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The best approximation of closed operators by bounded operators in Hilbert spaces

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We solve the problem of the best approximation of closed operators by linear bounded operators in Hilbert spaces under assumption that the operator transforms orthogonal basis in Hilbert space into an orthogonal system. As a consequence, sharp additive Hardy-Littlewood-Pólya type inequality for multiple closed operators is established. We also demonstrate application of these results in concrete situations: for the best approximation of powers of the Laplace-Beltrami operator on classes of functions defined on closed Riemannian manifolds, for the best approximation of differentiation operators on classes of functions defined on the period and on the real line with the weight e^{-x^2}, and for the best approximation of functions of self-adjoint operators in Hilbert spaces.

Key words and phrases: best approximation of operators, Stechkin problem, Kolmogorov-type inequalities, self-adjoint operator, Laplace-Beltrami operator, closed operator.

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1 Introduction

The problem of the best approximation of unbounded operator by bounded ones was formulated by S.B. Stechkin in [13, 14]. There he also established some important relations and obtained first solutions to this problem. We follow [14] to state the problem rigorously. Let X, Y be normed spaces, $A : X \to Y$ be an operator with domain $\mathcal{D}(A)$, $W \subset \mathcal{D}(A)$ be some class of elements, $\mathcal{L} = \mathcal{L}(X, Y)$ be the space of linear bounded operators $S : X \to Y$. For $S \in \mathcal{L}$, denote *the error of approximation of A by S* on W:

$$U(A, S; W) := \sup_{x\in W} \|Ax - Sx\|_Y.$$

Let $N \geq 0$ and $\mathcal{L}_N = \mathcal{L}_N(X, Y)$ be the set of operators $S \in \mathcal{L}$ such that $\|S\| \leq N$.

The problem of *the best approximation of operator A* by linear bounded operators on the class W, also known as *the Stechkin problem*, consists of finding the quantity

$$E_N(A; W) = \inf_{S\in\mathcal{L}_N} \sup_{x\in W} \|Ax - Sx\|_Y, \tag{1}$$

and extremal operators $S^* \in \mathcal{L}_N$ (if any exists) for which the inf in the right hand part of (1) is achieved.

The following simple yet powerful lower bound for (1) was found in [14].

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Proposition 1. *Let A be a homogeneous operator, $W \subset \mathcal{D}(A)$ be a centrally-symmetric convex set. Then, for every $N \geq 0$ and $\delta \geq 0$,*

$$E_N(A;W) \geq \sup_{x\in W}(\|Ax\|_Y - N\|x\|_X). \tag{2}$$

Moreover, if there exist $x_0 \in W$ and $S_0 \in \mathcal{L}$ such that $\|Ax_0\|_Y = U(A,S_0;W) + \|S_0\| \cdot \|x_0\|_X$, then S_0 is extremal operator in problem (1) *for $N = \|x_0\|_X$ and*

$$E_{\|S_0\|}(A;W) = U(A,S_0;W) = \|Ax_0\|_Y - \|S_0\| \cdot \|x_0\|_X.$$

For a good overview of known results on the Stechkin problem and discussion of related questions we refer the reader to surveys [1,3], book [6] and recent paper [2].

In this paper we will study problem (1) for operators acting in Hilbert spaces. Let us remind known results in this situation. Yu.N. Subbotin and L.V. Taikov [15] solved the problem of the best approximation of differential operator $A = D^k$ of order $k \in \mathbb{N}$ acting in the space $X = Y = L_2(\mathbb{R})$ on the class

$$W = \left\{x \in L_2(\mathbb{R}) \,:\, x^{(r-1)} \in AC_{loc}(\mathbb{R}),\, \|x^{(r)}\|_{L_2(\mathbb{R})} \leq 1\right\},$$

where $r \in \mathbb{N}$, $r > k$. V.F. Babenko and R.O. Bilichenko [5,10] generalized this result for integral powers $A = \Delta^k$ of self-adjoint and normal operators Δ, respectively, acting in Hilbert space H on the class

$$W = \{x \in \mathcal{D}(\Delta^r) \,:\, \|\Delta^r x\|_H \leq 1\}.$$

Later, V.F. Babenko, Yu.V. Babenko and N.A. Kriachko [4] obtained further generalization of above results by considering the function $A = \varphi(\Delta)$ of self-adjoint operator $\Delta : H \to H$ and the class

$$W = \{x \in H \,:\, \|\psi(\Delta)x\|_H \leq 1\}.$$

Here φ and ψ are continuous complex-valued functions on $\mathbb{R}$ such that $|\varphi(t)|$ and $|\psi(t)|$ are even and strictly increasing to $+\infty$ on $(0,\infty)$ and, in addition, $|\varphi(t)|^2 = F(|\psi(t)|^2)$, where F is a strictly increasing, concave function with $F(0) = 0$. Note that papers [4,5,10] imposed an additional assumption on operator Δ that effectively requires its spectrum to have non-empty continuous part (see proof of Corollary 6 below).

In this paper we solve the Stechkin problem in case A is closable operator acting between Hilbert spaces that preserves orthogonality between elements of orthogonal basis and class W is defined with the help of multiple closed operators (see Section 3). In Section 2 we will follow [7] to introduce necessary notations, definitions and auxiliary results. In Section 4 we demonstrate application of main result to the problem of finding sharp inequalities for the norms of derivatives and in several concrete situations: for the best approximation of powers of the Laplace-Beltrami operators on Riemannian manifolds, and for the best approximation of differentiation operator in spaces of periodic functions and spaces of functions defined on the real line with the weight e^{-x^2}. In addition, we will demonstrate that the results of this paper generalize previous results in [4,5,10,15].

