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δ-Almost Ricci soliton on 3-dimensional trans-Sasakian manifold

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In this paper, we consider δ-almost Ricci soliton on 3-dimensional trans-Sasakian manifold admitting η-parallel Ricci tensor. We give some conditions for $P \cdot \phi = 0$, $P \cdot S = 0$, $Q \cdot P = 0$. Also, we show that there is almost pseudo symmetric δ-almost Ricci soliton on 3-dimensional trans-Sasakian manifold admitting cyclic Ricci tensor. Finally, we give an example for verifying the obtained results.

Key words and phrases: Ricci soliton, δ-almost Ricci soliton, trans-Sasakian manifold.

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1 Introduction

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively. Ricci solitons have been studied by many authors (see [1–4, 11, 12]).

As a generalization of Ricci soliton, the notion of almost Ricci soliton by considering the constant λ as a smooth function introduced in [10]. Recently, J.N. Gomes et. al. extended the term of almost Ricci soliton to δ-almost Ricci soliton (briefly δ-ARS) on a complete Riemannian manifold by

$$\frac{\delta}{2}\pounds_V g + S + \lambda g = 0,$$

where $\delta : M \to \mathbb{R}$ is a smooth function [9]. In [8], δ-ARS was studied on K-contact metric manifolds.

Also, the projective curvature tensor P is given by

$$P(U,V)Z = R(U,V)Z - \frac{1}{n-1}\big(g(V,Z)QU - g(U,Z)QV\big), \tag{2}$$

where Q is the Ricci operator (see [14]).

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2 Preliminaries

Let M be a connected almost contact metric manifold of dimension $2n+1$ with an almost contact metric structure (ϕ, ξ, η, g) and g is a Riemannian metric such that

$$\phi^2(U) = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi U) = 0, \quad \phi\xi = 0, \tag{3}$$
$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),$$

for all vector fields U, V on M.

If there are smooth functions α, β on (M, ϕ, ξ, η, g) satisfying

$$(\nabla_U \phi)\, V = \alpha\big(g(U, V)\xi - \eta(V)U\big) + \beta\big(g(\phi U, V)\xi - \eta(V)\phi U\big), \tag{4}$$

then it is called a *trans-Sasakian manifold* (see [6]).

By using (3) with (4), we get

$$\nabla_U \xi = -\alpha\phi U + \beta\big(U - \eta(U)\xi\big). \tag{5}$$

If we consider α and β to be constant, then for a 3-dimensional trans-Sasakian manifold we get

$$\begin{aligned} R(U, V)Z &= \left(\frac{r}{2} - 2\left(\alpha^2 - \beta^2\right)\right)\left(g(V, Z)U - g(U, Z)V\right) \\ &\quad + \left(\frac{r}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\Big(g(U, Z)\eta(V)\xi - g(V, Z)\eta(U)\xi \\ &\qquad + \eta(U)\eta(Z)V - \eta(V)\eta(Z)U\Big), \\ S(U, V) &= \left(\frac{r}{2} - \left(\alpha^2 - \beta^2\right)\right)g(U, V) - \left(\frac{r}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\eta(U)\eta(V), \\ QU &= \left(\frac{r}{2} - \left(\alpha^2 - \beta^2\right)\right)U - \left(\frac{r}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\eta(U)\xi. \end{aligned} \tag{6}$$

Proposition 1. *If a 3-dimensioanl trans-Sasakian manifold M admits a δ-almost Ricci soliton, then $\lambda = -2\left(\alpha^2 - \beta^2\right)$.*

Proof. Let $(g, \xi, \delta, \lambda)$ be a δ-almost Ricci soliton on M. From (1), we can write

$$\frac{\delta}{2}(\pounds_\xi g)(U, Y) + S(U, Y) + \lambda g(U, Y) = 0. \tag{7}$$

Also, in view of (5), we get

$$\begin{aligned} (\pounds_\xi g)(U, Y) &= g(\nabla_U \xi, V) + g(U, \nabla_V \xi) \\ &= -\alpha g(\phi U, Y) + \beta g(U, Y) - \beta\eta(U)\eta(Y) \\ &\quad - \alpha g(U, \phi Y) + \beta g(U, Y) - \beta\eta(U)\eta(Y) \\ &= 2\beta\big(g(U, Y) - \eta(U)\eta(Y)\big). \end{aligned}$$

Using the above equality in (7), we arrive at

$$S(U, Y) = (-\lambda - \delta\beta)\, g(U, Y) + \delta\beta\eta(U)\eta(Y). \tag{8}$$

Taking $U = \xi = Y$ in (8) and using (6), we obtain

$$\lambda = -2\left(\alpha^2 - \beta^2\right). \tag{9}$$

□

Corollary 1. *A 3-dimensional trans-Sasakian manifold with δ-almost Ricci soliton is an η-Einstein manifold.*

3 δ-almost Ricci soliton on a 3-dimensional trans-Sasakian manifold with η-parallel Ricci tensor

Definition 1 ([13]). *A 3-dimensional trans-Sasakian manifold M is called to be η-parallel Ricci tensor if*

$$g\big((\nabla_U Q)Y, V\big) = 0$$

for any U, V, Y on M.

