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Approximation characteristics of the isotropic Nikol'skii-Besov functional classes

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In the paper, we investigates the isotropic Nikol'skii-Besov classes $B^r_{p,\theta}(\mathbb{R}^d)$ of non-periodic functions of several variables, which for $d=1$ are identical to the classes of functions with a dominating mixed smoothness $S^r_{p,\theta}B(\mathbb{R})$. We establish the exact-order estimates for the approximation of functions from these classes $B^r_{p,\theta}(\mathbb{R}^d)$ in the metric of the Lebesgue space $L_q(\mathbb{R}^d)$, by entire functions of exponential type with some restrictions for their spectrum in the case $1 \leqslant p \leqslant q \leqslant \infty$, $(p,q) \neq \{(1,1),(\infty,\infty)\}$, $d \geq 1$. In the case $2 < p = q < \infty$, $d=1$, the established estimate is also new for the classes $S^r_{p,\theta}B(\mathbb{R})$.

Key words and phrases: isotropic Nikol'skii-Besov classes, entire function of exponential type, support of the function, Fourier transform.

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Introduction

In this paper, we continue to study the approximative characteristics for the isotropic Nikol'skii-Besov classes $B^r_{p,\theta}(\mathbb{R}^d)$ of functions of many variables in the metric of the Lebesgue space $L_q(\mathbb{R}^d)$. In the case $1 \leqslant p \leqslant q \leqslant \infty$, $(p,q) \neq \{(1,1),(\infty,\infty)\}$, we established the order estimates of the approximation of functions from these classes by entire functions of exponential type (see, e.g., [9, Ch. 3]) with a spectrum concentrated on sets whose Lebesgue measure does not exceed M.

The isotropic spaces $B^r_{p,\theta}(\mathbb{R}^d)$ were introduced by S.M. Nikol'skii [11] in the case $\theta = \infty$ ($B^r_{p,\infty}(\mathbb{R}^d) \equiv H^r_p(\mathbb{R}^d)$) and O.V. Besov [3], when $1 \leqslant \theta < \infty$. In the mentioned works, the definitions of the spaces $H^r_p(\mathbb{R}^d)$ and $B^r_{p,\theta}(\mathbb{R}^d)$ were given in terms of certain restrictions on the modulus of smoothness of functions from those spaces. In what follows for the sake of convenience, we will use the equivalent definition of the spaces $B^r_{p,\theta}(\mathbb{R}^d)$, which was given by P.I. Lizorkin [5] and is based on the application of the Fourier transform. Note that S.M. Nikol'skii [12] and O.V. Besov [3] obtained a series of results concerning the embeddings of the spaces $H^r_p(\mathbb{R}^d)$ and $B^r_{p,\theta}(\mathbb{R}^d)$ with the parameters p, θ and r, respectively, and the extensions of functions from these spaces.

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1 Definition of functions classes

Let $\mathbb{R}^d$, $d \geqslant 1$, be a d-dimensional Euclidean space with elements $\boldsymbol{x} = (x_1, ..., x_d)$, $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + ... + x_d y_d$. Let $L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, be a space of functions $f(\boldsymbol{x}) = f(x_1, ..., x_d)$ measurable on $\mathbb{R}^d$ with finite norm

$$\|f\|_p := \|f\|_{L_p(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |f(\boldsymbol{x})|^p d\boldsymbol{x} \Big)^{1/p}, \quad 1 \leqslant p < \infty, \ \|f\|_\infty := \|f\|_{L_\infty(\mathbb{R}^d)} = \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathbb{R}^d} |f(\boldsymbol{x})|.$$

For $k \in \mathbb{N}$, $\boldsymbol{h} \in \mathbb{R}^d$, and $f \in L_p(\mathbb{R}^d)$, by $\Delta_{\boldsymbol{h}}^k f(\boldsymbol{x})$ we denote the multiple difference $\Delta_{\boldsymbol{h}}^k f(\boldsymbol{x}) = \Delta_{\boldsymbol{h}} \Delta_{\boldsymbol{h}}^{k-1} f(\boldsymbol{x})$, where $\Delta_{\boldsymbol{h}} f(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x})$ and $\Delta_{\boldsymbol{h}}^0 f(\boldsymbol{x}) = f(\boldsymbol{x})$.

The multiple difference $\Delta_{\boldsymbol{h}}^k f(\boldsymbol{x})$ can also be rewritten in the form

$$\Delta_{\boldsymbol{h}}^k f(\boldsymbol{x}) = \sum_{l=0}^{k} (-1)^{l+k} C_k^l f(\boldsymbol{x} + l\boldsymbol{h}).$$

The modulus of smoothness of order k for a function $f \in L_p(\mathbb{R}^d)$ denoted by $\omega_k(f, t)_p$ is given by the formula

$$\omega_k(f, t)_p = \sup_{|\boldsymbol{h}| \leqslant t} \|\Delta_{\boldsymbol{h}}^k f(\cdot)\|_p,$$

where $|\boldsymbol{h}| = \sqrt{h_1^2 + \ldots + h_d^2}$ is the Euclidean norm of the vector $\boldsymbol{h}$.

We say that a function $f \in L_p(\mathbb{R}^d)$ belongs to the isotropic space $B_{p,\theta}^r(\mathbb{R}^d)$, $1 \leqslant p, \theta \leqslant \infty$, $r > 0$, if

$$\Big(\int_0^\infty \big(t^{-r} \omega_k(f, t)_p\big)^\theta \frac{dt}{t} \Big)^{1/\theta} < \infty \quad \text{for} \quad 1 \leqslant \theta < \infty$$

and

$$\sup_{t > 0} \omega_k(f, t)_p t^{-r} < \infty \quad \text{for} \quad \theta = \infty.$$

Note that, in this case, the condition $k > r$ must be satisfied.