2 Preliminaries

Let H be a separable Hilbert space over $\mathbb{C}$ endowed with scalar product $(\cdot,\cdot)_H$, norm $\|\cdot\|_H$ and orthonormal basis $\{e_n\}_{n\in M}$, where M is a finite or countable set. For definiteness but without loss of generality, we fix the meaning of the sum of series indexed by the set M. Let

$\{M_N\}_{N=1}^{\infty}$ be any given sequence of nested finite subsets of M such that $M = \bigcup\limits_{N=1}^{\infty} M_N$. Then, for a sequence $\{a_n\}_{n\in M} \subset H$ or $\{a_n\}_{n\in M} \subset \mathbb{C}$, we set:

$$\sum_{n\in M} a_n := \lim_{N\to\infty} \sum_{n\in M_N} a_n$$

providing the limit in the right hand part exists. In what follows the choice of the sequence $\{M_N\}$ is not essential as only absolutely convergent series will be considered. For $x \in H$, by $x_n := (x, e_n)_H$, $n \in M$, we denote its Fourier coefficients with respect to the system $\{e_n\}_{n\in M}$ and by $S_N x := \sum_{n\in M_N} x_n e_n$, $N \in \mathbb{N}$, – partial sums of the Fourier series of x.

For convenience, for sequences $\{a_n\}_{n\in M}, \{b_n\}_{n\in M} \subset [0,\infty)$, we denote

$$\widetilde{\sum_{n\in M}} \frac{a_n}{b_n} := \begin{cases} \infty, & \text{if } \exists n_0 \in M \text{ s.t. } (a_{n_0} \neq 0) \wedge (b_{n_0} = 0), \\ \sum\limits_{n\in M \,:\, b_n \neq 0} \dfrac{a_n}{b_n}, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{\sup_{n\in M}} \frac{a_n}{b_n} := \begin{cases} \infty, & \text{if } \exists n_0 \in M \text{ s.t. } (a_{n_0} \neq 0) \wedge (b_{n_0} = 0), \\ \sup\limits_{n\in M \,:\, b_n \neq 0} \dfrac{a_n}{b_n}, & \text{otherwise.} \end{cases}$$

When the set $\{n \in M \,:\, b_n \neq 0\}$ is empty we define $\widetilde{\sum}_{n\in M} \frac{a_n}{b_n} := 0$ and $\widetilde{\sup}_{n\in M} \frac{a_n}{b_n} := 0$.

Next, we let H' be a Hilbert space over $\mathbb{C}$ endowed with scalar product $(\cdot,\cdot)_{H'}$ and norm $\|\cdot\|_{H'}$. For $m \in \mathbb{Z}_+$, consider linear operators $B_j : H \to H'$, $j = 0,\dots,m$, with domains $\mathcal{D}(B_j)$. We will require operators $B_0,\dots,B_m$ to satisfy conditions:

(B1) $\forall\, n \in M$ it follows that $e_n \in \bigcap_{j=0}^{m} \mathcal{D}(B_j) =: \mathcal{D}_{\mathbf{B}}$;

(B2) $\forall\, n', n'' \in M$, $n' \neq n''$, and $\forall\, j \in \{0,1,\dots,m\}$ it follows that $(B_j e_{n'}, B_j e_{n''})_{H'} = 0$;

(B3) $\exists\, j_0 \in \{0,\dots,m\}$ and $\exists\, n_0 \in M$ such that $B_{j_0} e_{n_0} \neq 0$;

(B4) $B_0, B_1, \dots, B_m$ are closed operators (see, e.g., [16, §2.6]).

For $j = 0,\dots,m$, we consider the subspace

$$H_j := \left\{ x \in H \,:\, \|x\|_{H_j}^2 := \sum_{n\in M} |x_n|^2 \|B_j e_n\|_{H'}^2 < \infty \right\},$$

set $H_{\mathbf{B}} := \bigcap\limits_{j=0}^{m} H_j$, and endow $H_{\mathbf{B}}$ with the (semi-)norm

$$\|x\|_{\mathbf{B}}^2 := \sum_{j=0}^{m} \sum_{n\in M} |x_n|^2 \cdot \|B_j e_n\|_{H'}^2 = \sum_{n\in M} |x_n|^2 \cdot b_n, \quad x \in H_{\mathbf{B}},$$

where

$$b_n := \sum_{j=0}^{m} \|B_j e_n\|_{H'}^2, \quad n \in M.$$

The next proposition provides constructive characterization of elements of $H_{\mathbf{B}}$, establishes embedding $H_{\mathbf{B}} \subset \mathcal{D}_{\mathbf{B}}$ and presents sufficient conditions for the coincidence of these sets.

