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New approach to timelike Bertrand curves in 3-dimensional Minkowski space

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In the theory of curves in Euclidean 3-space, it is well known that a curve β is said to be a Bertrand curve if for another curve $\beta^\star$ there exists a one-to-one correspondence between β and $\beta^\star$ such that both curves have common principal normal line. These curves have been studied in different spaces over a long period of time and found wide application in different areas. In this article, the conditions for a timelike curve to be Bertrand curve are obtained by using a new approach in contrast to the well-known classical approach for Bertrand curves in Minkowski 3-space. Related examples that meet these conditions are given. Moreover, thanks to this new approach, timelike, spacelike and Cartan null Bertrand mates of a timelike general helix have been obtained.

Key words and phrases: Bertrand curve, timelike curve, spacelike curve, Cartan null curve, Minkowski 3-space.

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Introduction

A classical problem in differential geometry, raised by French mathematician B. Saint-Venant in 1845 (see [32]), led to discovery of Bertrand curves in 1850 (see [3]). A Bertrand curve is a curve in the Euclidean space such that its principal normal is the principal normal of the second curve. J. Bertrand proved that a necessary and sufficient condition for the existence of such a second curve is required in fact a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, we have $\lambda k_1 + \mu k_2 = 1$, $\lambda, \mu \in \mathbb{R}$. Since 1850, after the paper of J. Bertrand, the pairs of curves like this have been called conjugate Bertrand curves, or more commonly Bertrand curves (see [13]).

The study of this kind of curves has been extended to many other ambient spaces. In [20], L.R. Pears studied this problem for curves in the n-dimensional Euclidean space $\mathbb{E}^n$, $n > 3$, he proved that either k_2 or k_3 must be zero. In other words, Bertrand curves in $\mathbb{E}^n$, $n > 3$, are degenerate, which means that a Bertrand curve in $\mathbb{E}^n$ must belong to a three-dimensional subspace $\mathbb{E}^3 \subset \mathbb{E}^n$. This result is restated to H. Matsuda and S. Yorozu [18]. They proved that there are not any special Bertrand curves in $\mathbb{E}^n$, $n > 3$. As a result of this fact, they defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski

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3-space and Minkowski space-time (see [1,2,7,11,24–26,28]) as well as in Euclidean space. In addition, in [27,30], the authors studied $(1,3)$-type Bertrand curves in semi-Euclidean 4-space with index 2.

Bertrand curves in the three-dimensional sphere $\mathbb{S}^3 \subset \mathbb{E}^4$ have been studied by P. Lucas and J.A. Ortega-Yagües in [15,16]. They considered the correspondence of the principal normal geodesics by using the tools of connection, and gave the relationship between $(1,3)$-type Bertrand curve in $\mathbb{E}^4$ and the Bertrand curve on 3-dimensional sphere $\mathbb{S}^3$. Also in [17], a new algorithm to construct Bertrand curves in three-dimensional semi-Euclidean space $\mathbb{E}^3_q$, by using an arc length parametrized curve in a totally umbilical surface $\mathbb{S}^2$, $\mathbb{S}^2_1$ or $\mathbb{H}^2$, is given by P. Lucas and J.A. Ortega-Yagües. They proved that every Bertrand curve in $\mathbb{E}^3_q$ can be obtained in this way. In [6], J.H. Choi et al. studied the Bertrand curves in 3-dimensional simply connected space forms, i.e. 3-dimensional Euclidean space $\mathbb{E}^3$, 3-dimensional sphere $\mathbb{S}^3$ and 3-dimensional hyperbolic space $\mathbb{H}^3$ by using the curvature functions of the curve. Moreover, it is done in other spaces, such as in Riemann-Otsuki spaces [34]. In [9], S. Honda and M. Takahashi studied Frenet type framed Bertrand curves in $\mathbb{E}^3$. These curves are studied in 3-space forms by H. Jie and D. Pei in [12].

The Bertrand curve can be regarded as the generalization of the helix. The helix, as a special kind of curve, has drawn the attention of scientists as well as mathematicians because of its various applications in science.

Bertrand curves have a wide range of applications. For instance, Bertrand curves represent particular examples of offset curves which are used in computer-aided design (*CAD*) and computer-aided manufacture (*CAM*) (see [8,19]). In [21], Bertrand trajectory ruled surfaces have been defined and a generalization of the theory of Bertrand curves has been presented for the Bertrand trajectory ruled surfaces based on line geometry.