Now, we consider δ-ARS on M with η-parallel Ricci tensor. From (8), we have

$$QY = (-\lambda - \delta\beta)\, Y + \delta\beta\eta(Y)\xi.$$

Taking covariant differentiation of the above equation with respect to U, we obtain

$$\begin{aligned}(\nabla_U Q)Y = &- (U\lambda)Y - (U\delta)\beta Y - (U\beta)\delta Y + (U\delta)\beta\eta(Y)\xi \\ &+ (U\beta)\delta\eta(Y)\xi + \delta\beta\Big(((\nabla_U\eta)Y)\xi - \eta(Y)\nabla_U\xi\Big).\end{aligned}$$

Using (5) in the above equation, we get

$$\begin{aligned}(\nabla_U Q)Y = &- (U\lambda)Y - (U\delta)\beta Y + (U\delta)\beta\eta(Y)\xi \\ &+ \delta\beta\Big(- \alpha g(\phi U, Y)\xi + \beta g(U, Y)\xi - \alpha\eta(Y)\phi U + \beta\eta(Y)U - 2\beta\eta(U)\eta(Y)\xi\Big).\end{aligned}$$

Taking inner product with V, we obtain

$$\begin{aligned}g\big((\nabla_U Q)Y, V\big) = &- (U\lambda)g(Y, V) - (U\delta)\beta g(Y, V) + (U\delta)\beta\eta(Y)\eta(V) \\ &+ \delta\beta\Big(- \alpha g(\phi U, Y)\eta(V) + \beta g(U, Y)\eta(V) - \alpha g(\phi U, V)\eta(Y) \\ &+ \beta g(U, V)\eta(Y) - 2\beta\eta(U)\eta(Y)\eta(V)\Big) = 0.\end{aligned}$$

Putting $Y = \xi$ in the above equation, we get

$$-(U\lambda)\eta(V) + \delta\beta\big(- \alpha g(\phi U, V) + \beta g(U, V) - \beta\eta(U)\eta(V)\big) = 0. \tag{10}$$

Again taking $V = \xi$ in (10), we obtain

$$U\lambda = 0, \tag{11}$$

which gives that U is constant. By use of (11) in (10), we can state

$$\delta\beta\big(- \alpha g(\phi U, V) + \beta g(U, V) - \beta\eta(U)\eta(V)\big) = 0.$$

Putting $U = V = e_i$, $1 \leq i \leq 3$, where e_i denotes a set of orthonormal vector fields of M, we get $\delta\beta^2 = 0$, which yields $\delta = 0$. Thus we arrive at

$$S(U, Y) = -\lambda g(U, Y).$$

So, we have following result.

Theorem 1. *If a 3-dimensional trans-Sasakian manifold admits a δ-ARS with η-parallel Ricci tensor then it becomes an Einstein manifold.*

4 The curvature conditions on a 3-dimensional trans-Sasakian manifold admitting a δ-almost Ricci soliton

Firstly, we give the curvature condition $P \cdot \phi = 0$ on M admitting a δ-ARS. We know that if $(P \cdot \phi)(X, Y)U = 0$, then we have

$$P(X, Y)\phi U - \phi P(X, Y)U = 0.$$

If we take $U = \xi$ in the above equation, we get

$$\phi P(X, Y)\xi = 0.$$

Using (2) in the latter equality, we obtain

$$\left(\left(\alpha^2 - \beta^2\right) + \frac{\lambda + \delta\beta}{2}\right)\left(\eta(Y)\phi X - \eta(X)\phi Y\right) = 0. \tag{12}$$

Replacing X by ϕX in (12) and using (3), we get

$$\left(\left(\alpha^2 - \beta^2\right) + \frac{\lambda + \delta\beta}{2}\right)\left(- X + \eta(X)\xi\right)\eta(Y) = 0.$$

Putting $Y = \xi$ and replacing X by ϕX in the above equation, we arrive at

$$\left(\left(\alpha^2 - \beta^2\right) + \frac{\lambda + \delta\beta}{2}\right)\phi X = 0. \tag{13}$$

Taking inner product with W in (13), we obtain

$$\left(\left(\alpha^2 - \beta^2\right) + \frac{\lambda + \delta\beta}{2}\right) g(\phi X, W) = 0.$$

It follows that $\lambda + \delta\beta = -2\left(\alpha^2 - \beta^2\right)$, which gives $\delta = 0$. Using this value in (1), we obtain

$$S(U, V) = -\lambda g(U, V).$$

So, we get the following assertion.