Lemma 1. *Let $B_0, B_1, \ldots, B_m$ satisfy conditions* (B1), (B2) *and* (B4). *Then $H_{\mathbf{B}} \subset \mathcal{D}_{\mathbf{B}}$ and, for every $x \in H_{\mathbf{B}}$,*

$$B_j x = \sum_{n\in M} x_n \cdot B_j e_n, \quad j = 0, 1, \ldots, m, \qquad \|x\|_{\mathbf{B}}^2 = \sum_{j=0}^{m} \|B_j x\|_{H'}^2.$$

Furthermore, if $B_j e_n \in \mathcal{D}(B_j^*)$ (for the definition of the adjoint operator see, e.g., [16, §7.1]) for every $n \in M$ and $j = 0, 1, \ldots, m$, then $\mathcal{D}_{\mathbf{B}} = H_{\mathbf{B}}$.

Finally we consider linear operator $A : H \to H'$ with domain $\mathcal{D}(A)$. We will require A to satisfy the conditions below:

(A1) $\forall\, n \in M$ it follows that $e_n \in \mathcal{D}(A)$;

(A2) $\exists\, n_0 \in M$ such that $Ae_{n_0} \neq 0$;

(A3) A is closable (see, e.g., [16, §2.6]).

Theorem 6 in [7] gives the mean-squared version of the Hardy-Littlewood-Pólya inequality.

Proposition 2. *Let $m \in \mathbb{Z}_+$, operators $B_0, B_1, \ldots, B_m$ satisfy conditions* (B1) – (B4), *operator A satisfys conditions* (A1) – (A3), *and*

$$\mathcal{Q}^2 := \sup_{\substack{f\in H':\\ \|f\|_{H'}\le 1}} \widetilde{\sum_{n\in M}} \frac{|(f, Ae_n)_{H'}|^2}{b_n} < \infty.$$

Then, for $x \in \mathcal{D}(A) \cap H_{\mathbf{B}}$, there holds true sharp inequality

$$\|Ax\|_{H'} \le \mathcal{Q} \cdot \|x\|_{\mathbf{B}}.$$

Furthermore, if A is closed then above inequality holds true for every $x \in H_{\mathbf{B}}$.

In what follows we additionally assume that the system $\{Ae_n\}_{n\in M}$ is orthogonal. In this case Proposition 2 can be simplified as follows.

Corollary 1. *Let $m \in \mathbb{Z}_+$, operators $B_0, B_1, \ldots, B_m$ satisfy conditions* (B1) – (B4), *operator A satisfy conditions* (A1) – (A3), *the system $\{Ae_n\}_{n\in M}$ be orthogonal, and*

$$\mathcal{Q} := \widetilde{\sup_{n\in M}} \frac{\|Ae_n\|_{H'}}{\sqrt{b_n}} < \infty. \tag{3}$$

Then, for $x \in \mathcal{D}(A) \cap H_{\mathbf{B}}$ (for $x \in H_{\mathbf{B}}$ if A is closed), there holds true sharp inequality

$$\|Ax\|_{H'} \le \mathcal{Q} \cdot \|x\|_{\mathbf{B}}.$$

3 Main results

Let $p, q \in \mathbb{Z}_+$, $p + q = m - 1$, and split operators $B_1, \ldots, B_m$ into two groups: operators $C_j : H \to H'$, $j = 0, 1, \ldots, p$, and $D_k : H \to H'$, $k = 0, 1, \ldots, q$. Let us consider the problem of the best approximation of operator $A : H_{\mathbf{C}} \to \mathbb{C}$ by linear bounded functionals on the class

$$W_{\mathbf{D}} := \{x \in \mathcal{D}(A) \cap H_{\mathbf{C}} \cap H_{\mathbf{D}} \,:\, \|x\|_{\mathbf{D}} \le 1\}.$$

For $N \geq 0$, consider operator $S_N : H_{\mathbf{C}} \to H'$ defined as follows

$$S_N x = \sum_{n \in M} x_n \cdot \min\left\{N\sqrt{c_n}, \|Ae_n\|_{H'}\right\} \cdot \phi_n, \quad x \in H_{\mathbf{C}},$$

where $\phi_n = Ae_n / \|Ae_n\|_{H'}$ if $\|Ae_n\|_{H'} \neq 0$ and $\phi_n = 0$ otherwise.

Let $C^* = \{n \in M \,:\, c_n \neq 0\}$, set $R_N = \{n \in C^* \,:\, N\sqrt{c_n} < \|Ae_n\|_{H'}\}$ and define

$$N^* := \inf\left\{N > 0 \,:\, \widetilde{\sup_{n \in R_N}} \frac{\|Ae_n\|_{H'} - N\sqrt{c_n}}{\sqrt{d_n}} < \infty\right\}.$$

Observe that $N^* \leq \mathcal{Q}$. Indeed, for every $N > \mathcal{Q}$ and $n \in R_N$, by condition (3) we have

$$\|Ae_n\|_{H'} \leq \mathcal{Q} \cdot \sqrt{b_n} \leq \mathcal{Q} \cdot (\sqrt{c_n} + \sqrt{d_n}) < N\sqrt{c_n} + N\sqrt{d_n},$$

leading to the estimate $N^* \leq N$ and, hence, to the desired inequality $N^* \leq \mathcal{Q}$.