A Razzaboni surface is a surface which is generated by a one-parameter family of geodesic Bertrand curves. In [23], W.K. Schief gave a modern accessible overview of Razzaboni's classical works (1898–1903) on such surfaces and their transformations (see [22]). Some known results related to Bertrand curves and Razzaboni surfaces in Euclidean 3-space generalized to Minkowski 3-space $\mathbb{E}^3_1$ by C. Xu et al. in [33].

In [29], the authors studied the timelike Bertrand curves in Minkowski 3-space. They obtained the necessary and sufficient conditions for timelike curves to have a timelike, spacelike or Cartan null Bertrand mate curve, separately. Moreover, in [5], the auhors gave a new approach for Bertrand curves in 3-dimensional Euclidean space. Also they showed that there exists general helix except circular helix which is Bertrand curve in $\mathbb{E}^3$.

In this paper, the conditions for a timelike curve to be Bertrand curve are obtained by using a new approach in contrast to the well-known classical approach for Bertrand curves in Minkowski 3-space $\mathbb{E}^3_1$. Related examples that meet these conditions are given. Moreover, thanks to this new approach, timelike, spacelike and Cartan null Bertrand mates of a timelike general helix have been obtained.

1 Preliminaries

Minkowski space $\mathbb{E}^3_1$ is a three-dimensional affine space endowed with an indefinite flat metric g with signature $(-,+,+)$. This means that metric bilinear form can be written as

$$g(x,y) = -x_1y_1 + x_2y_2 + x_3y_3$$

for any two vectors $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. Recall that a vector $v \in \mathbb{E}^3_1 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Two vectors v and w are said to be orthogonal if $g(v,w) = 0$ (see [31]). An arbitrary curve $\alpha(s)$ in $\mathbb{E}^3_1$ can be locally *spacelike*, *timelike* or *null* (*lightlike*) if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [13]. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. A spacelike or a timelike curve $\alpha(s)$ has unit speed if $g(\alpha'(s), \alpha'(s)) = \pm 1$ (see [4, 14]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}^3_1$, consisting of the tangent, the principal normal and the binormal vector fields, respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case I. If α is a non-null curve, the Frenet equations are given by (see [13])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

where k_1 and k_2 are the first and the second curvature of the curve, respectively. Moreover, the following conditions hold:

$$g(T,T) = \epsilon_1 = \pm 1, \quad g(N,N) = \epsilon_2 = \pm 1, \quad g(B,B) = \epsilon_3 = \pm 1$$

and

$$g(T,N) = g(T,B) = g(N,B) = 0.$$

Case II. If α is a null curve, the Frenet equations are given by (see [4, 10])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2}$$

where $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, \quad g(N,N) = g(T,B) = 1.$$

2 A new approach to timelike Bertrand curves in Minkowski 3-space

In this section, we will reconsider the Bertrand curves in Minkowski 3-space $\mathbb{E}^3_1$. In the following definition, we give the well-known definition of Bertrand curve in $\mathbb{E}^3_1$.

Definition 1. *A curve $\beta : I \subset \mathbb{R} \to \mathbb{E}^3_1$ with non-zero curvatures is a Bertrand curve if there is a curve $\beta^\star : I^\star \to \mathbb{E}^3_1$ and a bijection $\varphi : \beta \to \beta^\star$ such that the principal normal vectors of $\beta(s)$ and $\beta^\star(s^\star)$ at $s \in I$, $s^\star \in I^\star$ coincide. In this case, $\beta^\star(s^\star)$ is called the Bertrand mate of $\beta(s)$.*

Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^3_1$ be a timelike Bertrand curve with the Frenet frame $\{T(s), N(s), B(s)\}$ and non-zero curvatures k_1, k_2, and $\beta^\star : I \to \mathbb{E}^3_1$ be a Bertrand mate curve of β with the Frenet frame $\{T^\star(s), N^\star(s), B^\star(s)\}$ and non-zero curvatures $k^\star_1, k^\star_2$. Then $\beta^\star$ can be written as

$$\beta^\star(s^\star) = \beta^\star(f(s)) = \beta(s) + u(s)T(s) + v(s)N(s) + w(s)B(s),$$

where $u(s), v(s)$ and $w(s)$ are differentiable functions on I. Since the principal normal vector of the timelike curve β is spacelike, the Bertrand mate curve $\beta^\star$ can be a timelike curve, a spacelike curve with spacelike principal normal or a Cartan null curve. We will consider all cases separately in the following theorem.