If the norm in the space $B_{p,\theta}^r(\mathbb{R}^d)$ is given by the formulas

$$\|f\|_{B_{p,\theta}^r(\mathbb{R}^d)} = \|f\|_p + \Big(\int_0^\infty (t^{-r} \omega_k(f, t)_p)^\theta \frac{dt}{t} \Big)^{1/\theta}, \quad 1 \leqslant \theta < \infty,$$

and

$$\|f\|_{B_{p,\infty}^r(\mathbb{R}^d)} = \|f\|_p + \sup_{t > 0} \omega_k(f, t)_p t^{-r},$$

then this is a Banach space.

As already noted, the space $B_{p,\theta}^r(\mathbb{R}^d)$ was introduced by O.V. Besov in [3] and $B_{p,\infty}^r(\mathbb{R}^d) = H_p^r(\mathbb{R}^d)$, where $H_p^r(\mathbb{R}^d)$ is the space introduced by S.M. Nikol'skii in [11]. In what follows, unless otherwise specified, the term "$B_{p,\theta}^r(\mathbb{R}^d)$ classes" stands for unit balls in the space $B_{p,\theta}^r(\mathbb{R}^d)$, namely,

$$B_{p,\theta}^r(\mathbb{R}^d) := \{f \in L_p \colon \|f\|_{B_{p,\theta}^r(\mathbb{R}^d)} \leqslant 1\}.$$

In establishing results, an important role is played by the property of expansion of the spaces $B_{p,\theta}^r(\mathbb{R}^d)$ with increase in the parameter θ (see, e.g. [9, p. 277]), i.e.

$$B_{p,1}^r(\mathbb{R}^d) \subset B_{p,\theta}^r(\mathbb{R}^d) \subset B_{p,\theta'}^r(\mathbb{R}^d) \subset B_{p,\infty}^r(\mathbb{R}^d) = H_p^r(\mathbb{R}^d), \quad 1 \leqslant \theta < \theta' \leqslant \infty.$$

We now present P.I. Lizorkin's result, that enables us to define the norm of functions from the spaces $B_{p,\theta}^r(\mathbb{R}^d)$ in another form, which allows one to use the Fourier transforms in the theory of these spaces.

Theorem 1 ([5]). *A function f belongs to the space $B^r_{p,\theta}(\mathbb{R}^d)$, $r > 0$, $1 \leqslant p, \theta \leqslant \infty$, if and only if it can be represented by a convergent series in the metric L_p*

$$f(\boldsymbol{x}) = \sum_{s=0}^{\infty} P_{\boldsymbol{a}^s}(\boldsymbol{x}), \quad P_{\boldsymbol{a}^s}(\boldsymbol{x}) = P_{a_1^s,\ldots,a_1^s}(\boldsymbol{x}), \tag{1}$$

where $P_{\nu_1,\ldots,\nu_d}(\boldsymbol{x})$ are entire functions whose powers do not exceed $\nu_1, \ldots, \nu_d$ in each variable $x_1, \ldots, x_d$, respectively, and the following condition

$$\left(\sum_{s=0}^{\infty} b^{s\theta} \| P_{\boldsymbol{a}^s}(\cdot) \|_p^{\theta} \right)^{1/\theta} < \infty \tag{2}$$

is satisfied, where $b = a_1^r > 1$. Moreover, the following estimate is true

$$\| f(\cdot) \|_{B^r_{p,\theta}(\mathbb{R}^d)} \leqslant C_1 \left(\sum_{s=0}^{\infty} b^{s\theta} \| P_{\boldsymbol{a}^s}(\cdot) \|_p^{\theta} \right)^{1/\theta}, \quad C_1 > 0. \tag{3}$$

In addition, if the partial sums of order n of series (1) *realize the best approximation or give an order of the best approximation, then the expression on the left-hand side of* (2) *and $\| f(\cdot) \|_{B^r_{p,\theta}(\mathbb{R}^d)}$ are equivalent, i.e. together with* (3)*, the following estimate is true*

$$\left(\sum_{s=0}^{\infty} b^{s\theta} \| P_{\boldsymbol{a}^s}(\cdot) \|_p^{\theta} \right)^{1/\theta} \leqslant C_2 \| f(\cdot) \|_{B^r_{p,\theta}(\mathbb{R}^d)}, \quad C_2 > 0.$$

Note that P.I. Lizorkin proved this theorem in a more general case where the parameter r in the definition of the Nikol'skii-Besov spaces is a vector with different coordinates, i.e. for the so-called anisotropic Nikol'skii-Besov spaces.

On the basis of Theorem 1, we now give equivalent definitions of the norm of functions from the isotropic spaces $B^r_{p,\theta}(\mathbb{R}^d)$, depending on the value of the parameter p, which are used in what follows. To this end, we recall the definitions of the Fourier transform (see, e.g. [2, Ch. 11], [6], [16, Ch. 2]) and of the de-la-Vallée-Poussin sums [9, Ch. 8].

Let $S = S(\mathbb{R}^d)$ be the Schwarz space of test complex-valued functions φ infinitely differentiable on $\mathbb{R}^d$ and decreasing at infinity together with their derivatives faster than any power of the function $(x_1^2 + \ldots + x_d^2)^{-1/2}$, considered in the appropriate topology. By S' we denote the space of linear continuous functionals on S. The elements of the space S' are generalized functions. If $f \in S'$, then $\langle f, \varphi \rangle$ denotes the value of a functional f on the test function $\varphi \in S$. Denote by $\mathfrak{F}\varphi$ and $\mathfrak{F}^{-1}\varphi$ the Fourier transform and the inverse Fourier transform of functions φ from the spaces S and S'.

For any function φ continuous on $\mathbb{R}^d$, the closure of the set of all points $\boldsymbol{x} \in \mathbb{R}^d$ such that $\varphi(\boldsymbol{x}) \neq 0$ is called the support of the function φ and denoted by $\operatorname{supp} \varphi$.