Theorem 1. *Let $p, q \in \mathbb{Z}_+$, both groups of operators $C_0, \ldots, C_p$ and $D_0, \ldots, D_q$ satisfy conditions* (B1) – (B4)*, operator A satisfy conditions* (A1) – (A3)*, the system $\{Ae_n\}_{n \in M}$ be orthogonal and condition* (3) *for operators $C_0, \ldots, C_p, D_0, \ldots, D_q$ be fulfilled. Then, for $N > N^*$,*

$$E_N(A; W_{\mathbf{D}}) = \max\left\{\widetilde{\sup_{n \in M \setminus C^*}} \frac{\|Ae_n\|_{H'}}{\sqrt{d_n}};\ \widetilde{\sup_{n \in R_N}} \frac{\|Ae_n\|_{H'} - N\sqrt{c_n}}{\sqrt{d_n}}\right\},$$

and the operator S_N is extremal in problem (1).

Proof. Since operators C_j's satisfy condition (B2), we have $C^* \neq \varnothing$. Let us denote

$$N^{**} := \sup_{n \in C^*} \frac{\|Ae_n\|_{H'}}{\sqrt{c_n}}.$$

Clearly, N^{**} is well defined and $N^{**} \geq \mathcal{Q} \geq N^*$. First, we calculate the norm of S_N:

$$\begin{aligned}
\|S_N\|^2 &= \sup_{\|x\|_{\mathbf{C}} \leq 1} \sum_{n \in M} |x_n|^2 \cdot \min\left\{N^2 c_n, \|Ae_n\|_{H'}^2\right\} \\
&= \sup_{\|x\|_{\mathbf{C}} \leq 1} \sum_{n \in C^*} c_n |x_n|^2 \cdot \min\left\{N^2, \frac{\|Ae_n\|_{H'}^2}{c_n}\right\} \\
&= \sup_{n \in C^*} \min\left\{N^2, \frac{\|Ae_n\|_{H'}^2}{c_n}\right\} = \min\left\{N^2, (N^{**})^2\right\}.
\end{aligned}$$

Hence, $\|S_N\| \leq N$. Now, we estimate the deviation of S_N from operator A on the class $W_{\mathbf{D}}$. Using closability of A, we obtain

$$\begin{aligned}
U(A, S_N; W_{\mathbf{D}}) &= \sup_{x \in W_{\mathbf{D}}} \sqrt{\sum_{n \in M \setminus C^*} |x_n|^2 \cdot \|Ae_n\|_{H'}^2 + \sum_{n \in R_N} |x_n|^2 \left(\|Ae_n\|_{H'} - N\sqrt{c_n}\right)^2} \\
&= \widetilde{\sup_{n \in (M \setminus C^*) \cup R_N}} \frac{\|Ae_n\|_{H'} - N\sqrt{c_n}}{\sqrt{d_n}} = \max\left\{\widetilde{\sup_{n \in M \setminus C^*}} \frac{\|Ae_n\|_{H'}}{\sqrt{d_n}};\ \widetilde{\sup_{n \in R_N}} \frac{\|Ae_n\|_{H'} - N\sqrt{c_n}}{\sqrt{d_n}}\right\}.
\end{aligned}$$

The first sup in the later relation is finite by condition (3) and the second sup is finite for $N > N^*$ by definition of N^*.

Let us now prove that $E_N(A; W_{\mathbf{D}}) \geq U(A, S_N, W_{\mathbf{D}})$. Assume that

$$U(A, S_N; W_{\mathbf{D}}) = \widetilde{\sup_{n \in (M \setminus C^*) \cup R_N}} \frac{\|Ae_n\|_{H'} - N\sqrt{c_n}}{\sqrt{d_n}} > 0.$$

Then, for every $\varepsilon > 0$, there exists $n_0 \in (M \setminus C^*) \cup R_N$ such that

$$\frac{\|Ae_{n_0}\|_{H'} - N\sqrt{c_{n_0}}}{\sqrt{d_{n_0}}} > U(A, S_N; W_{\mathbf{D}}) - \varepsilon.$$

Combining above inequality with the lower estimate (2), we obtain

$$\begin{aligned} E_N(A; W_{\mathbf{D}}) &\geq \left\| A\left(e_{n_0}/\sqrt{d_{n_0}}\right)\right\|_{H'} - N\left\|\left(e_{n_0}/\sqrt{d_{n_0}}\right)\right\|_{H_{\mathbf{C}}} \\ &= \frac{\|Ae_{n_0}\|_{H'} - N\sqrt{c_{n_0}}}{\sqrt{d_{n_0}}} > U(A; S_N, W_{\mathbf{D}}) - \varepsilon, \end{aligned}$$

which leads to the desired estimate as $\varepsilon \to 0^+$. Case when $U(A, S_N; W_{\mathbf{D}}) = 0$ is trivial. □

Remark 1. *Under assumptions of Theorem 1, it follows directly from the definition of N^* that $E_N(A; W_{\mathbf{D}}) = \infty$ when $N < N^*$.*

Remark 2. *Under assumptions of Theorem 1, in the case $N = N^*$ the quantity $E_N(A; W_{\mathbf{D}})$ can be either finite or infinite. Indeed, let $H = H' = \ell_2$; $F_\alpha x = (1^\alpha x_1, 2^\alpha x_2, 3^\alpha x_3, \ldots)$, $\alpha \geq 0$, for $x \in \ell_2$; $A_\alpha = F_1 + F_\alpha$, with $\alpha \in [0, 1)$ and domain $\mathcal{D}(A_\alpha) = \{x \in \ell_2 : \sum_{j=1}^\infty j^2 |x_j|^2 < \infty\}$; $m = 1$ and $p = q = 0$; $C_0 = B_0 = F_1$ and $D_0 = B_1 = F_0$. Evidently, $N^* = 1$ and $R_1 = \mathbb{N}$, and $E_1(A_\alpha; W_{\mathbf{D}}) = 1$ if $\alpha = 0$ and $E_1(A_\alpha; W_{\mathbf{D}}) = +\infty$ if $\alpha \in (0, 1)$.*