Theorem 1. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed timelike curve with the non-zero curvatures k_1, k_2. Then the curve β is a Bertrand curve with Bertrand mate $\beta^\star$ if and only if one of the following conditions hold.*

(i) There exist differentiable functions u, v and w satisfying

$$v^{'} + uk_1 = wk_2 \quad \text{and} \quad w^{'} + vk_2 = 0 \tag{3}$$

or there exist differentiable functions u, v, w and a real number h satisfying

$$\begin{gathered} v^{'} + uk_1 = wk_2, \quad w^{'} + vk_2 \neq 0, \quad 1 + u^{'} + vk_1 = h\big(w^{'} + vk_2\big), \\ k_1 - hk_2 \neq 0, \quad hk_1 - k_2 \neq 0, \quad h^2 - 1 > 0. \end{gathered} \tag{4}$$

In this case, the Bertrand mate curve $\beta^\star$ is a timelike curve.

(ii) There exist differentiable functions u, v, w and a real number h satisfying

$$\begin{gathered} v^{'} + uk_1 = wk_2, \quad w^{'} + vk_2 \neq 0, \quad 1 + u^{'} + vk_1 = h\big(w^{'} + vk_2\big), \\ k_1 - hk_2 \neq 0, \quad hk_1 - k_2 \neq 0, \quad h^2 - 1 < 0. \end{gathered}$$

In this case, the Bertrand mate curve $\beta^\star$ is a spacelike curve with spacelike principal normal.

(iii) There exist differentiable functions u, v, w and real numbers γ, $h = \pm 1$ satisfying

$$\begin{gathered} v^{'} + uk_1 = wk_2, \quad w^{'} + vk_2 \neq 0, \quad 1 + u^{'} + vk_1 = h\left(w^{'} + vk_2\right), \\ hk_1 - k_2 \neq 0, \quad |w^{'} + vk_2| = \gamma^2 \left|hk_1 - k_2\right|, \quad hk_1 + k_2 \neq 0. \end{gathered} \tag{5}$$

In this case, the Bertrand mate curve $\beta^\star$ is a Cartan null curve.

Proof. Assume that β is a timelike Bertrand curve parametrized by arc length s with non-zero curvatures k_1, k_2 and the curve $\beta^\star$ is the Bertrand mate curve of β parametrized by with arc length or pseudo arc $s^\star$. Then we can write the curve $\beta^\star$ as

$$\beta^\star\left(s^\star\right) = \beta^\star\big(f(s)\big) = \beta(s) + u(s)T(s) + v(s)N(s) + w(s)B(s) \tag{6}$$

for all $s \in I$, where $u(s), v(s)$ and $w\left(s\right)$ are differentiable functions on I.

(i) Let $\beta^\star$ be a timelike curve. Then differentiating (6) with respect to s and using the Frenet equations (1), we get

$$f^{'}T^\star = \big(1 + u^{'} + vk_1\big)T + \big(v^{'} + uk_1 - wk_2\big)N + \big(w^{'} + vk_2\big)B. \tag{7}$$

By taking the scalar product of (7) with N, we obtain

$$wk_2 = v^{'} + uk_1. \tag{8}$$

Substituting (8) in (7), we find

$$f^{'}T^{\star}=\left(1+u^{'}+vk_1\right)T+\left(w^{'}+vk_2\right)B. \tag{9}$$

By taking the scalar product of (9) with itself, we obtain

$$\left(f^{'}\right)^2=\left(1+u^{'}+vk_1\right)^2-\left(w^{'}+vk_2\right)^2. \tag{10}$$

If we denote

$$\delta=\frac{1+u^{'}+vk_1}{f^{'}} \qquad \text{and} \qquad \gamma=\frac{w^{'}+vk_2}{f^{'}}, \tag{11}$$

we get $T^{\star}=\delta T+\gamma B$. Differentiating this equation with respect to s and using the Frenet equations (1), we find

$$f^{'}k_1^{\star}N=\delta^{'}T+\left(\delta k_1-\gamma k_2\right)N+\gamma^{'}B. \tag{12}$$

By taking the scalar product of (12) with N, we get

$$\delta^{'}=0 \quad \text{and} \quad \gamma^{'}=0. \tag{13}$$

Firstly, we assume that $\gamma=0$. Then we have $w^{'}+vk_2=0$. Now we assume that $\gamma\neq 0$. Then we have $1+u^{'}+vk_1=h\left(w^{'}+vk_2\right)$, where $h=\delta/\gamma$. Substituting (13) in (12), we find

$$f^{'}k_1^{\star}N=\left(\delta k_1-\gamma k_2\right)N. \tag{14}$$

By taking the scalar product of (14) with itself, using (10) and (11), we obtain

$$\left(f^{'}\right)^2\left(k_1^{\star}\right)^2=\frac{\left(hk_1-k_2\right)^2}{h^2-1},$$