Theorem 2. *If a 3-dimensional trans-Sasakian manifold admits a δ-ARS and satisfies $P \cdot \phi = 0$, then the manifold is an Einstein manifold.*

Now, we examine the curvature conditions $P \cdot S = 0$ on M with a δ-ARS. In this case, we have

$$S\big(P(\xi, Y)Z, V\big) + S\big(Z, P(\xi, Y)V\big) = 0.$$

In view of (2) in the above equation, we get

$$\left(\alpha^2 - \beta^2\right)\Big(S(\xi, V)g(Y, Z) - S(Y, V)\eta(Z) + S(\xi, Z)g(Y, V) - S(Y, Z)\eta(V)\Big) - \frac{1}{2}\Big(\delta\beta\left(g(Y, V)\eta(Z) + g(Y, Z)\eta(V)\right) - 2\delta\beta\eta(Z)\eta(V)\eta(Y)\Big) = 0.$$

Taking $V = \xi$ in the latter equality and using (8), we obtain

$$\left(\alpha^2 - \beta^2\right) S(Y, Z) = \left(-\lambda\left(\alpha^2 - \beta^2\right) - \frac{\delta\beta}{2}\right) g(Y, Z) + \frac{\delta\beta}{2}\eta(Y)\eta(Z).$$

Therefore we can state following result.

Theorem 3. *If a 3-dimensional trans-Sasakian manifold admits a δ-ARS and satisfies $P \cdot S = 0$, then the manifold is an η-Einstein manifold.*

Finally, we consider the curvature condition $Q \cdot P = 0$ on M with a δ-ARS. Thus, we get

$$(Q \cdot P)(X, V)W = 0.$$

The latter equation implies

$$Q\big(P(X, V)W\big) - P(QX, V)W - P(X, QV)W - P(X, V)QW = 0.$$

Using (8) in the above equation, we obtain

$$-2\big((\lambda + \delta\beta)P(X, V)W\big) + \delta\beta\Big(\eta\big(P(X, V)W\big)\xi - P(\xi, V)W\eta(X) \\ - P(X, \xi)W\eta(V) - P(X, V)\xi\eta(W)\Big) = 0.$$

Taking $W = \xi$ in the previous equality, we get

$$\big(- 2(\lambda + \delta\beta) - \delta\beta\big)P(X, V)\xi + \delta\beta\Big(\eta\big(P(X, V)\xi\big)\xi - P(\xi, V)\xi\eta(X) - P(X, \xi)\xi\eta(V)\Big) = 0.$$

Now, using (2) in the above equation, we obtain

$$-2\big((\lambda + \delta\beta) + \delta\beta\big)\left(\left(\alpha^2 - \beta^2\right) + \frac{\lambda + \delta\beta}{2}\right)\big(\eta(V)X - \eta(X)V\big) = 0. \tag{14}$$

Replacing X by ϕX and taking $V = \xi$ in (14), we get

$$-2\big((\lambda + \delta\beta) + \delta\beta\big)\left(\left(\alpha^2 - \beta^2\right) + \frac{\lambda + \delta\beta}{2}\right)\phi X = 0.$$

Taking inner product with Y in the above equation, we find

$$-2\big((\lambda + \delta\beta) + \delta\beta\big)\left(\left(\alpha^2 - \beta^2\right) + \frac{\lambda + \delta\beta}{2}\right)g(\phi X, Y) = 0,$$

from which we arrive at $\lambda = 0$ or $\lambda + \delta\beta = -2\left(\alpha^2 - \beta^2\right)$. In view of (9), we get $\delta = 0$. Using this equation in (1), we get following result.

Theorem 4. *If a 3-dimensional trans-Sasakian manifold admits a δ-ARS and satisfies $Q \cdot P = 0$, then the Ricci soliton steady or the manifold is an Einstein manifold.*

5 Almost pseudo Ricci symmetric δ-almost Ricci soliton

Definition 2 ([5])**.** *A 3-dimensional trans-Sasakian manifold M is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S is not identically zero and satisfies*

$$(\nabla_U S)(Y, V) = \big(A(U) + B(U)\big)S(Y, V) + A(Y)S(U, V) + A(V)S(U, Y), \tag{15}$$

where A and B are two non-zero 1-forms given by

$$A(U) = g(U, \rho_1), \quad B(U) = g(U, \rho_2). \tag{16}$$

In view of (15), we can write

$$(\nabla_U S)(Y,V) + (\nabla_Y S)(V,U) + (\nabla_V S)(Y,U)$$
$$= \big(3A(U) + B(U)\big)S(Y,V) + \big(3A(Y) + B(Y)\big)S(V,U) + \big(3A(V) + B(V)\big)S(Y,U).$$