The generalized function f vanishes in an open set G when $\langle f, \varphi \rangle = 0$ for all $\varphi \in S$ and $\operatorname{supp} \varphi \subset G$. The union of all neighborhoods where f is equal to zero is an open set and called the null set of the generalized function f. It is denoted by G_f. The complement of the largest open set G_f to $\mathbb{R}^d$ is called the support of the generalized function f, i.e. $\operatorname{supp} f$ equal to $\bar{G}_f$ is a closed set.

According to the formula

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(\boldsymbol{x}) \varphi(\boldsymbol{x}) d\boldsymbol{x}, \quad \varphi \in S,$$

each function $f \in L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, defines a linear continuous functional on S and, therefore, is an element of S' in this sense. Hence, the Fourier transform of a function $f \in L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, can be regarded as the Fourier transform of the generalized function $\langle f, \varphi \rangle$.

Further, we consider a continuous analog of the de-la-Vallée-Poussin kernel

$$V_{2^s}(\boldsymbol{x}) = \frac{1}{2^{sd}} \prod_{j=1}^{d} \frac{\cos 2^s x_j - \cos 2^{s+1} x_j}{x_j^2}, \quad j = \overline{1,d}, \quad s \in \mathbb{N} \cup \{0\}.$$

This kernel has the following properties (see [9, p. 358]):

1) $V_{2^s}(\boldsymbol{z}) = V_{2^s}(z_1, \ldots, z_d)$ is an entire function of the exponential type of power 2^{s+1} in each variable z_j, $j = \overline{1,d}$, bounded and summable on $\mathbb{R}^d$;

2)
$$\left(\frac{2}{\pi}\right)^{d/2} \tilde{V}_{2^s} = \frac{1}{\pi^d} \int_{\square_{2^s}} V_{2^s}(\boldsymbol{t}) e^{-i\boldsymbol{t}\boldsymbol{x}} d\boldsymbol{t}, \qquad \text{where } \square_{2^s} = \left\{ |x_j| \leqslant 2^s,\ j = \overline{1,d} \right\};$$

3)
$$\frac{1}{\pi^d} \int_{\mathbb{R}^d} V_{2^s}(\boldsymbol{t}) d\boldsymbol{t} = 1;$$

4)
$$\frac{1}{\pi^d} \int_{\mathbb{R}^d} |V_{2^s}(\boldsymbol{t})|\, d\boldsymbol{t} \leqslant C_3 < \infty.$$

Note that the following equality is true

$$\tilde{V}_{2^s} = \mu_{2^s}(\boldsymbol{x}) = \prod_{j=1}^{d} \mu_{2^s}(x_j),$$

where

$$\mu_{2^s}(x_j) = \sqrt{\frac{\pi}{2}} \begin{cases} 1, & |x_j| < 2^s, \\ (2^{s+1} - x_j)/2^s, & 2^s < |x_j| < 2^{s+1}, \\ 0, & 2^{s+1} < |x_j|. \end{cases}$$

For functions $g_1 \in L_1(\mathbb{R}^d)$ and $g_2 \in L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, we define their convolution by the relation (see, e.g. [9, p. 52])

$$\big(g_1 * g_2\big)(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}} \int g_1(\boldsymbol{x} - \boldsymbol{u}) g_2(\boldsymbol{u})\, d\boldsymbol{u}.$$

In this case, the following inequality is true

$$\|g_1 * g_2\|_p \leqslant \frac{1}{(2\pi)^{d/2}} \|g_1\|_1 \|g_2\|_p\,.$$

Let $f \in L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$. In this case, we set

$$\sigma_{2^s}(f, \boldsymbol{x}) = \left(\frac{2}{\pi}\right)^{d/2} \big(V_{2^s} * f\big)(\boldsymbol{x}) = \frac{1}{\pi^d} \int V_{2^s}(\boldsymbol{x} - \boldsymbol{u}) f(\boldsymbol{u})\, d\boldsymbol{u}.$$

This function is an analog of the de-la-Vallée-Poussin sum of the periodic function of order 2^s. Moreover, $\sigma_{2^s}(f, \boldsymbol{x}) \in L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, is an entire function of the exponential type 2^{s+1}

in each variable x_j, $j = \overline{1,d}$. In terms of the Fourier transform, the function $\sigma_{2^s}(f, \boldsymbol{x})$ can be represented in the form [9, p. 359]

$$\sigma_{2^s}(f, \boldsymbol{x}) = \sigma_{2^s}(f) = \mathfrak{F}^{-1}(\mu_{2^s} \cdot \mathfrak{F}f).$$

Further, we associate each function $f \in L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, with the series

$$f = \sigma_{2^0}(f) + \sum_{s=1}^{\infty} \left(\sigma_{2^s}(f) - \sigma_{2^{s-1}}(f)\right), \tag{4}$$

which converges to f in the metric of the space $L_p(\mathbb{R}^d)$ [10]. This space is called the expansion of the function f in the de-la-Vallée-Poussin-type sums.

We introduce the notation

$$q_0(f) = \sigma_{2^0}(f), \quad q_s(f) = \sigma_{2^s}(f) - \sigma_{2^{s-1}}(f), \quad s \in \mathbb{N}. \tag{5}$$

According to relation (5), equality (4) for f can be rewritten in the form $f = \sum_{s=0}^{\infty} q_s(f)$.

Recall that the approximation of a function $f \in L_p$, $1 \leqslant p \leqslant \infty$, with the use of $\sigma_{2^s}(f)$ has the same order as the best approximation of this function with the help of functions of the exponential type 2^s.

Thus, by using Theorem 1, we can give the following definition of the spaces $B^r_{p,\theta}(\mathbb{R}^d)$.