where $hk_1-k_2\neq 0$ and $h^2-1>0$. If we put $\lambda=\frac{\delta k_1-\gamma k_2}{f^{'}k_1^{\star}}$, we get $N^{\star}=\lambda N$. Differentiating the last equation with respect to s and using Frenet equations (1), we find

$$f'k_2^{\star}B^{\star}=\lambda k_1T+\lambda^{'}N+\lambda k_2B-f'k_1^{\star}T^{\star},$$

where $\lambda^{'}=0$. Rewriting the above equation by using (9), we get $f'k_2^{\star}B^{\star}=P(s)T+Q(s)B$, where

$$P(s)=\frac{\left(hk_1-k_2\right)\left(w^{'}+vk_2\right)\left(hk_2-k_1\right)}{\left(f'\right)^2k_1^{\star}(h^2-1)}, \qquad Q(s)=\frac{\left(hk_1-k_2\right)\left(w^{'}+vk_2\right)\left(hk_2-k_1\right)h}{\left(f'\right)^2k_1^{\star}(h^2-1)},$$

which implies that $hk_2-k_1\neq 0$.

Conversely, let β be a timelike curve parametrized by arc length s with non-zero curvatures k_1,k_2. Firstly assume that β satisfies the conditions (3) for differentiable functions u,v and w. Then we can define a curve $\beta^{\star}$ as

$$\beta^{\star}\left(s^{\star}\right)=\beta^{\star}\left(f(s)\right)=\beta(s)+u(s)T(s)+v(s)N(s)+w(s)B(s).$$

Differentiating the above equation with respect to s, we find

$$\frac{d\beta^{\star}}{ds}=\left(1+u^{'}+vk_1\right)T.$$

This implies

$$f' = \left\| \frac{d\beta^\star}{ds} \right\| = m_1 \left(1 + u' + v k_1\right) > 0,$$

where $m_1 = sgn\left(1 + u' + v k_1\right)$. Then we can easily obtain

$$T^\star = m_1 T, \quad N^\star = m_1 m_2 N, \quad B^\star = m_1 m_2 m_3 B$$

and

$$k_1^\star = \frac{m_2 k_1}{f'}, \quad k_2^\star = \frac{m_3 k_2}{f'},$$

where $m_2, m_3 = \pm 1$. Therefore the curve β is a Bertrand curve and the curve $\beta^\star$ is a timelike Bertrand mate curve of the curve β. Also, there exists a homothety map between β and $\beta^\star$.

Now, assume that β satisfies the conditions (4) for differentiable functions u, v, w and a real number h. Then we can define a curve $\beta^\star$ as

$$\beta^\star\left(s^\star\right) = \beta^\star\left(f(s)\right) = \beta(s) + u(s)T(s) + v(s)N(s) + w(s)B(s).$$

Differentiating the above equation with respect to s, we find

$$\frac{d\beta^\star}{ds} = \left(1 + u' + v k_1\right)T + \left(w' + v k_2\right)B, \tag{15}$$

which leads to

$$f' = \left\| \frac{d\beta^\star}{ds} \right\| = n_1 \left(w' + v k_2\right)\sqrt{h^2 - 1},$$

where $n_1 = sgn\left(w' + v k_2\right)$. Rewriting (15), we obtain

$$T^\star = \frac{n_1}{\sqrt{h^2 - 1}}\left(hT + B\right), \quad g\left(T^\star, T^\star\right) = -1. \tag{16}$$

Differentiating (16) with respect to s, we get

$$\frac{dT^\star}{ds^\star} = \frac{n_1\left(hk_1 - k_2\right)}{f'\sqrt{h^2 - 1}} N, \tag{17}$$

which causes that

$$k_1^\star = \left\| \frac{dT^\star}{ds^\star} \right\| = \frac{n_2\left(hk_1 - k_2\right)}{f'\sqrt{h^2 - 1}},$$

where $n_2 = sgn\left(hk_1 - k_2\right)$. Now, we can find $N^\star$ as

$$N^\star = n_1 n_2 N, \quad g\left(N^\star, N^\star\right) = 1. \tag{18}$$

Differentiating (18) with respect to s, using (16) and (17), we get

$$\frac{dN^\star}{ds^\star} - k_1^\star T^\star = \frac{n_1 n_2\left(hk_2 - k_1\right)}{f'\left(h^2 - 1\right)}\left(T + hB\right),$$

which leads to

$$k_2^\star = \frac{n_3\left(hk_2 - k_1\right)}{f'\sqrt{h^2 - 1}},$$

where $n_3 = sgn\big(hk_2 - k_1\big)$. Lastly, we define $B^\star$ as

$$B^\star = \frac{n_1 n_2 n_3}{\sqrt{h^2-1}} \left(T + hB\right), \quad g\left(B^\star, B^\star\right) = 1.$$

Then $\beta^\star$ is a timelike curve and a Bertrand mate curve of β. Thus, β is a Bertrand curve.