Remark 3. *The quantity $E_N(A; W_{\mathbf{D}})$ is non-decreasing in N and tends to 0 as $N \to \infty$ only if $\widetilde{\sup}_{n \in M \setminus C^*} \frac{\|Ae_n\|_{H'}}{\sqrt{d_n}} = 0$.*

Remark 4. *In the case A is closed operator and under conditions of Theorem 1, $W_{\mathbf{D}} = \{x \in H_{\mathbf{C}} \cap H_{\mathbf{D}} : \|x\|_{\mathbf{D}} \leq 1\}$.*

4 Applications

Additive Hardy-Littlewood-Pólya type inequalities. S.B. Stechkin proposed problem (1) as an approach for solving the problem of finding sharp inequalities for the norms of derivatives. Let X, Y, Z be Banach spaces, $T : X \to Y$ and $S : X \to Z$ be operators with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, respectively, such that $\mathcal{D}(B) \subset \mathcal{D}(A)$. The problem of finding sharp additive inequalities between the norms of images of operators T and S consists of finding the set Γ of pairs of non-negative numbers (P, Q) such that sharp inequality holds

$$\|Tx\|_Y \leq P \cdot \|x\|_X + Q \cdot \|Bx\|_Z, \qquad x \in \mathcal{D}(B). \tag{4}$$

Here sharpness is understood in the sense that $Q = Q(P)$ is the lowest possible constant in inequality (4) under given $P \geq 0$. Consider the class $W = \{x \in \mathcal{D}(B) : \|Bx\|_Z \leq 1\}$. Proposition 1 provides a simple connection between problems (1) and (4): for every $P \geq 0$, $Q \leq E_P(A; W)$. We refer the reader to books [6, 12] on thorough overview of these inequalities of type (4) and related inequalities.

As a direct consequence from Theorem 1 and Remarks 1–2 we obtain.

Corollary 2. *Under conditions of Theorem 1, there holds sharp inequality*

$$\|Ax\|_{H'} \le N \cdot \|x\|_{\mathbf{C}} + E_N(A; W_{\mathbf{D}}) \cdot \|x\|_{\mathbf{D}}, \qquad x \in \mathcal{D}(A) \cap H_{\mathbf{C}} \cap H_{\mathbf{D}}.$$

Moreover, if $E_{N^*}(A; W_{\mathbf{D}}) < \infty$ *then* Γ *consists of all pairs* $(N, E_N(A; W_{\mathbf{D}}))$, $N \ge N^*$. *Otherwise,* Γ *consists of all pairs* $(N, E_N(A; W_{\mathbf{D}}))$, $N > N^*$.

In addition, let us compare N^* with the analogue of the sharp constant in the Markov-Nikolskii type inequality:

$$M^* := \sup_{x \in \cap_{j=0}^{q} \ker D_j \setminus \{\theta\}} \frac{\|Ax\|_{H'}}{\|x\|_{\mathbf{C}}} = \widetilde{\sup_{n \in M\,:\,d_n = 0}} \frac{\|Ae_n\|_{H'}}{\sqrt{c_n}}.$$

In case the first sup above is taken over empty set we set $M^* := 0$. From Corollary 2, for every $x \in \cap_{j=0}^{q} \ker D_j \setminus \{\theta\}$ and $N > N^*$, we have $\|Ax'\|_{H'} \le N\|x\|_{\mathbf{C}}$. Hence, $M^* \le N^*$. Under some additional assumptions, we can show that $N^* = M^*$.

Remark 5. *If* $\widetilde{\sup_{n \in M}} \frac{c_n}{d_n} < \infty$ *then* $N^* = M^*$.

Remark 6. *Let us give an example when* $M^* < N^*$. *Let* $H = H' = \ell_2$, $m = 1$, $p = q = 0$, $D_0 = B_1$ *with* $D_0 x = (0, x_2, x_3, \ldots)$; $Ax = (x_1/2, 2x_2, 3x_3, \ldots)$; $Cx = (x_1, (2 - \sqrt{2})x_2, (3 - \sqrt{3})x_3, \ldots)$. *Then* $M^* = 1/2$ *and* $N^* = 1$.

Best approximation of powers of the Laplace-Beltrami operator. Let $d \in \mathbb{N}$ and $(\mathcal{M}, g)$ be a d-dimensional C^∞ compact Riemannian manifold without boundary with volume element μ_g. Following notations in [9], we denote by Δ the extension of the negative Laplace-Beltrami operator in the space

$$L_2(\mathcal{M}) = \left\{ x : \mathcal{M} \to \mathbb{C} \,:\, \|x\|^2_{L_2(\mathcal{M})} := \int_{\mathcal{M}} |x(t)|^2 \, \mathrm{d}\mu_g(t) < \infty \right\}.$$

