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On wavelet type Bernstein operators

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This paper deals with construction and studying wavelet type Bernstein operators by using the compactly supported Daubechies wavelets of the given function f. The basis used in this construction is the wavelet expansion of the function f instead of its rational sampling values $f\left(\frac{k}{n}\right)$. After that, we investigate some properties of these operators in some function spaces.

Key words and phrases: Bernstein polynomial, interpolation, wavelet, compactly supported Daubechies wavelet, approximation.

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Introduction

For a bounded real valued function $f \in B[0,1]$ defined on the interval $[0,1]$, the Bernstein operators $B_n(f)$, $n \geq 1$, are defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad n \geq 1, \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $0 \leq x \leq 1$, is the Bernstein basis.

In approximation theory, these operators and some of their modifications are very well-known. Especially, since the classical Bernstein operators (1) cannot be used for $L^p[0,1]$, $1 \leq p < \infty$, approximation, to obtain some positive results for these functions by using the Bernstein operators, their Kantorovich and Durrmeyer type versions were considered.

The goal of this study is to find a positive solution to the approximation (or superposition) problem for operators in some general function spaces by using the effects and relations between different function spaces provided by the wavelets. In other words, we will propose a generalization and extension of the theory of approximation by introducing an integral operator, called wavelet type operators (see [1,4–7]). The new operators are more flexible than the previous ones and they are at least a natural extension of the classical Bernstein operators, and their Kantorovich and Durrmeyer type modifications.

1 Preliminaries and auxiliary results

In this section, we recall some notations and background material, namely Daubechies' compactly supported wavelets [2,3], used throughout the paper.

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As usual, we denote by $C[0,1]$ the Banach space of continuous functions $u : [0,1] \to \mathbb{R}$ with norm

$$\|u\| = \sup \{|u(x)| : x \in [0,1]\},$$

$L^p[0,1]$, $1 \leq p \leq \infty$, denotes the space of Lebesgue measurable functions f satisfying some conditions related with the pth power. The norms of $f \in L^p[0,1]$, $1 \leq p \leq \infty$, are given by

$$\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt \right)^{1/p} < \infty, \qquad \|f\|_\infty = \operatorname{ess\,sup} \{|f(x)| : x \in [0,1]\}.$$

Definition 1 (boxcar function)**.** *Let $I \subset \mathbb{R}$ be a single interval and A is a fixed positive constant. A function $\phi : \mathbb{R} \to \{0, A\}$, defined as*

$$\phi(x) := \begin{cases} A, & x \in I, \\ 0, & x \notin I, \end{cases}$$

is called boxcar function.

Definition 2 (scale function)**.** *A scale function is a special boxcar function defined as*

$$\phi(x) = \begin{cases} 1, & x \in [0,1), \\ 0, & x \notin [0,1). \end{cases} \tag{2}$$

Clearly, a scale function can be also define by using Heaviside unit step function

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

namely

$$\phi(x) = H(x) - H(x-1).$$

Definition 3 ([2, 3])**.** *A multiresolution analysis (MRA) is a sequence $(Vj)_{j \in Z}$ of closed subspaces of $L^2(\mathbb{R})$ such that the following statements hold:*

i) *V_j is a set of all $f \in L^2(\mathbb{R})$ which are constant on 2^{-j} length intervals and*

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \subset .. \subset L_2(\mathbb{R}),$$

$$\overline{\bigcup_j V_j} = L_2(\mathbb{R});$$

ii)

$$\begin{aligned} &\forall \; j,k \in \mathbb{Z}, \;\; f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}, \\ &\forall \; k \in \mathbb{Z}, \;\; f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0, \\ &\forall \; j,k \in \mathbb{Z}, \;\; f(x) \in V_j \Longleftrightarrow f(x - 2^{-j}k) \in V_j; \end{aligned}$$

iii)

$$\bigcap_j V_j = \{0\}.$$

Definition 4 (Wavelet)**.** *A wavelet is a small wave which oscillates and decays in the time domain. A wavelet basis set starts with two orthogonal functions: the scaling function or father wavelet $\phi(t)$ and the wavelet function or mother wavelet $\psi(t)$. By scaling and translation of these two orthogonal functions we obtain a complete basis set. The scaling and wavelet functions, respectively, satisfy*

$$\int_{-\infty}^{\infty} \phi(t)\, dt = 1, \qquad \int_{-\infty}^{\infty} \psi(t)\, dt = 0.$$

These two functions have finite energy, namely $\phi, \psi \in L^2(\mathbb{R})$, and orthogonal.