Definition 1 ([10]). *A function f belongs to the space $B^r_{p,\theta}(\mathbb{R}^d)$, $1 \leqslant p, \theta \leqslant \infty$, $r > 0$, if, for this function, the quantities*

$$\left(\sum_{s=0}^{\infty} 2^{sr\theta} \|q_s(\cdot)\|_p^{\theta}\right)^{1/\theta},$$

for $1 \leqslant \theta < \infty$ and $\sup_{s \geqslant 0} 2^{sr} \|q_s(\cdot)\|_p$, for $\theta = \infty$ are finite. Moreover, according to Theorem 1, the norm $\|f(\cdot)\|_{B^r_{p,\theta}(\mathbb{R}^d)}$, $1 \leqslant \theta \leqslant \infty$, of the function f satisfies the relations

$$\|f(\cdot)\|_{B^r_{p,\theta}(\mathbb{R}^d)} \asymp \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|q_s(\cdot)\|_p^{\theta}\right)^{1/\theta}, \quad \textit{for} \quad 1 \leqslant \theta < \infty \tag{6}$$

and

$$\|f(\cdot)\|_{B^r_{p,\infty}(\mathbb{R}^d)} \asymp \sup_{s \geqslant 0} 2^{sr} \|q_s(\cdot)\|_p. \tag{7}$$

Here and below, for positive quantities a and b, the notation $a \asymp b$ means that there exist positive constants C_4 and C_5 that do not depend on an essential parameter in the values a and b (e.g., C_4 and C_5 in the expressions (6) and (7) do not depend on the function f) such that $C_4 a \leq b$ (in this case, we write $a \ll b$) and $C_5 a \geq b$ (in this case, we write $a \gg b$). In the present paper, all constants C_i, $i = 1, 2, \ldots$, depend only on the parameters contained in the definition of the function class, the metric in which we estimate the error of approximation, and the dimension of the space $\mathbb{R}^d$.

In the case $1 < p < \infty$, the definition of the spaces $B^r_{p,\theta}(\mathbb{R}^d)$ can be given in a different form.

Let f be a function represented by the Fourier integral

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \tilde{f}(\boldsymbol{\lambda}) e^{i(\boldsymbol{\lambda}, \boldsymbol{x})} d\boldsymbol{\lambda}.$$

Then the expression
$$S_{2^s}(f) = \frac{1}{(2\pi)^{d/2}} \int_{-2^s}^{2^s} \cdots \int_{-2^s}^{2^s} \tilde{f}(\boldsymbol{\lambda}) e^{i(\boldsymbol{\lambda},\boldsymbol{x})} d\boldsymbol{\lambda},$$
where $\tilde{f}(\boldsymbol{\lambda})$ is the Fourier transform of the function $f \in L_p(\mathbb{R}^d)$, is called a segment of the Fourier integral for the function f.

Let $D_{2^s} = D_{2^s,\ldots,2^s}$ be a parallelepiped, $|\lambda_j| < 2^s$, $j = \overline{1,d}$, $s \geqslant 0$, and let $\Gamma_{2^s} = D_{2^s} - D_{2^{s-1}}$ for $s \geqslant 1$ and $\Gamma_{2^0} = D_{2^0}$. We set
$$f_{(s)} = f_{2^s} = S_{2^s}(f) - S_{2^{s-1}}(f) = \int_{\Gamma_{2^s}} \tilde{f}(\boldsymbol{\lambda}) e^{i(\boldsymbol{\lambda},\boldsymbol{x})} d\boldsymbol{\lambda}, \quad s \geqslant 1,$$
and
$$f_{(0)} = f_{2^0} = S_{2^0}(f) = \int_{\Gamma_{2^0}} \tilde{f}(\boldsymbol{\lambda}) e^{i(\boldsymbol{\lambda},\boldsymbol{x})} d\boldsymbol{\lambda},$$
where the $f_{(s)}(\boldsymbol{x})$ are entire functions from $L_p(\mathbb{R}^d)$, $1 < p < \infty$ (see, e.g. [6]). The Fourier transform of $f_{(s)}$ is concentrated in $\Gamma_{2^s} = \left\{2^{s-1} \leqslant \max_{j=\overline{1,d}} |\lambda_j| \leqslant 2^s\right\}$ and coincides there with $\widetilde{f}$.

Definition 2 ([5]). *A function f belongs to the space $B^r_{p,\theta}(\mathbb{R}^d)$, $r > 0$, $1 < p < \infty$, $1 \leqslant \theta \leqslant \infty$, if, for this function, the quantities*
$$\left(\sum_{s=0}^{\infty} 2^{sr\theta} \|f_{(s)}(\cdot)\|_p^{\theta} \right)^{1/\theta} < \infty,$$
for $1 \leqslant \theta < \infty$ and $\sup_{s\geqslant 0} 2^{sr} \|f_{(s)}(\cdot)\|_p < \infty$, if $\theta = \infty$ are finite. Moreover, according to Theorem 1, the norm $\|f\|_{B^r_{p,\theta}(\mathbb{R}^d)}$ of the functions f satisfies the relations
$$\|f(\cdot)\|_{B^r_{p,\theta}(\mathbb{R}^d)} \asymp \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|f_{(s)}(\cdot)\|_p^{\theta} \right)^{1/\theta},$$
for $1 \leqslant \theta < \infty$ and
$$\|f(\cdot)\|_{B^r_{p,\infty}(\mathbb{R}^d)} \asymp \sup_{s\geqslant 0} 2^{sr} \|f_{(s)}(\cdot)\|_p. \tag{8}$$

2 Approximation by entire functions of exponential type

We now give the definition of the approximating characteristic used in what follows.

Let $\mathcal{L}$ be a finite set of numbers $s \in \mathbb{Z}_+$, $\mathfrak{M} = \mathfrak{M}(\mathcal{L}) = \bigcup_{s\in\mathcal{L}} \Gamma_{2^s}$. For any $f \in L_q(\mathbb{R}^d)$, $1 \leqslant q \leqslant \infty$, we put
$$S_{\mathfrak{M}}(f,\boldsymbol{x}) = \sum_{s\in\mathcal{L}} f_{(s)}(\boldsymbol{x}). \tag{9}$$

Note that $S_{\mathfrak{M}}(f,\boldsymbol{x})$ is an entire function that belongs to the space $L_q(\mathbb{R}^d)$ (see, e.g. [6]) and the support of its Fourier transform is concentrated in $\mathfrak{M}$, i.e. $\operatorname{supp} S_{\mathfrak{M}}(f,\boldsymbol{x}) \subseteq \mathfrak{M} = \bigcup_{s\in\mathcal{L}} \Gamma_{2^s}$.