Δ has a discrete spectrum $0 = \mu_0^2 < \mu_1^2 \le \mu_2^2 \le \ldots$ without accumulations, where each eigenvalue occurs as many times as its multiplicity. We denote by $\{\varphi_j\}_{j=0}^\infty$ an L_2-orthonormal basis of C^∞-real eigenfunctions associated with the μ_j's. For $s \in \mathbb{R}$, define the Sobolev space $H^s(\mathcal{M})$ as follows:

$$H^s(\mathcal{M}) := \left\{ x : \mathcal{M} \to \mathbb{C} \,:\, \|x\|^2_{H^s(\mathcal{M})} := \sum_{j=1}^{\infty} |a_j|^2 \mu_j^{2s} < \infty \right\},$$

where $a_j = (x, \varphi_j)_{L_2(\mathcal{M})} = \int_{\mathcal{M}} x(t)\varphi_j(t)\, \mathrm{d}\mu_g(t)$, and set

$$H_0^s(\mathcal{M}) := \{x \in H^s(\mathcal{M}) \,:\, a_0 = 0\}.$$

Define the fractional power of Δ. For $x \in H_0^{2s}(\mathcal{M})$, set

$$\Delta^s x := \sum_{j=1}^{\infty} a_j \mu_j^{2s} \varphi_j.$$

Consider $H = H' = L_2(\mathcal{M})$, $m = 1$, $A = \Delta^k$, $B_0 = \mathrm{id}$, $B_1 = \Delta^r$ with $k, r \in \mathbb{R}$ such that $0 < k < r$. Denote by $0 < \lambda_1^2 < \ldots < \lambda_n^2 < \ldots$ distinct eigenvalues $\mu_1^2, \mu_2^2, \ldots$ of Δ such that $\lambda_1^2 = \mu_1^2$. Clearly,

$$W_{\mathbf{D}} = W_0^{2r}(\mathcal{M}) = \left\{ x \in H_0^{2r}(\mathcal{M}) \,:\, \|x\|_{H^{2r}(\mathcal{M})} \le 1 \right\}.$$

Taking into account the fact that the function $\frac{x^{2k} - N}{x^{2r}}$ attains its maximal value on $[N, +\infty)$ at the point $\left(\frac{r \cdot N}{r - k}\right)^{1/k}$, we obtain the following result.

Corollary 3. *Under above notations, $N^* = 0$ and for $N > 0$,*

$$E_N(\Delta^k; W_0^{2r}(\mathcal{M})) = \sup_{n\in\mathbb{N}\,:\,\lambda_n^{2k}>N} \frac{\lambda_n^{2k} - N}{\lambda_n^{2r}} = \max\left\{\frac{\lambda_{n_N}^{2k} - N}{\lambda_{n_N}^{2r}}; \frac{\lambda_{n_N+1}^{2k} - N}{\lambda_{n_N+1}^{2r}}\right\},$$

where $n_N = \max\left\{n \in \mathbb{N} \,:\, \lambda_n \le \left(\frac{r\cdot N}{r-k}\right)^{1/k}\right\}$.

Furthermore, for every $x \in H^{2r}(\mathcal{M})$, there holds sharp inequality

$$\|\Delta^k x\|_{L_2(\mathcal{M})} \le N \cdot \|x\|_{L_2(\mathcal{M})} + \max\left\{\frac{\lambda_{n_N}^{2k} - N}{\lambda_{n_N}^{2r}}; \frac{\lambda_{n_N+1}^{2k} - N}{\lambda_{n_N+1}^{2r}}\right\} \cdot \left\|\Delta^{2r} x\right\|_{L_2(\mathcal{M})},$$

which turns into equality on either φ_{n_N} or φ_{n_N+1}.

Remark 7. *In the similar way we can solve the problem on the best approximation of powers of positive self-adjoint unbounded operators with discrete spectrum or on the best approximation of integral powers of any self-adjoint unbounded operators with discrete spectrum.*

Best approximation of differentiation operator on the period. In case $\mathcal{M} = \mathbb{T}$ is the period of length 2π, Corollary 3 can be simplified to the following.

Corollary 4. *Under conditions of Corollary 3, for $N > 0$,*

$$E_N(D^k; W_0^r(\mathbb{T})) = \sup_{n\in\mathbb{N}\,:\,n^k>N} \frac{n^k - N}{n^r} = \max\left\{\frac{n_N^k - N}{n_N^r}; \frac{(n_N+1)^k - N}{(n_N+1)^r}\right\},$$

where $n_N = \left[\left(\frac{r\cdot N}{r-k}\right)^{1/k}\right]$.

Furthermore, for every $x \in H^r(\mathbb{T})$, there holds sharp inequality

$$\|x^{(k)}\|_{L_2(\mathbb{T})} \le N \cdot \|x\|_{L_2(\mathbb{T})} + \max\left\{\frac{n_N^k - N}{n_N^r}; \frac{(n_N+1)^k - N}{(n_N+1)^r}\right\} \cdot \left\|x^{(r)}\right\|_{L_2(\mathbb{T})},$$

which turns into equality on either $e^{in_N(\cdot)}$ or $e^{i(n_N+1)(\cdot)}$.

Note that sharp multiplicative inequalities for $\|\cdot\|_{L_2(\mathbb{T})}$-norms of derivatives of periodic functions were obtained in [11].