In general, the wavelets refer to the set of family of orthonormal functions of the form

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \quad a > 0,\ b \in \mathbb{R}, \tag{3}$$

where ψ is the basic wavelet.

Example 1 (Haar wavelet)**.** *The simplest wavelet is known as the Haar wavelet defined as*

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \textit{otherwise} \end{cases}$$

with the scaling function (2).

Clearly, Haar wavelets constitutes an orthonormal system for the space of square-integrable functions on the real line. Since Haar wavelet is not continuous and therefore not differentiable, it is suitable for representing discrete signals not for representing smooth signals or functions.

In the present study, we consider orthonormal bases of wavelets in $L^2(\mathbb{R})$, and assume that there is a scaling function (father wavelet) $\phi(t)$ whose translates $\{\phi(t-n)\}$ are orthogonal and the mother wavelet $\psi(t)$ based on the father wavelet $\phi(t)$ gives rise to the orthonormal basis $\psi_{j,k}(t)$ of $L^2(\mathbb{R})$, where

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k). \tag{4}$$

Hence, by using a multiresolution analysis (MRA), each $f \in L^2(\mathbb{R})$ has the following representation

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} b_{j,k}\psi_{j,k}(x),$$

called wavelet expansion of $f \in L^2(\mathbb{R})$, where $b_{j,k}$ are wavelet coefficients defined by

$$b_{j,k} = \langle f(x), \psi_{j,k}(x)\rangle = \int_{\mathbb{R}} f(x)\overline{\psi_{j,k}(x)}dx = 2^{j/2}\int_{\mathbb{R}} f(x)\overline{\psi\left(2^j x - k\right)}dx.$$

Some convergence results about wavelet expansions are descibed in [5–7].

From (3) and (4) with some special cases of a and b one can obtain different types of wavelets, such as Haar wavelet, Franklin system, Strömberg wavelet, Meyer wavelets, etc.

Definition 5 (Compactly supported Daubechies wavelet [2,3])**.**

Owing to the above definitions, we introduce the compactly supported Daubechies wavelets considered in this paper. Let us assume that $\psi \in L_\infty(\mathbb{R})$ *satisfies:*

(a) ψ *is compactly supported, namely there is a real constant* $\lambda > 0$ *such that* $\operatorname{supp}\psi \subset [0,\lambda]$,

(b) $\displaystyle\int_{-\infty}^{\infty} \psi(x)dx = 1$,

(c) the first N moments satisfy

$$\int_{-\infty}^{\infty} x^j \psi(x)dx = 0, \quad j = 1, \ldots, N.$$

Definition 6. *Let* $f \in B[0,1]$, *and let* $\psi \in L_\infty(\mathbb{R})$ *be a father wavelet satisfying (a), (b), (c). Then the wavelet type Bernstein operators, constructed by using the compactly supported Daubechies wavelets, are defined by*

$$\begin{aligned}(WB_n f)(t) &:= n \sum_{k=0}^{n} p_{n,k}(t) \int_{\mathbb{R}} f(x)\, \psi(nx-k)\, dx = \sum_{k=0}^{n} p_{n,k}(t) \int_{\mathbb{R}} f\left(\frac{x+k}{n}\right) \psi(x)\, dx \\ &= \sum_{k=0}^{n} p_{n,k}(t) \int_{0}^{\lambda} f\left(\frac{x+k}{n}\right) \psi(x)\, dx, \quad t \in \mathbb{R}.\end{aligned} \tag{5}$$