(ii) Let $\beta^\star$ be a spacelike curve with spacelike principal normal. In this case, we omit the proof since it is similar to the case when $\beta^\star$ is timelike.

(iii) Let $\beta^\star$ be a Cartan null curve. Then differentiating (6) with respect to s and using the Frenet equations (1) and (2), we get

$$f^{'} T^\star = \big(1 + u^{'} + vk_1\big) T + \big(v^{'} + uk_1 - wk_2\big) N + \big(w^{'} + vk_2\big) B. \tag{19}$$

By taking the scalar product of (19) with N, we obtain $wk_2 = v^{'} + uk_1$. Substituting this in (19), we find

$$f^{'} T^\star = \big(1 + u^{'} + vk_1\big) T + \big(w^{'} + vk_2\big) B. \tag{20}$$

By taking the scalar product of (20) with itself, we obtain

$$\big(1 + u^{'} + vk_1\big)^2 = \big(w^{'} + vk_2\big)^2 \tag{21}$$

and $1 + u^{'} + vk_1 = h\big(w^{'} + vk_2\big)$, where $h = \pm 1$. If we denote

$$\delta = \frac{w^{'} + vk_2}{f^{'}}, \tag{22}$$

we get $T^\star = \delta\left(hT + B\right)$. Differentiating this equation with respect to s and using the Frenet equations (1) and (2), we find

$$f^{'} N^\star = \delta^{'} \left(hT + B\right) + \delta \left(hk_1 - k_2\right) N. \tag{23}$$

From (23), we get

$$\delta^{'} = 0 \quad \text{and} \quad hk_1 - k_2 \neq 0. \tag{24}$$

Substituting (24) in (23), we find

$$f^{'} N^\star = \delta \left(hk_1 - k_2\right) N. \tag{25}$$

By taking the scalar product of (25) with itself, using (21) and (22), we obtain

$$\left|w^{'} + vk_2\right| = \delta^2 \left|hk_1 - k_2\right|.$$

Also, since $N^\star = \pm N$, we have $k_2^\star T^\star - B^\star = \pm\left(k_1 T + k_2 B\right)$ and $-2k_2^\star = k_2^2 - k_1^2$, which implies that $|k_1| \neq |k_2|$ or $hk_1 + k_2 \neq 0$.

Conversely, let β be a timelike curve parametrized by arc length s with non-zero curvatures k_1, k_2. Assume that β satisfies the conditions (5) for differentiable functions u, v, w and real number $h = \pm 1$. Then we can define a curve $\beta^\star$ as

$$\beta^\star\left(s^\star\right) = \beta^\star\big(f\left(s\right)\big) = \beta(s) + u(s)T(s) + v(s)N(s) + w(s)B(s).$$

Differentiating the above equation with respect to s, we find

$$\frac{d\beta^\star}{ds} = \big(w^{'} + vk_2\big)\left(hT + B\right) \tag{26}$$

and
$$\frac{d^2\beta^\star}{ds^2} = \left(w' + vk_2\right)' (hT + B) + \left(w' + vk_2\right)(hk_1 - k_2) N,$$
which leads to
$$f' = \sqrt{m_2\left(w' + vk_2\right)}\sqrt{m_3\left(hk_1 - k_2\right)},$$
where $m_2 = sgn\left(w' + vk_2\right)$ and $m_3 = sgn\left(hk_1 - k_2\right)$. Rewriting (26), we obtain
$$T^\star = m_4\delta\left(hT + B\right), \quad g\left(T^\star, T^\star\right) = 0, \tag{27}$$
where $m_4 = sgn\left(\delta\right)$. Differentiating (27) with respect to s, we get
$$\frac{dT^\star}{ds^\star} = \frac{m_4\delta\left(hk_1 - k_2\right)}{f'} N = m_3 m_4 N \quad \text{and} \quad k_1^\star = 1.$$
Now, we can find $N^\star$ as $N^\star = m_3 m_4 N$, $g\left(N^\star, N^\star\right) = 1$. So we can obtain
$$B^\star = \frac{m_4}{2\delta}\left(-hT + B\right), \quad g\left(B^\star, B^\star\right) = 0 \quad \text{and} \quad g\left(T^\star, B^\star\right) = 1.$$
Lastly, we can get
$$k_2^\star = g\left(\frac{dN^\star}{ds^\star}, B^\star\right) = \frac{m_3\left(hk_1 + k_2\right)}{2f'\delta} \neq 0.$$
Then $\beta^\star$ is a Cartan null curve and a Bertrand mate curve of β. Thus, β is a Bertrand curve. The proof is completed. □

The following examples confirm the above theorem. They are new in the literature.