Best approximation of differentiation operator in spaces with weights. Denote by $L_{2,e^{-t^2}}(\mathbb{R})$ the space of Lebesgue measurable functions $x : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|x\| := \left(\int_{\mathbb{R}} |x(t)|^2 e^{-t^2} \mathrm{d}t\right)^{1/2} < \infty.$$

It is well known that the system of Hermite polynomials $\{H_k\}_{k=1}^\infty$ is orthonormal basis in $L_{2,e^{-t^2}}(\mathbb{R})$. By $L_{2,e^{-t^2}}^r(\mathbb{R})$ we denote the space of functions $x \in L_{2,e^{-t^2}}(\mathbb{R})$ having locally absolutely continuous derivative $x^{(r-1)}$ and such that $x^{(r)} \in L_{2,e^{-t^2}}(\mathbb{R})$. It is well-known that $H_n^{(k)} = \beta_{n,k} H_{n-k}$, where $\beta_{n,k} = (2^k n(n-1)\dots(n-k+1))^{1/2}$.

Consider $H = H' = L_{2,e^{-t^2}}(\mathbb{R})$, $m = 1$, $k, r \in \mathbb{N}$, $k < r$, $A = D^k$, $B_0 = C_0 = \mathrm{id}$, $B_1 = D_0 = D^r$. Then $N^* = M^* = \beta_{r-1,k}$ and $W_{\mathbf{D}} = W_{2,e^{-t^2}}^r(\mathbb{R}) = \left\{x \in L_{2,e^{-t^2}}^r(\mathbb{R}) \,:\, \|x^{(r)}\| \le 1\right\}$.

It is not difficult to see that the function

$$g_N(x) = \frac{\beta_{x,k} - N}{\beta_{x,r}}, \qquad x > 0,$$

has a unique positive maximum on $(0,\infty)$, which is attained at some point ξ_N. Denote by $n_N = n_{N;k,r} := [\xi_N]$ the integer part of ξ_N.

Corollary 5. *Under above assumptions, for $N \geq \beta_{r-1,k}$,*

$$E_N(D^k; W^r_{2,e^{-t^2}}(\mathbb{R})) = \sup_{n\in\mathbb{N}\,:\,\beta_{n,k}>N} \frac{\beta_{n,k} - N}{\beta_{n,r}} = \max\{g_N(n_N); g_N(n_N+1)\}.$$

Furthermore, for every $x \in L^r_{2,e^{-t^2}}(\mathbb{R})$, there holds sharp inequality

$$\|x^{(k)}\| \leq N \cdot \|x\| + \max\{g_N(n_N); g_N(n_N+1)\} \cdot \|x^{(r)}\|, \tag{5}$$

which turns into equality on one of the Hermite polynomials H_{n_N} or H_{n_N+1}.

Note that sharp multiplicative analogues of inequality (5) under some additional assumptions were obtained in [8].

Best approximation of functions of self-adjoint operators. Let us follow notations and assumptions of paper [4]. Let $\Delta : H \to H$ be a self-adjoint unbounded operator with domain $\mathcal{D}(\Delta)$. From the spectral theorem (see [16, Ch. XI, §5, 6]) it follows that there exists a partition of unity $\{E_t\}$, $t \in \mathbb{R}$, such that

$$\Delta = \int_{-\infty}^{+\infty} t\,\mathrm{d}E_t.$$

If $\varphi : \mathbb{R} \to \mathbb{C}$ is a continuous function then the operator $\varphi(\Delta)$ is well-defined and can be represented as

$$\varphi(\Delta) = \int_{-\infty}^{+\infty} \varphi(t)\,\mathrm{d}E_t.$$

Now, consider the functions $A = \varphi(\Delta)$ and $D_0 = B_1 = \psi(\Delta)$ of operator Δ, where φ and ψ are continuous complex-valued functions on $\mathbb{R}$ such that $|\varphi(t)|$ and $|\psi(t)|$ are even and strictly increasing to $+\infty$ on $(0,\infty)$. Assume $|\varphi(t)|^2 = F(|\psi(t)|^2)$, where F is a strictly increasing, concave function with $F(0) = 0$. In addition, let Δ be such that, for every $0 \leq s < t \leq \infty$,

$$(E_t - E_s)\mathcal{D}(\psi(\Delta)) \neq \{\theta\}. \tag{6}$$

Let also $m = 1$, $p = q = 0$, $C_0 = B_0 = \mathrm{id}_H$, and $W_{\mathbf{D}} = \{x \in H \,:\, \|\psi(\Delta)x\|_H \leq 1\}$.

For any $b > 0$, consider the function

$$\varphi_b(t) = \begin{cases} \varphi(t) - \frac{\varphi(b)}{\psi(b)} \cdot \psi(t), & |t| \leq b, \\ 0, & |t| > b, \end{cases} \qquad \text{and set } N(b) = \max_t |\varphi_b(t)|.$$

Corollary 6. *Under above notations let $\frac{|\varphi(t))|}{|\psi(t)|}$ be non-increasing. Then, for every $b > 0$,*

$$E_{N(b)}(A; W_{\mathbf{D}}) = \frac{|\varphi(b)|}{|\psi(b)|},$$

and $Q_b = \varphi_b(\Delta)$ is the operator of the best approximation of A on the class $W_{\mathbf{D}}$ with $\|Q_b\| \leq N(b)$.