Remark 1. *If we choose the father wavelet* $\psi(x)$ *as the Haar scaling function, namely* $\psi(x) = \chi_{[0,1]}(x)$, *then clearly our wavelet type operators reduce to the Kantorovich form of the Bernstein operators*

$$\begin{aligned}(WB_n f)(t) &= n \sum_{k=0}^{n} p_{n,k}(t) \int_{\mathbb{R}} f(x)\, \psi(nx-k)\, dx = \sum_{k=0}^{n} p_{n,k}(t) \int_{\mathbb{R}} f\left(\frac{u+k}{n}\right) \psi(u)\, du \\ &= \sum_{k=0}^{n} p_{n,k}(t) \int_{0}^{1} f\left(\frac{u+k}{n}\right) \psi(u)\, dx.\end{aligned}$$

This means that our operators constructed by wavelets are natural extensions of the Kantorovich type of the Bernstein operators and also Durrmeyer type operators.

2 Fundamental and some convergence properties

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems. This section provides the main approximation results of the paper.

We are now ready to establish one of the first main results of this study, which gives a strong relation between Bernstein operators (1) and our new operators (5) constructed by wavelets.

Theorem 1. *Let* $f \in B[0,1]$ *and let* $\psi \in L_\infty(\mathbb{R})$ *be a father wavelet satisfying (a), (b), (c). Then the moments of wavelet type Bernstein operators, constructed by using the compactly supported Daubechies wavelets* (5) *and the Bernstein operators* (1) *are the same, namely*

$$(WB_n x^s)(t) = (B_n x^s)(t), \quad s = 0, 1, \ldots, K$$

holds true.

Proof. In view of the definition of the operators (5), we have

$$\begin{aligned}(WB_n x^s)(t) &= n\sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}} x^s\psi(nx-k)\,dx = \sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}}\left(\frac{u+k}{n}\right)^s\psi(u)\,du\\ &= \frac{1}{n^s}\sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}}(u+k)^s\,\psi(u)\,du = \frac{1}{n^s}\sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}}\left[\sum_{i=0}^{s}\binom{s}{i}u^i k^{s-i}\right]\psi(u)\,du.\end{aligned}$$

In view of *(c)*, for $i \neq 0$ one has

$$\int_{\mathbb{R}}\left[\sum_{i=0}^{s}\binom{s}{i}u^i k^{s-i}\right]\psi(u)du = 0$$

and from *(b)* for $i = 0$ we get

$$(WB_n x^s)(t) = \frac{1}{n^s}\sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}} k^s\psi(u)du = \sum_{k=0}^{n}\frac{k^s}{n^s}p_{n,k}(t) = (B_n x^s)(t).$$

□

Remark 2. *Moreover, the central moments of the wavelet type Bernstein operators* (5) *are the same as of the classical Bernstein operators* (1). *Indeed, as in the previous Theorem 1, we get*

$$\begin{aligned}\left(WB_n(x-t)^\beta\right)(t) &= n\sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}}(x-t)^\beta\,\psi(nx-k)\,dx\\ &= \sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}}\left(\frac{u+k}{n}-t\right)^\beta\psi(u)\,du\\ &= \frac{1}{n^\beta}\sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}}(u+k-nt)^\beta\,\psi(u)\,du\\ &= \frac{1}{n^\beta}\sum_{k=0}^{n} p_{n,k}(t)\int_{\mathbb{R}}\left[\sum_{i=0}^{\beta}\binom{\beta}{i}u^i\,(nt-k)^{\beta-i}\right]\psi(u)\,du.\end{aligned}$$

Again by the properties of the compactly supported Daubechies wavelets, namely (c) and (b), we get

$$\left(WB_n(x-t)^\beta\right)(t) = \frac{1}{n^\beta}\sum_{k=0}^{n} p_{n,k}(t)\,(k-nt)^\beta = \left(B_n(x-t)^\beta\right)(t).$$

Throughout this work, as in the case of the Bernstein operators, we assume that the first two central moments of the Bernstein operators, constructed by using the compactly supported Daubechies wavelets (5), satisfy

$$\begin{aligned}m_0(\varphi) &:= (WB_n 1)(t) = 1,\\ m_1(\varphi) &:= \left(WB_n(x-t)\right)(t) = 0,\\ m_2(\varphi) &:= \left(WB_n(x-t)^2\right)(t) = \frac{x(1-x)}{n}.\end{aligned}\tag{6}$$

Theorem 2. *Let $f \in B[0,1]$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfying (a), (b), (c). Then*

$$\lim_{n\to\infty}(WB_n f)(t_0) = f(t_0)$$

holds true at each point t_0 of continuity of f.