For $f \in L_q(\mathbb{R}^d)$, let us consider the following approximative characteristic
$$e^{\mathfrak{F}}_M(f)_q = \inf_{\mathfrak{M}:\ \mathrm{mes}\mathfrak{M}\leqslant M} \|f(\cdot) - S_{\mathfrak{M}}(f,\cdot)\|_q,$$
where $M > 0$ and $\mathrm{mes}A$ denotes the Lebesgue measure of the set A.

If $F \subset L_q(\mathbb{R}^d)$, then we set
$$e^{\mathfrak{F}}_M(F)_q = \sup_{f\in F} e^{\mathfrak{F}}_M(f)_q. \tag{10}$$

We first formulate the following statements essentially used in our subsequent presentation.

Theorem 2. *Let* $1 \leqslant p \leqslant p' \leqslant \infty$, $\theta' \geqslant \theta$, $\kappa = 1 - \left(\frac{1}{p} - \frac{1}{p'}\right) \sum\limits_{j=1}^{d} \frac{1}{r_j} > 0$. *If* $f \in B^{\boldsymbol{r}}_{p,\theta}(\mathbb{R}^d)$, *then* $f \in B^{\boldsymbol{\rho}}_{p',\theta'}(\mathbb{R}^d)$, *where* $\rho_j = r_j\kappa$, $j = \overline{1,d}$. *In this case, the inequality*

$$\|f(\cdot)\|_{B^{\boldsymbol{\rho}}_{p',\theta'}(\mathbb{R}^d)} \leqslant C_6 \|f(\cdot)\|_{B^{\boldsymbol{r}}_{p,\theta}(\mathbb{R}^d)}$$

holds, where C_6 *is some constant independent of* f.

Theorem 2 was obtained by O.V. Besov [3, Theorem 2.1].

We also give one more assertion for entire functions of the exponential type which was obtained by S.M. Nikol'skii [11] (see also [9, p. 150]).

Theorem 3. *If* $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, *then any entire function of the exponential type* $g = g_\nu \in L_p(\mathbb{R}^d)$ *satisfies the "inequality of different metrics"*

$$\|g_\nu\|_{L_{p_2}(\mathbb{R}^d)} \leqslant 2^d \left(\prod_{j=1}^{d} \nu_k\right)^{1/p_1 - 1/p_2} \|g_\nu\|_{L_{p_1}(\mathbb{R}^d)}. \tag{11}$$

Theorem 4. *Let* $1 \leqslant p \leqslant q \leqslant \infty$, $(p,q) \neq \{(1,1),(\infty,\infty)\}$, $1 \leqslant \theta \leqslant \infty$. *If* $r > d(1/p - 1/q)$, *then the order relation*

$$e^{\mathfrak{F}}_M(B^r_{p,\theta}(\mathbb{R}^d))_q \asymp M^{-r/d+1/p-1/q} \tag{12}$$

is true.

Note that the fulfillment of the condition $r > d(1/p - 1/q)$, according to Theorem 2, ensures that functions $f \in B^r_{p,\theta}(\mathbb{R}^d)$ belong to the space $L_q(\mathbb{R}^d)$.

Proof. First, we will get the upper estimate in (12). Since, for $1 \leqslant \theta < \infty$, the embedding $B^r_{p,\theta}(\mathbb{R}^d) \subset H^r_p(\mathbb{R}^d)$ is true, it suffices to get the required estimate for $e^{\mathfrak{F}}_M\big(H^r_p(\mathbb{R}^d)\big)_q$. We now consecutively consider several possible relations between the parameters p and q.

Let $1 \leqslant p < q \leqslant \infty$. Then, for given $M \in \mathbb{N}$, we find a number $n(M)$ such that $2^n \asymp M^{1/d}$ and, for $f \in H^r_p(\mathbb{R}^d)$ consider its approximation by a sum of the form $S_n(f, \boldsymbol{x}) = \sum_{s=0}^{n} f_{(s)}(\boldsymbol{x})$.

Let q_0 be a number satisfying the condition $p < q_0 < q$. Further, for $f \in H^r_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, according to (7), we can write $\|q_s(f,\cdot)\|_p \ll 2^{-sr}$. By using the Minkowski inequality and twice using the Nikol'skii "inequality of different metrics" (11), we can write

$$\begin{aligned}
e^{\mathfrak{F}}_M(f)_q &\ll \|f(\cdot) - S_n(f,\cdot)\|_q = \Bigg\| \sum_{s=n+1}^{\infty} f_{(s)}(\cdot) \Bigg\|_q \leqslant \sum_{s=n+1}^{\infty} \|f_{(s)}(\cdot)\|_q \\
&\ll \sum_{s=n+1}^{\infty} 2^{sd(1/q_0 - 1/q)} \|f_{(s)}(\cdot)\|_{q_0} \asymp \sum_{s=n+1}^{\infty} 2^{sd(1/q_0 - 1/q)} \|q_s(\cdot)\|_{q_0} \\
&\ll \sum_{s=n+1}^{\infty} 2^{sd(1q_0 - 1q)} 2^{sd(1/p - 1/q_0)} \|q_s(\cdot)\|_p = \sum_{s=n+1}^{\infty} 2^{sd(1/p-1/q)} \|q_s(\cdot)\|_p \\
&\leqslant \sum_{s=n+1}^{\infty} 2^{sd(1/p-1/q)} 2^{-sr} = \sum_{s=n+1}^{\infty} 2^{-sd(r/d - 1/p + 1/q)} \\
&\ll 2^{-nd(r/d-1/p+1/q)} \asymp M^{-r/d+1/p-1/q}.
\end{aligned}$$

Consider the case $1 < p = q < \infty$. Then for $f \in H^r_p(\mathbb{R}^d)$, taking into account (8) and the Minkowski inequality we have

$$e^{\mathfrak{F}}_M(f)_q \ll \|f(\cdot) - S_n(f,\cdot)\|_q = \Big\| \sum_{s=n+1}^{\infty} f_{(s)}(\cdot) \Big\|_p \leqslant \sum_{s=n+1}^{\infty} \|f_{(s)}(\cdot)\|_p$$
$$\leqslant \sum_{s=n+1}^{\infty} 2^{-sr} \ll 2^{-nr} \asymp M^{-r/d}.$$

Hence, we obtained the upper estimate for the quantity $e^{\mathfrak{F}}_M\big(B^r_{p,\theta}(\mathbb{R}^d)\big)_q$.