Proof. From (6) we conclude that $E_t \neq E_s$, for every $0 \le s < t \le \infty$. Since $\{E_t\}$ are projectors, their images $R_t = E_t(H)$ are closed subspaces in H and are such that $R_s \subset R_t$ and $R_s \neq R_t$. Hence, for every $h > 0$ and $n \in \mathbb{Z}_+$, there exists $z_n \in R_{(n+1)h} \cap R_{nh}^{\perp}$ having unit norm $\|z_n\|_H = 1$. Note that $(E_t z_n, z_n)_H = 0$, for every $t \le n$, and $(E_t z_n, z_n)_H = 1$, for every $t \ge n+1$. Evidently, A is well-defined on every z_n, as

$$\|Az_n\|_H^2 = \int_{-\infty}^{+\infty} |\varphi(t)|^2 \,\mathrm{d}(E_t z_n, z_n)_H = \int_{nh}^{(n+1)h} |\varphi(t)|^2 \,\mathrm{d}(E_t z_n, z_n)_H \le |\varphi((n+1)h)|^2.$$

Similar arguments apply for operator B. Note also that $\{Az_n\}$ and $\{Bz_n\}$ are orthogonal.

For $h > 0$, denote by H_h the subspace of H having orthonormal basis $\{z_n\}_{n\in\mathbb{Z}_+}$ and set $W_{\mathbf{D},h} = W_{\mathbf{D}} \cap H_h$. Then, by Theorem 1,

$$E_{N(b)}(A; W_{\mathbf{D}}) \ge E_{N(b)}(A; W_{\mathbf{D},h}) = \sup_{n\in R_{N(b)}} \frac{\|Az_n\|_H - N(b)}{\|Bz_n\|_H} \ge \sup_{n\in R_{N(b)}} \frac{|\varphi(nh)| - N(b)}{|\psi((n+1)h)|}.$$

For every sufficiently small $\varepsilon > 0$, let $y_\varepsilon > 0$ be such that $|\varphi_b(y_\varepsilon)| > N(b) - \varepsilon$. Choose $h = h(\varepsilon)$ and $n = n(\varepsilon) \in \mathbb{Z}_+$ to be such that $y_\varepsilon = nh$ and $|\psi((n+1)h)| \le |\psi(y_\varepsilon)| + \varepsilon$. Then

$$E_{N(b)}(A; W_{\mathbf{D}}) \ge \frac{|\varphi(y_\varepsilon)| - N(b)}{|\psi(y_\varepsilon)| + \varepsilon} \ge \frac{|\varphi(y_\varepsilon) - \varphi_b(y_\varepsilon)| - \varepsilon}{|\psi(y_\varepsilon)| + \varepsilon} = \frac{\frac{|\varphi(b)|}{|\psi(b)|} \cdot |\psi(y_\varepsilon)| - \varepsilon}{|\psi(y_\varepsilon)| + \varepsilon}.$$

Letting $\varepsilon \to 0$, we obtain the desired lower estimate $E_{N(b)}(A; W_{\mathbf{D}}) \ge \frac{|\varphi(b)|}{|\psi(b)|}$.

The extremality of operator S_b was proved in [4, Theorem 5]. Alternatively, the operator

$$Q_b' = \int_{-\infty}^{+\infty} \frac{\varphi(t)}{|\varphi(t)|} \cdot \min\{N(b), |\varphi(t)|\} \,\mathrm{d}E_t,$$

where $\frac{\varphi(t)}{|\varphi(t)|} := 0$ when $\varphi(t) = 0$, is also extremal and its construction in some sense similar to the construction of the extremal operator in Theorem 1. □

Remark 8. *Assertion of Corollary 6 remains true is we relax the assumption that* $|\varphi(t)|^2 = F(|\psi(t)|^2)$, *where* F *is a strictly increasing, concave function with* $F(0) = 0$ *to only requiring that* $\operatorname{supp}\psi \subset \operatorname{supp}\varphi$. *Furthermore, the assumption of continuity of functions* φ *and* ψ *can be relaxed to right-continuity of function* ψ.

Note that Corollary 6 contains the classical result on the best approximation of differentiation operator of integral order in space $L_2(\mathbb{R})$ by Yu.N. Subbotin and L.V. Taikov [15] and generalizes [5, Theorem 3] and [4, Theorem 5]. Also, in a similar way as Corollary 6, the result of paper [10] for the best approximation of integral powers of normal operators can be obtained from Theorem 1.

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Бабенко В.Ф., Парафінович Н.В., Скороходов Д.С. *Найкраще наближення замкнених операторів обмеженими операторами в гільбертовому просторі* // Карпатські матем. публ. — 2022. — Т.14, №2. — C. 453–463.

Розв'язана задача найкращого наближення замкнених операторів лінійними обмеженими операторами в гільбертовому просторі у припущенні, що оператори зберігають ортогональність базису гільбертового простору. Як наслідок, отримана точна адитивна нерівність типу Харді-Літльвуда-Пойа для декількох замкнених операторів. Наведені застосування цих результатів у конкретних ситуаціях: для найкращого наближення степенів оператора Лапласа-Бельтрамі на класах функцій, визначених на замкнених ріманових многовидах, для найкращого наближення операторів диференціювання на класах періодичних функцій та функцій, визначених на дійсній осі з вагою e^{-x^2}, для найкращого наближення самоспряжених операторів в гільбертових просторах.

Ключові слова і фрази: найкраще наближення операторів, проблема Стечкіна, нерівності типу Колмогорова, самоспряжений оператор, оператор Лапласа-Бельтрамі, замкнений оператор.