Example 1. *Let us consider a timelike curve in $\mathbb{E}_1^3$ with the equation*
$$\beta(s) = \left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, s\right)$$
with the curvatures $k_1 = \sqrt{2}$, $k_2 = -1$ and the Frenet frame
$$T(s) = \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, 1\right), \quad N(s) = \left(\sinh s, \cosh s, 0\right), \quad B(s) = \left(\cosh s, \sinh s, \sqrt{2}\right).$$

(i) If we take $u = -1, v = \sqrt{2}, w = \sqrt{2}$ and $h = -\frac{3}{\sqrt{2}}$ in (i) of Theorem 1, then we get the curve $\beta^\star$ as follows
$$\beta^\star(s) = \left(2\sqrt{2}\sinh s, 2\sqrt{2}\cosh s, s + 1\right).$$
By straight calculations, we get
$$T^\star(s) = \left(\frac{2\sqrt{2}}{\sqrt{7}}\cosh s, \frac{2\sqrt{2}}{\sqrt{7}}\sinh s, \frac{1}{\sqrt{7}}\right), \quad N^\star(s) = \left(\sinh s, \cosh s, 0\right),$$
$$B^\star(s) = \left(-\frac{1}{\sqrt{7}}\cosh s, -\frac{1}{\sqrt{7}}\sinh s, -\frac{2\sqrt{2}}{\sqrt{7}}\right)$$
and
$$k_1^\star = \frac{2\sqrt{2}}{7}, \quad k_2^\star = \frac{1}{7}.$$
It can be easily seen that the curve $\beta^\star$ is a timelike Bertrand mate curve of the curve β.

(ii) *If we take* $u = 2, v = -\frac{\sqrt{2}}{3}, w = -2\sqrt{2}$ and $h = \frac{1}{\sqrt{2}}$ *in (ii) of Theorem 1, then we get the curve* $\beta^\star$ *as follows*

$$\beta^\star(s) = \left(\frac{2\sqrt{2}}{3}\sinh s, \frac{2\sqrt{2}}{3}\cosh s, s-2\right).$$

By straight calculations, we get

$$T^\star(s) = \left(2\sqrt{2}\cosh s, 2\sqrt{2}\sinh s, 3\right), \quad N^\star(s) = (\sinh s, \cosh s, 0),$$

$$B^\star(s) = \left(3\cosh s, 3\sinh s, 2\sqrt{2}\right)$$

and

$$k_1^\star = 6\sqrt{2}, \quad k_2^\star = -9.$$

It can be easily seen that the curve $\beta^\star$ *is a spacelike Bertrand mate curve of the curve* β.

(iii) *If we take* $u = 2, v = -1-\sqrt{2}, w = -2\sqrt{2}$ and $h = -1$ *in (iii) of Theorem 1, then we get the curve* $\beta^\star$ *as follows*

$$\beta^\star(s) = (-\sinh s, -\cosh s, s-2).$$

By straight calculations, we get

$$T^\star(s) = (-\cosh s, -\sinh s, 1), \quad N^\star(s) = (-\sinh s, -\cosh s, 0),$$

$$B^\star(s) = \left(\frac{\cosh s}{2}, \frac{\sinh s}{2}, \frac{1}{2}\right)$$

and

$$k_1^\star = 1, \quad k_2^\star = 1/2.$$

It can be easily seen that the curve $\beta^\star$ *is a Cartan null Bertrand mate curve of the curve* β.