We now establish the lower estimate in (12). Since the embedding $B^r_{p,1}(\mathbb{R}^d) \subset B^r_{p,\theta}(\mathbb{R}^d)$, $1 < \theta \leqslant \infty$, is true, it is sufficient to find the required estimate for the quantity $e^{\mathfrak{F}}_M(B^r_{p,1}(\mathbb{R}^d))_q$.

For the functions $f \in L_q(\mathbb{R}^d)$ and $g \in L_{q'}(\mathbb{R}^d)$, we will use the well-known relation (see, e.g. [4, p. 22])

$$\|f\|_q = \sup_{\|g\|_{q'} \leqslant 1} \int_{\mathbb{R}^d} |f(\boldsymbol{x})\, g(\boldsymbol{x})|\, d\boldsymbol{x}, \tag{13}$$

where $1/q + 1/q' = 1$.

Consider the case $1 < p \leqslant q < \infty$. Let $f \in B^r_{p,\theta}(\mathbb{R}^d)$ and $S_{\mathfrak{M}}(f, \boldsymbol{x})$ is an entire function, the support of the Fourier transform of which is concentrated on the set $\mathfrak{M} = \bigcup_{s \in \mathcal{L}} \Gamma_{2^s}$, $\operatorname{mes} \mathfrak{M} \leqslant M$.

According to the given relation (13), we can write

$$\|f(\cdot) - S_{\mathfrak{M}}(f,\cdot)\|_q = \sup_{\|g\|_{q'} \leqslant 1} \int_{\mathbb{R}^d} |(f(\boldsymbol{x}) - S_{\mathfrak{M}}(f,\boldsymbol{x}))g(\boldsymbol{x})|\, d\boldsymbol{x}. \tag{14}$$

For $\boldsymbol{k} \in \mathbb{N}^d$, we consider the function $D_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^d D_{k_j}(x_j)$, where

$$D_{k_j}(x_j) = \sqrt{\frac{2}{\pi}} \Big(2 \sin \frac{x_j}{2} \cos \frac{2k_j+1}{2} x_j\Big) \frac{1}{x_j}, \quad D_{1/2}(x_j) = D_0(x_j) := \sqrt{\frac{2}{\pi}}\, \frac{\sin x_j}{x_j}.$$

Then, for the Fourier transform of the function $D_{\boldsymbol{k}}(\boldsymbol{x})$, we can write [17]

$$\mathfrak{F} D_{\boldsymbol{k}}(\boldsymbol{x}) = \chi_{\boldsymbol{k}}(\boldsymbol{\lambda}) = \prod_{j=1}^d \chi_{k_j}(\lambda_j),$$

where

$$\chi_{k_j}(\lambda_j) = \begin{cases} 1, & k_j < |\lambda_j| < k_j + 1, \\ 1/2, & |\lambda_j| = k_j \quad \text{or} \quad |\lambda_j| = k_j + 1, \\ 0, & \text{otherwise}, \end{cases} \quad \chi_0(x_j) = \begin{cases} 1, & |\lambda_j| < 1, \\ 1/2, & |\lambda_j| = 1, \\ 0, & |\lambda_j| > 1. \end{cases}$$

Hence, for the inverse transform, we obtain $\mathfrak{F}^{-1}\chi_{\boldsymbol{k}}(\boldsymbol{t}) = D_{\boldsymbol{k}}(\boldsymbol{x})$.

To use relation (14), we construct the corresponding functions. For given $M \in \mathbb{N}$, we choose $n(M) \in \mathbb{N}$ from the inequalities $2^{(n-2)d} \leqslant M < 2^{(n-1)d}$.

Consider a function

$$F_n(\boldsymbol{x}) = \prod_{j=1}^d \sum_{k_j = 2^n}^{2^{n+1}-1} D_{k_j}(x_j).$$

Since $F_n(\boldsymbol{x})$ satisfies the relation (see, e.g. [17])

$$\|F_n(\cdot)\|_q \asymp 2^{nd(1-1/q)}, \quad 1 < q < \infty, \tag{15}$$

then

$$\|F_n(\cdot)\|_{B^r_{p,1}(\mathbb{R}^d)} = \sum_s 2^{sr}\|(F_n)_{(s)}(\cdot)\|_p \asymp 2^{(n+1)r}\|F_n(\cdot)\|_p \asymp 2^{(n+1)r}2^{nd(1-1/p)} \asymp 2^{nd(r/d+1-1/p)}.$$

Hence, it follows from above relations, that the function $f_1(\boldsymbol{x}) = C_7\, 2^{-nd(r/d+1-1/p)}F_n(\boldsymbol{x})$, belongs to the class $B^r_{p,1}(\mathbb{R}^d)$ with a constant $C_7 > 0$.

Further, according to (15) we have $\|F_n(\cdot)\|_{q'} \asymp 2^{nd/q}$, $1 < q < \infty$, $1/q + 1/q' = 1$. Therefore, the function $g_1(\boldsymbol{x}) = C_8 2^{-nd/q}F_n(\boldsymbol{x})$ satisfies the condition $\|q_1(\cdot)\|_{q'} \leqslant 1$ with a constant $C_8 > 0$.