Proof. In view of the definition of the operators (5), one has

$$\begin{aligned}(WB_nf)(t_0)-f(t_0)&=n\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\mathbb{R}}f(x)\,\psi(nx-k)\,dx-f(t_0)\\&=\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\mathbb{R}}f\left(\frac{u+k}{n}\right)\psi(u)\,du-f(t_0).\end{aligned}$$

By Theorem 1, we know that

$$(WB_n1)(t)=(B_n1)(t)=1. \tag{7}$$

Hence we can write

$$\begin{aligned}\left|(WB_nf)(t_0)-f(t_0)\right|&=\left|n\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\mathbb{R}}(f(x)-f(t_0))\psi(nx-k)\,dx\right|\\&\le\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\mathbb{R}}\left|f\left(\frac{u+k}{n}\right)-f(t_0)\right|\left|\psi(u)\right|du.\end{aligned}$$

Let us divide the last term into two parts as follows

$$\left|(WB_nf)(t_0)-f(t_0)\right|\le P_1+P_2,$$

where

$$\begin{aligned}P_1&=\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\mathbb{R}}\left|f\left(\frac{u+k}{n}\right)-f(t_0)\right|\left|\psi(u)\right|du\\&=\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|<\delta}\left|f\left(\frac{u+k}{n}\right)-f(t_0)\right|\left|\psi(u)\right|du\end{aligned}$$

and

$$P_2=\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|\ge\delta}\left|f\left(\frac{u+k}{n}\right)-f(t_0)\right|\left|\psi(u)\right|du.$$

Since t_0 is a continuity point of f, then clearly

$$\left|f(t)-f(t_0)\right|<\epsilon$$

whenever $|t-t_0|<\delta$, hence we can write

$$P_1=\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|<\delta}\left|f\left(\frac{u+k}{n}\right)-f(t_0)\right|\left|\psi(u)\right|du\le\epsilon\left\|\psi\right\|_\infty.$$

On the other hand, since

$$\left|f(t)-f(t_0)\right|\le 2\left\|f\right\|$$

whenever $|t-t_0|\ge\delta$, we get

$$\begin{aligned}P_2&=\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|\ge\delta}\left|f\left(\frac{u+k}{n}\right)-f(t_0)\right|\left|\psi(u)\right|du\\&\le 2\left\|f\right\|\sum_{k=0}^{n}p_{n,k}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|\ge\delta}\left|\psi(u)\right|du\\&\le 2\left\|f\right\|\frac{m_2(\varphi)}{\delta^2n^2}\left\|\psi\right\|_\infty=O\left(n^{-2}\right).\end{aligned}$$

Collecting these estimates we have

$$\lim_{n\to\infty}(WB_nf)(t_0)=f(t_0).$$

This completes the proof. □

Actually, Daubechies wavelets have strong relations with the properties of continuity and differentability. Namely, for an arbitrary fixed integer $N \geq 1$, compactly supported Daubechies wavelet ψ is supported with $[0, 2N-1]$, in addition there exists a constant $r > 0$ such that for $N \geq 2$, $\psi \in C^{rN}(\mathbb{R})$ and it have a given number of vanishing moments.

In particular, when $N = 1$, then the first Daubechies wavelet ψ will be the classical Haar basis. As N increases, the regularity of the wavelets increase (see [2,3]). This means that if we want to use Daubechies wavelets to reconstruct a function, it is more convenient to choose or construct wavelets based on the continuity or differentiability properties of the given function (see Example 1).

Using Weierstrass criterion, as a consequence of the Theorem 1 we have the following uniform convergence result.

