subgroup of D^*. Then M is the normal closure in D^* of the set*

$$S := \left\{\alpha = \sum_{i=n}^{\infty} a_i y^i : \alpha \in M, n \in \mathbb{Z}, y > 1\right\}.$$

Moreover, if $v(M) \neq G$, then M is the normal closure in D^* of the set

$$\left\{\alpha = \sum_{i=0}^{\infty} a_i y^i : \alpha \in M, y > 1\right\} \bigcup v(M).$$

Proof. Let N be the normal closure in D^* of S. It is obvious that $N \subseteq M$. To show the reverse inclusion, we may assume that α is an element of M. Put $y = v(\alpha)$.

Case 1. Let $y > 1$. By Lemma 3, there exists $\beta \in D^*$ such that $\beta\alpha\beta^{-1} = \sum_{i=n}^{+\infty} a_i y^i$, where $n \in \mathbb{Z}$ and $a_i \in F$ for every $i > n$, $a_n \in F^*$. Since $\alpha \in M$ and M is normal in $F\big((G)\big)^*$, $\beta\alpha\beta^{-1} \in M$. Since $y > 1$, we have $\beta\alpha\beta^{-1} = \sum a_i y^i \in S \subseteq N$. This leads to

$$\alpha = \beta^{-1}\left(\sum a_i y^i\right)\beta \in N.$$

Case 2. Let $y < 1$. Then $v\left(\alpha^{-1}\right) > 1$. By repeating the arguments in the proof of Case 1 for α^{-1}, one has $\alpha^{-1} \in N$, which also deduces that $\alpha \in N$.

Case 3. Let $y = 1$. By Corollary 1, either $v(M) = G$ or $v(M)$ is maximal in G. Since G is a noncyclic free group, $v(M) \neq \{1\}$. Select $\beta \in M$ such that $v(\beta) \neq 1$. Then $\alpha\beta \in M$ and $v(\alpha\beta) \neq 1$. According to the two above cases, $\alpha\beta \in N$ and $\beta \in N$. Thus, $\alpha = (\alpha\beta)\beta^{-1} \in N$.

The three cases prove that $M = N$, that is, M is the normal closure of S.

Now, assume that $v(M) \neq G$. Put

$$T = \left\{\alpha = \sum_{i=0}^{\infty} a_i y^i : \alpha \in M, y > 1\right\}.$$

By Corollary 1, $v(M) \subseteq M$. Since M is normal in D^*, the normal closure of $T \cup v(M)$ is contained in M. For $\alpha = \sum\limits_{i=n}^{\infty} a_i y^i \in S$, we have

$$\alpha = \left(\sum_{i=0}^{\infty} a_{i+n} y^i\right) y^n \in \langle T, v(M) \rangle.$$

It follows that $\langle S \rangle \leq \langle T, v(M) \rangle$. Since M is the normal closure of S, M is contained in the normal closure of $T \cup v(M)$. Hence, M is the normal closure of $T \cup v(M)$. □

Acknowledgments. This work was written while the second author was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). The second author would like to thank the institute for providing a fruitful research environment and working condition.

Funding. The first author is funded by Vietnam Ministry of Education and Training under Grant No. B2024-CTT-02.

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Received 01.08.2022
Revised 31.01.2024

Дан Л.К., Део Т.Т. *Замітка про нормальні максимальні підгрупи у кільцях з діленнями Мальцева-Неймана* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 587–592.

Метою цієї роботи є опис нормальних максимальних підгруп одиничних груп кілець з діленнями Мальцева-Неймана. Як наслідок, ми ствердно відповідаємо на гіпотезу, висунуту в [Akbari S., Mahdavi-Hezavehi M. *On the existence of normal maximal subgroups in division rings.* J. Pure Appl. Algebra 2002, **171** (2–3), 123–131], стосовно кілець з діленнями Мальцева-Неймана для нециклічних вільних груп.

Ключові слова і фрази: кільце з діленням, кільце з діленням Мальцева-Неймана, максимальна підгрупа, нормальна максимальна підгрупа.