So, using (14), we get

$$\begin{aligned}
\|f_1(\cdot) - S_{\mathfrak{M}}(f_1,\cdot)\|_q &= \sup_{\|g_1\|_{q'}\leqslant 1} \int_{\mathbb{R}^d} |(f_1(\boldsymbol{x}) - S_{\mathfrak{M}}(f_1,\boldsymbol{x}))g_1(\boldsymbol{x})|\, d\boldsymbol{x} \\
&\gg \int_{\mathbb{R}^d} |(2^{-nd(r/d+1-1/p)}F_n(\boldsymbol{x}) - S_{\mathfrak{M}}(f_1,\boldsymbol{x}))2^{-nd/q}F_n(\boldsymbol{x})|\, d\boldsymbol{x} \\
&= 2^{-nd(r/d+1-1/p)}2^{-nd/q}\int_{\mathbb{R}^d} |(F_n(\boldsymbol{x}) - S_{\mathfrak{M}}(F_n,\boldsymbol{x}))F_n(\boldsymbol{x})|\, d\boldsymbol{x} \\
&\gg 2^{-nd(r/d+1-1/p)}2^{\frac{nd}{q}}(2^{nd} - M) \geqslant 2^{-nd(r/d-1/p+1/q)}2^{nd}\Big(1 - \frac{1}{2^d}\Big) \\
&\asymp 2^{-nd(r/d-1/p+1/q)} \asymp M^{-r/d+1/p-1/q}.
\end{aligned}$$

We now consider the case where $1 \leqslant p < \infty$ and $q = \infty$. For given M, we select a number $n(M) \in \mathbb{N}$ such that the relations $2^{nd} \asymp M$ and $2^{nd} \geqslant 4M$ are true and set

$$v_{n+1}(\boldsymbol{x}) = \prod_{j=1}^{d}\big(V_{2^{n+1}}(x_j) - V_{2^n}(x_j)\big), \quad n \in \mathbb{N}, \quad x_j \in \mathbb{R}, \quad j = \overline{1,d}.$$

Note that for $\|v_{n+1}(\cdot)\|_p$, $1 \leqslant p \leqslant \infty$, the estimate

$$\|v_{n+1}\|_p \asymp 2^{nd(1-1/p)}, \quad 1 \leqslant p \leqslant \infty. \tag{16}$$

is true [7,18].

Consider a function $f_2(\boldsymbol{x}) = C_9 2^{-nd(r/d+1-1/p)}v_{n+1}(\boldsymbol{x})$, which, as shown in [18], for a certain choice of the constant C_9 belongs to class $B^r_{p,1}(\mathbb{R}^d)$.

Further, let $S_{\mathfrak{M}}(f_2,\boldsymbol{x})$ be an entire function of the form (9). Since for $p = \infty$ according to (16) $\|v_{n+1}(\cdot)\|_\infty \asymp 2^{nd}$, then

$$\begin{aligned}
\|f_2(\cdot) - S_{\mathfrak{M}}(f_2,\cdot)\|_\infty &\geqslant |\|f_2(\cdot)\|_\infty - \|S_{\mathfrak{M}}(f_2,\cdot)\|_\infty| \gg 2^{-nd(r/d+1-1/p)}(2^{nd} - M) \\
&\asymp 2^{-nd(r/d+1-1/p)}2^{nd} = 2^{-nd(r/d-1/p)} \asymp M^{r/d+1/p}.
\end{aligned}$$

Finally, let $p = 1$ and $1 < q < \infty$. In this case, we will again use relation (13) and, as the functions $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, we will choose the functions $f_2(\boldsymbol{x})$ at $p = 1$, namely $f_3(\boldsymbol{x}) = C_{10}2^{-nr}v_{n+1}(\boldsymbol{x})$, $C_{10} > 0$, and $g_2(\boldsymbol{x}) = C_{11}2^{-nd/q}v_{n+1}(\boldsymbol{x})$, $C_{11} > 0$. We also assume that the relations $2^{nd} \asymp M$ and $2^{nd} \geqslant 4M$ are true.

We now show that for the proper choice of the constant C_{11} the function $g_2(\boldsymbol{x})$ satisfies the conditions of relation (13) for the function $g(\boldsymbol{x})$, we have

$$\|g_2(\cdot)\|_{q'} \asymp 2^{-nd/q}\|v_{n+1}(\cdot)\|_{q'} \asymp 2^{-nd(1-1/q')}2^{nd(1-1/q')} = 1.$$

Thus, by applying relation (13) to the functions $f_3(\boldsymbol{x})$ and $g_2(\boldsymbol{x})$, we find

$$e_M^{\mathfrak{F}}(f_3)_q \gg 2^{-nr}2^{-nd/q}(\|v_{n+1}\|_2^2 - M) \gg 2^{-nd(r/d+1/q)}2^{nd} \asymp M^{r/d+1-1/q}.$$

This means that the required lower estimates are established in all cases of the theorem. The Theorem 4 is proved. □

At the end of the work we will make some comments on the obtained results.

In the one-dimensional case $d = 1$, the isotropic Nikol'skii-Besov classes $B^r_{p,\theta}(\mathbb{R})$ are identical to the classes with mixed smoothness $S^r_{p,\theta}B(\mathbb{R})$ [1]. In the cases $1 \leqslant p < q \leqslant \infty$ and $1 < p = q \leqslant 2$, $d = 1$, the exact-order estimates of quantity (10) for the classes $S^r_{p,\theta}B(\mathbb{R}^d)$ are found in [8, 19–21]. In the case $2 < p = q < \infty$, $d = 1$, the estimate of quantity (10) is also new for the classes $S^r_{p,\theta}B(\mathbb{R})$.

The quantity (10) is a non-periodic analogue of the best orthogonal approximation and for isotropic Nikol'skii-Besov classes of the periodic functions of many variables were studied in [13]. Isotropic Nikol'skii-Besov classes of the periodic functions also studied in [14, 15].