Corollary 1. *The same arguments of Theorem 2 apply to the case when $f \in C[0,1]$. In this case the convergence is uniform with respect to $x \in [0,1]$, and hence one has*

$$\lim_{n\to\infty} \|(WB_n f) - f\|_\infty = 0.$$

Theorem 3. *Let $f \in C[0,1]$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfying (a), (b), (c). Then*

$$\|WB_n f\|_\infty \leq K \|f\|_\infty$$

holds true, where $K = \lambda \|\psi\|_\infty$.

Proof. For the operators (5) we have

$$\left|(WB_n f)(t)\right| = \left|\sum_{k=0}^{n} p_{n,k}(t) \int_0^\lambda f\left(\frac{x+k}{n}\right) \psi(x)\,dx\right| \leq \sum_{k=0}^{n} p_{n,k}(t) \int_0^\lambda \left|f\left(\frac{x+k}{n}\right)\right| |\psi(x)|\,dx.$$

By taking the norm of the functions f and ψ and considering (6), one has

$$\left|(WB_n f)(t)\right| \leq \sum_{k=0}^{n} p_{n,k}(t) \|f\|_\infty \|\psi\|_\infty \lambda \leq \|f\|_\infty \|\psi\|_\infty \lambda.$$

Hence we get

$$\|WB_n f\|_\infty \leq \lambda \|\psi\|_\infty \|f\|_\infty.$$

□

Note 1. *Since the compactly supported Daubechies wavelets are also an unconditional orthonormal base of $L^p(\mathbb{R})$, this allows us to investigate the convergence problem on $L^p(\mathbb{R})$ by means of our wavelet type Bernstein operators* (5).

For $C[0,1]$, let us consider the following Peetre's K-functional

$$K_2(f,\delta) := \inf_{g\in W^2} \{ \|f-g\|_\infty + \delta \|g''\|_\infty \}, \tag{8}$$

where $\delta > 0$ and $W^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}$. Then there exists an absolute constant $C > 0$ such that

$$C^{-1}\omega_2(f,\sqrt{\delta}) \leq K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \tag{9}$$

where

$$\omega_2(f,\sqrt{\delta}) := \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,1]} \left|f(x+2h) - 2f(x+h) + f(x)\right| \tag{10}$$

is the second order modulus of smoothness of f.

Theorem 4. *Let $f \in C[0,1]$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfying (a), (b), (c). Then*

$$\lim_{n\to\infty} (WB_n f)(x) = f(x)$$

and

$$\left|(WB_n f)(x) - f(x)\right| \le (K+1)\, K_2\left(f; \frac{m_2(\varphi)+\lambda^2}{n^2}\right),$$

where $K = \lambda \|\psi\|_\infty$ and $K_2(f;\delta)$ is the Peetre's K-functional.

Proof. Let $g \in W^2$. By Taylor's theorem, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-v) g''(v)\, dv, \quad t \in [0,1].$$

In view of Remark 2 and (7), applying WS_n to the both sides of the above equation, we have

$$\begin{aligned}
\left|(WB_n g)(x) - g(x)\right| &= \left|\left(WB_n\left(g'(x)(t-x) + \int_x^t (t-v) g''(v)\, dv\right)\right)(x) - g(x)\right| \\
&\le \sum_{k=0}^{n} p_{n,k}(x) \int_{\mathbb{R}} \left| \int_x^{\frac{u+k}{n}} \left(\frac{u+k}{n} - v\right) g''(v)\, dv\right| |\psi(u)|\, du \\
&\le \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\lambda} \left[\int_x^{\frac{u+k}{n}} \left|\frac{u+k}{n} - v\right| |g''(v)|\, dv\right] |\psi(u)|\, du \\
&\le \lambda \|\psi\|_\infty \|g''\|_\infty \sum_{k=0}^{n} p_{n,k}(x) \left(\frac{\lambda+k}{n} - x\right)^2 \\
&= \lambda \|\psi\|_\infty \|g''\|_\infty \sum_{k=0}^{n} p_{n,k}(x) \left[\left(\frac{k}{n} - x\right)^2 + \frac{\lambda^2}{n^2} + 2\frac{\lambda}{n}\left(\frac{k}{n} - x\right)\right] \\
&\le \lambda \|\psi\|_\infty \|g''\|_\infty \left[\frac{m_2(\varphi)}{n^2} + \frac{\lambda^2}{n^2} + 2\frac{\lambda}{n}\frac{m_1(\varphi)}{n}\right] \\
&= \frac{\lambda \|\psi\|_\infty \|g''\|_\infty}{n^2} \left[m_2(\varphi) + \lambda^2\right].
\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in W^2$ and using (8), we get