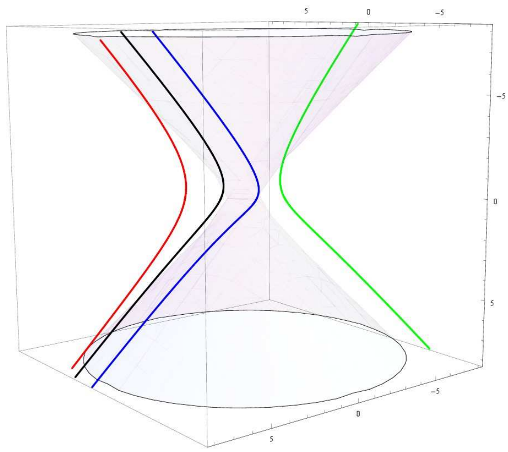

Figure 1. The black graphic is β, the red graphic is the timelike Bertand mate curve, the blue graphic is the spacelike Bertand mate curve and the green graphic is the null Bertand mate curve in Example 1

Example 2. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a timelike Bertrand curve with the curvatures k_1, k_2. Then the conditions of Theorem 1 are satisfied. Assume that $v = v_0 \in \mathbb{R}$. Then we can find*

$$uk_1 = wk_2 \quad \text{and} \quad 1 + u' + v_0 k_1 = h\left(w' + v_0 k_2\right),$$

which implies that

$$w = \frac{k_1\left(s - v_0 \int (hk_2 - k_1)\, ds\right)}{hk_1 - k_2} \quad \text{and} \quad u = \frac{k_2\left(s - v_0 \int (hk_2 - k_1)\, ds\right)}{hk_1 - k_2}.$$

Thus we obtain the Bertrand mate curve $\beta^\star$ as

$$\beta^\star = \beta + \frac{k_2\left(s - v_0 \int (hk_2 - k_1)\, ds\right)}{hk_1 - k_2} T + v_0 N + \frac{k_1\left(s - v_0 \int (hk_2 - k_1)\, ds\right)}{hk_1 - k_2} B.$$

Here the Bertrand mate curve $\beta^\star$ is spacelike, timelike or null if $h^2 > 1$, $h^2 < 1$ or $h^2 = 1$, respectively.

In what follows, we give the examples for timelike general helices, which are Bertrand curves. We know that the timelike general helices do not satisfy the conditions of the theory for classical Bertrand curves (see [29]). So these examples are so important for Bertrand curves.

Example 3. *Let us consider a timelike general helix in $\mathbb{E}_1^3$ with the equation*

$$\beta(s) = \left(\frac{3s^8 - 5}{24s^3}, \frac{3s^8 + 5}{24s^3}, \frac{3s}{4}\right)$$

with the curvatures $k_1 = 5/s$, $k_2 = 3/s$ and the Frenet frame

$$T(s) = \left(\frac{5\left(s^8 + 1\right)}{8s^4}, \frac{5\left(s^8 - 1\right)}{8s^4}, \frac{3}{4}\right), \quad N(s) = \left(\frac{s^8 - 1}{2s^4}, \frac{s^8 + 1}{2s^4}, 0\right),$$

$$B(s) = \left(-\frac{3\left(s^8 + 1\right)}{8s^4}, -\frac{3\left(s^8 - 1\right)}{8s^4}, -\frac{5}{4}\right).$$

(i) *If we take $u = 3\left(s - \ln s\right)/7$, $v = 1$, $w = 5\left(s - \ln s\right)/7$ and $h = 2$ in (i) of Theorem 1, then we get the curve $\beta^\star$ from Example 2 as follows*

$$\beta^\star(s) = \left(\frac{-12 - 5s + 12s^8 + 3s^9}{24s^4}, \frac{12 + 5s + 12s^8 + 3s^9}{24s^4}, \frac{5s}{28} + \frac{4\ln s}{7}\right).$$

By straight calculations, we get

$$T^\star(s) = \left(\frac{7\left(s^8 + 1\right)}{8\sqrt{3}s^4}, \frac{7\left(s^8 - 1\right)}{8\sqrt{3}s^4}, \frac{1}{4\sqrt{3}}\right), \quad N^\star(s) = \left(\frac{s^8 - 1}{2s^4}, \frac{s^8 + 1}{2s^4}, 0\right),$$

$$B^\star(s) = \left(-\frac{s^8 + 1}{8\sqrt{3}s^4}, -\frac{s^8 - 1}{8\sqrt{3}s^4}, -\frac{7}{4\sqrt{3}}\right)$$

and

$$k_1^\star = \frac{49}{48 + 15s}, \quad k_2^\star = \frac{7}{48 + 15s}.$$

It can be easily seen that the curve $\beta^\star$ is a timelike Bertrand mate curve of the curve β.