If M admits a cyclic Ricci tensor, the above equation reduces to

$$\big(3A(U) + B(U)\big)S(Y,V) + \big(3A(Y) + B(Y)\big)S(V,U) + \big(3A(V) + B(V)\big)S(Y,U) = 0. \quad (17)$$

Taking $V = \xi$ in (17) and using (8) with (16), we get

$$-\lambda\big(3A(U) + B(U)\big)\eta(Y) - \lambda\big(3A(Y) + B(Y)\big)\eta(U) + \big(3\eta(\rho_1) + \eta(\rho_2)\big)S(Y,U) = 0. \quad (18)$$

Similarly taking $Y = \xi$ in (18) and by use of (8) with (16), we find

$$-\lambda\big(3A(U) + B(U)\big) - 2\lambda\big(3\eta(\rho_1) + \eta(\rho_2)\big)\eta(U) = 0. \quad (19)$$

Substituting $U = \xi$ in (19) and using (8) with (16), we obtain

$$3\eta\,(\rho_1) + \eta\,(\rho_2) = 0.$$

Thus, we have following result.

Theorem 5. *There is no almost pseudo Ricci symmetric δ-ARS on a 3-dimensional trans-Sasakian manifold M admitting cyclic Ricci tensor, unless $3A + B$ vanishes everywhere on M.*

Example 1. *Let us consider 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) standard coordinates in $\mathbb{R}^3$. Define vector fields $\{\omega_i : 1 \leq i \leq 3\}$ on M given [7] by*

$$\omega_1 = e^{2z}\partial x, \quad \omega_2 = e^{2z}\partial y, \quad \omega_3 = \partial z.$$

If we consider $\omega_3 = \xi$ such that

$$\eta(U) = g(U,\omega_3),$$

then we have $\eta(\xi) = 1$. Also $\phi(\omega_1) = \omega_2$, $\phi(\omega_2) = \omega_1$, $\phi(\omega_3) = 0$. Thus, (g,ϕ,ξ,η) defines an almost contact metric structure. Now, we have

$$[\omega_1,\omega_1] = 0, \quad [\omega_1,\omega_3] = -2\omega_1, \quad [\omega_2,\omega_3] = -2\omega_2.$$

Using Kozsul's formula we obtain

$$\begin{array}{lll} \nabla_{\omega_1}\omega_1 = 2\omega_3, & \nabla_{\omega_1}\omega_2 = 0, & \nabla_{\omega_1}\omega_3 = -2\omega_1 \\ \nabla_{\omega_2}\omega_1 = 0, & \nabla_{\omega_2}\omega_2 = 2\omega_3, & \nabla_{\omega_2}\omega_3 = -2\omega_2 \\ \nabla_{\omega_3}\omega_1 = 0, & \nabla_{\omega_3}\omega_2 = 0, & \nabla_{\omega_3}\omega_3 = 0. \end{array}$$

In view of this equation for Riemannian curvature tensor R and Ricci tensor S, we arrive at

$$\begin{array}{ll} R(\omega_1,\omega_2)\omega_2 = -4\omega_1, & R(\omega_1,\omega_3)\omega_3 = -4\omega_1, \\ R(\omega_2,\omega_3)\omega_3 = -4\omega_2, & R(\omega_3,\omega_1)\omega_1 = -4\omega_2, \\ R(\omega_3,\omega_2)\omega_2 = 4\omega_2, & R(\omega_2,\omega_1)\omega_1 = 4\omega_3, \end{array}$$

and

$$S(\omega_1,\omega_1) = 0, \quad S(\omega_2,\omega_2) = 0, \quad S(\omega_3,\omega_3) = -8.$$

By use of above values on (7), finally we get $\lambda = 8$ and $\delta = -8$. Hence we can state $(g,\xi,8,-8)$ defines a δ-ARS on trans-Sasakian manifold of type $(\alpha,1)$.

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Асет Т. *δ-Майже солітон Річчі на тривимірному транс-Сасакяновому многовиді* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 558–564.

У цій роботі ми розглядаємо δ-майже солітон Річчі на тривимірному транс-Сасакяновому многовиді, який допускає η-паралельний тензор Річчі. Ми наводимо деякі умови для $P \cdot \phi = 0$, $P \cdot S = 0$, $Q \cdot P = 0$. Також показано, що на тривимірному транс-Сасакяновому многовиді, який допускає циклічний тензор Річчі, існує майже псевдосиметричний δ-майже солітон Річчі. Насамкінець, ми наводимо приклад для перевірки отриманих результатів.

Ключові слова і фрази: солітон Річчі, δ-майже солітон Річчі, транс-Сасакяновий многовид.