$$\begin{aligned}
\left|(WB_n f)(x) - f(x)\right| &\le \inf_{g\in W^2} \left\{ \|WB_n (f-g)\|_\infty + \|f-g\|_\infty + \left|(WB_n g)(x) - g(x)\right| \right\} \\
&\le \inf_{g\in W^2} \left\{ (\lambda \|\psi\|_\infty + 1) \|f-g\|_\infty + \frac{\lambda \|\psi\|_\infty \left[m_2(\varphi)+\lambda^2\right]}{n^2} \|g''\|_\infty \right\} \\
&\le (K+1) \inf_{g\in W^2} \left\{ \|f-g\|_\infty + \frac{m_2(\varphi)+\lambda^2}{n^2} \|g''\|_\infty \right\} \\
&= (K+1)\, K_2\left(f; \frac{m_2(\varphi)+\lambda^2}{n^2}\right),
\end{aligned}$$

where $K = \lambda \|\psi\|_\infty$. □

Theorem 5. *Let $f \in C[0,1]$, $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies (a), (b), (c) and $\alpha \in (0,2)$ be fixed real number. Then*

$$\omega_2(f;t) = \mathcal{O}(t^\alpha) \Longrightarrow \left|(WB_n f)(x) - f(x)\right| = \mathcal{O}(1/n)^\alpha$$

holds true.

Proof. In view of the relation (9) between modulus of smoothess (10) and Peetre's K-functional (8), from Theorem 4 we have

$$\begin{aligned}\left|(WB_n f)(x)-f(x)\right| &\leq (K+1)K_2\left(f;\frac{m_2(\varphi)+\lambda^2}{n^2}\right)\\ &\leq (K+1)\,C\omega_2\left(f;\sqrt{\frac{m_2(\varphi)+\lambda^2}{n^2}}\right)\leq (K+1)\,C\left(\frac{m_2(\varphi)+\lambda^2}{n^2}\right)^{\alpha/2}.\end{aligned}$$

□

Theorem 6. *Let $f\in L^1[0,1]$ and let $\psi\in L_\infty(\mathbb{R})$ be a father wavelet satisfying (a), (b), (c). Then*

$$\|WB_n f\|_1\leq K\|f\|_1$$

holds true, where $K=nh\|\psi\|_\infty\|p_{n,k}\|_1$ and $h:=\lfloor\lambda\rfloor+1$. Here $\lfloor x\rfloor$ denotes the floor function of the real number x.

Proof. We have

$$\begin{aligned}\int_0^1|(WB_n f)(t)|\,dt &= \int_0^1\left|\sum_{k=0}^n p_{n,k}(t)\int_0^\lambda f\left(\frac{x+k}{n}\right)\psi(x)dx\right|dt\\ &\leq \int_0^1\sum_{k=0}^n p_{n,k}(t)\int_0^\lambda\left|f\left(\frac{x+k}{n}\right)\right||\psi(x)|\,dx\,dt\\ &\leq n\|\psi\|_\infty\sum_{k=0}^n\int_{\frac{k}{n}}^{\frac{\lambda+k}{n}}|f(u)|\,du\left(\int_0^1 p_{n,k}(t)\,dt\right)\\ &\leq n\|\psi\|_\infty\|p_{n,k}\|_1\sum_{k=0}^n\int_{\frac{k}{n}}^{\frac{\lambda+k}{n}}|f(u)|\,du.\end{aligned}$$