(ii) If we take $u = -6s - 21\ln s$, $v = 1$, $w = -10s - 35\ln s$ and $h = 1/2$ in (ii) of Theorem 1, then we get the curve $\beta^\star$ from Example 2 as follows

$$\beta^\star(s) = \left(\frac{-12 - 5s + 12s^8 + 3s^9}{24s^4}, \frac{12 + 5s + 12s^8 + 3s^9}{24s^4}, \frac{35s}{4} + 28\ln s\right).$$

By straight calculations, we get

$$T^\star(s) = \left(\frac{s^8+1}{8\sqrt{3}s^4}, \frac{s^8-1}{8\sqrt{3}s^4}, \frac{7}{4\sqrt{3}}\right), \quad N^\star(s) = \left(\frac{s^8-1}{2s^4}, \frac{s^8+1}{2s^4}, 0\right),$$

$$B^\star(s) = \left(-\frac{7\left(s^8+1\right)}{8\sqrt{3}s^4}, -\frac{7\left(s^8-1\right)}{8\sqrt{3}s^4}, -\frac{1}{4\sqrt{3}}\right)$$

and

$$k_1^\star = \frac{1}{48+15s}, \quad k_2^\star = \frac{7}{48+15s}.$$

It can be easily seen that the curve $\beta^\star$ is a spacelike Bertrand mate curve of the curve β.

(iii) If we take $u = \frac{2s}{3} + \frac{3\ln s}{4}$, $v = \frac{1}{4} - \frac{s}{3}$, $w = s + \frac{5\ln s}{4}$ and $h = \gamma = 1$ in (iii) of Theorem 1, then we get the curve $\beta^\star$ as follows

$$\beta^\star(s) = \left(\frac{s^8-1}{8s^4}, \frac{s^8+1}{8s^4}, -\ln s\right).$$

By straight calculations, we get

$$T^\star(s) = \left(\frac{s^8+1}{4s^4}, \frac{s^8-1}{4s^4}, -\frac{1}{2}\right), \quad N^\star(s) = \left(\frac{s^8-1}{2s^4}, \frac{s^8+1}{2s^4}, 0\right),$$

$$B^\star(s) = \left(-\frac{s^8+1}{2s^4}, \frac{1-s^8}{2s^4}, -1\right)$$

and

$$k_1^\star = 1, \quad k_2^\star = 2.$$

It can be easily seen that the curve $\beta^\star$ is a Cartan null Bertrand mate curve of the curve β.

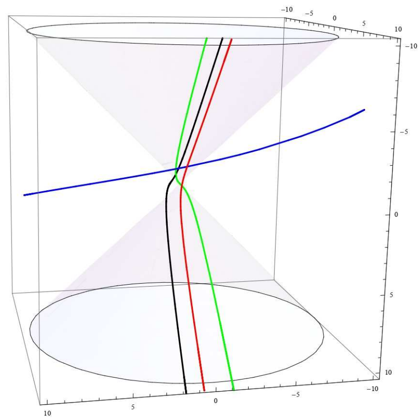

Figure 2. The black graphic is β, the red graphic is the timelike Bertand mate curve, the blue graphic is the spacelike Bertand mate curve and the green graphic is the null Bertand mate curve in Example 3

Remark 1. *If we take the functions u and w as $u = w = 0$ in Theorem 1, we obtain the theorems in [29]. In addition to [29], in this paper, we give the necessary and sufficient conditions for timelike curves in Minkowski* 3*-space to have a Cartan null Bertrand mate curve given by*

$$\beta^\star\left(s^\star\right) = \beta^\star\big(f(s)\big) = \beta(s) + v(s)N(s). \tag{28}$$

So we complete all cases for timelike Bertrand curves whose Bertrand mate curve is given by (28)*.*

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Ердем Х.А., Учум А., Іларслан К., Камчі Ч. *Новий підхід до часоподібних кривих Бертрана у тривимірному просторі Мінковського* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 482–494.

Добре відомо в теорії кривих в евклідовому тривимірному просторі, що криву β називають кривою Бертрана, якщо для іншої кривої $\beta^\star$ існує така бієкція між β та $\beta^\star$, що обидві криві мають спільну головну нормаль. Такі криві вивчалися в різних просторах протягом тривалого періоду часу і знайшли широке застосування в різних областях. У цій статті отримано умови для того, щоб часоподібна крива була кривою Бертрана. Ці умови отримані за допомогою нового підходу на відміну від добре відомого класичного підходу для кривих Бертрана у тривимірному просторі Мінковського. Наведено відповідні приклади, які відповідають цим умовам. Крім того, завдяки цьому новому підходу було отримано часоподібні, простороподібні та картанівські вироджені пари Бертрана часоподібної загальної спіралі.

Ключові слова і фрази: крива Бертрана, часоподібна крива, простороподібна крива, вироджена крива Картана, тривимірний простір Мінковського.