For a real number x, $\lfloor x\rfloor=\max\{n\in\mathbb{Z}\mid n\leq x\}$ denotes the integer part function. Now, we set $h:=\lfloor\lambda\rfloor+1$. Hence we have

$$\int_0^1|(WB_n f)(t)|dt\leq n\|\psi\|_\infty\|p_{n,k}\|_1\sum_{k=0}^n\int_{\frac{k}{n}}^{\frac{h+k}{n}}|f(u)|du\leq nh\|\psi\|_\infty\|p_{n,k}\|_1\|f\|_1=:K\|f\|_1,$$

here $K=nh\|\psi\|_\infty\|p_{n,k}\|_1$ and this completes the proof. □

Theorem 7. *Let $f\in L^p[0,1]$, $1\leq p\leq\infty$, and let $\psi\in L_\infty(\mathbb{R})$ be a father wavelet satisfying (a), (b), (c). Then*

$$\|WB_n f\|_p\leq K_p\|f\|_p$$

holds true, where $K_p=n\|\psi\|_\infty\|p_{n,k}\|_1^{1/p}h^{1/p}>0$ and $h:=\lfloor\lambda\rfloor+1$.

Proof. We have

$$\begin{aligned}\left(\int_0^1|(WB_n f)(t)|^p\,dt\right)^{1/p} &= \left(\int_0^1\left|n\sum_{k=0}^n p_{n,k}(t)\int_{\mathbb{R}}f(x)\psi(nx-k)dx\right|^p dt\right)^{1/p}\\ &\leq n\left(\int_0^1\left(\sum_{k=0}^n p_{n,k}(t)\int_{\mathbb{R}}|f(x)\psi(nx-k)|\,dx\right)^p dt\right)^{1/p}=:W.\end{aligned}$$

Applying Jensen inequality, generalized Minkowsky inequality, and the change of variable $nx-k=u$, we obtain

$$\begin{aligned} W^p &\leq n\int_{\mathbb{R}}\sum_{k=0}^{n} p_{n,k}(t)\left(\int_{\mathbb{R}}\left|f\left(x\right)\psi(nx-k)\right|dx\right)^p dt \\ &\leq n\int_{\mathbb{R}}\sum_{k=0}^{n} p_{n,k}(t)\left(\int_{\mathbb{R}}\left|f(x)\right|^p\left|\psi(nx-k)\right|^p dx\right)dt \\ &= n\int_{\mathbb{R}}\left(\sum_{k=0}^{n}\left|f(x)\right|^p\left|\psi(nx-k)\right|^p\int_{\mathbb{R}} p_{n,k}(t)dt\right)dx \\ &\leq n\left\|\psi\right\|_\infty\left\|p_{n,k}\right\|_1\sum_{k=0}^{n}\int_{\frac{k}{n}}^{\frac{\lambda+k}{n}}\left|f\left(x\right)\right|^p dx. \end{aligned}$$

Since $h=\lfloor\lambda\rfloor+1$, we have $W^p\leq n^p\left\|\psi\right\|_\infty^p\left\|p_{n,k}\right\|_1 h\left\|f\right\|_p^p$. This implies

$$\left\|WB_n f\right\|_p\leq\left(n^p\left\|\psi\right\|_\infty^p\left\|p_{n,k}\right\|_1 h\left\|f\right\|_p^p\right)^{1/p}=K_p\left\|f\right\|_p,$$

where $K_p=n\left\|\psi\right\|_\infty\left\|p_{n,k}\right\|_1^{1/p}h^{1/p}$. This completes the proof. □

Note 2. *By Riesz-Thorin Theorem, Theorem 7 is a natural consequence of Theorems 3 and 6.*

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Ця стаття присвячена побудові та вивченню операторів Бернштейна вейвлетного типу з використанням вейвлетів Добеші з компактними носіями заданої функції f. Основою, яка використовується в цій конструкції, є вейвлет-розклад функції f замість її значень $f\left(\frac{k}{n}\right)$. Після цього ми досліджуємо деякі властивості цих операторів у деяких функціональних просторах.

Ключові слова і фрази: поліном Бернштейна, інтерполяція, вейвлет, вейвлет Добеші з компактним носієм, апроксимація.