References

[1] Amanov T.I. *Representation and imbedding theorems for function spaces $S^{(r)}_{p,\theta}B(\mathbb{R}_n)$ and $S^{(r)*}_{p,\theta}B$, $(0 \leq x_j \leq 2\pi;$ $j = 1, \ldots, n)$*. Tr. Mat. Inst. Steklova 1965, **77**, 5–34. (in Russian)

[2] Berezansky Yu.M., Sheftel Z.G., Us G.F. Functional Analysis. Vol. 1. Institute of Matematics of NAS of Ukraine, Kyiv, 2010.

[3] Besov O.V. *Investigation of one family of functional spaces in connection with the embedding and continuation theorems*. Tr. Mat. Inst. Steklova 1961, **60**, 42–81. (in Russian)

[4] Besov O.V., Il'in V.P., Nikol'skii S.M. Integral Representations of Functions and Embedding Theorems. Nauka, Moscow, 1996. (in Russian)

[5] Lizorkin P.I. *Generalized Hölder spaces $B^{(r)}_{p,\theta}$ and their relation with the Sobolev spaces $L^{(r)}_p$*. Sib. Math. J. 1968, **9** (5), 837–858. doi:10.1007/BF01041168 (translation of Sibirsk. Mat. Zh. 1968, **9** (5), 1127–1152. (in Russian))

[6] Lizorkin P.I. *Generalized Liouville differentiation and the multiplier method in the theory of imbeddings of classes of differentiable functions*. Proc. Steklov Inst. Math. 1969, **105**, 105–202. (translation of Tr. Mat. Inst. Steklova 1969, **105**, 89–167. (in Russian))

[7] Myroniuk V.V. *Approximation of functions from the isotropic classes $B^{\Omega}_{1,\theta}(\mathbb{R}^d)$ by entire functions of exponential type*. Zb. Pr. Inst. Mat. NAN Ukr., Collection of Works "Approximation Theory of Functions and Related Problems" 2013, **10** (1), 169–183. (in Ukrainian)

[8] Myroniuk V.V., Yanchenko S.Ya. *Approximation of functions from generalized Nikol'skii-Besov classes by entire functions in Lebesgue spaces*. Mat. Stud. 2013, **39** (2), 190–202. (in Ukrainian)

[9] Nikol'skii S.M. Approximation of Functions of Many Variables and Imbedding Theorems. Nauka, Moscow, 1969. (in Russian)

[10] Nikol'skii S.M. *Embedding theorems for the classes of generalized functions*. Sib. Math. J. 1968, **9** (5), 821–837. doi:10.1007/BF01041167 (translation of Sibirsk. Mat. Zh. 1968, **9** (5), 1107–1126. (in Russian))

[11] Nikol'skii S.M. *Inequalities for entire functions of finite power and their application to the theory of differentiable functions of many variables*. Tr. Mat. Inst. Steklova 1951, **38**, 244–278. (in Russian)

[12] Nikol'skii S.M. *On one family of functional spaces*. Uspekhi Mat. Nauk 1956, **11** (6 (72)), 203–212. (in Russian)

[13] Romanyuk A.S. *Approximative characteristics of the isotropic classes of periodic functions of many variables*. Ukrainian Math. J. 2009, **61** (4), 613–626. doi:10.1007/s11253-009-0232-y (translation of Ukrain. Mat. Zh. 2009, **61** (4), 513–523. (in Russian))

[14] Romanyuk A.S. *Approximation of the isotropic classes $B^r_{p,\theta}$ of periodic functions of many variables in the space L_q*. Zb. Pr. Inst. Mat. NAN Ukr., Collection of Works "Approximation Theory of Functions and Related Problems" 2008, **5** (1), 263–278 (in Russian).

[15] Romanyuk A.S., Romanyuk V.S. *Trigonometric and orthoprojection widths of classes of periodic functions of many variables*. Ukrainian Math. J. 2009, **61** (10), 1589–1609. doi:10.1007/s11253-010-0300-3 (translation of Ukrain. Mat. Zh. 2009, **61** (10), 1348–1366. (in Russian))

[16] Vladimirov V.S. Equations of Mathematical Physics. Nauka, Moscow, 1981. (in Russian)

[17] Heping W., Yongsheng S. *Approximation of multivariate functions with a certain mixed smoothness by entire functions*. Northeast. Math. J. 1995, **11** (4), 454–466.

[18] Yachenko S.Ya. *Approximation of functions from the isotropic Nikol'skii-Besov classes in the uniform and integral metrics*. Ukrainian Math. J. 2016, **67** (10), 1599–1610. doi:10.1007/s11253-016-1175-8 (translation of Ukrain. Mat. Zh. 2015, **67** (10), 1423–1433. (in Ukrainian))

[19] Yanchenko S.Ya. *Approximation of the classes $S^r_{p,\theta}B(\mathbb{R}^d)$ of functions of many variables by entire functions of a special form*. Ukrainian Math. J. 2011, **62** (8), 1307–1325. doi:10.1007/s11253-011-0431-1 (translation of Ukrain. Mat. Zh. 2010, **62** (8), 1124–1138 (in Ukrainian))

[20] Yanchenko S.Ya. *Approximation of the Nikol'skii-Besov functional classes by entire functions of a special form*. Carpathian Math. Publ. 2020, **12** (1), 148–156. doi:10.15330/cmp.12.1.148-156

[21] Yanchenko S.Ya. *Estimates for approximative characteristics of classes $S^r_{p,\theta}B(\mathbb{R}^d)$ of functions in the uniform metric*. Zb. Pr. Inst. Mat. NAN Ukr., Collection of Works "Approximation Theory of Functions and Related Problems" 2013, **10** (1), 328–340. (in Ukrainian)

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У статті досліджуються ізотропні класів Нікольського-Бєсова $B^r_{p,\theta}(\mathbb{R}^d)$ неперіодичних функцій багатьох змінних, які при $d = 1$ тотожні класам функцій з домінуючою мішаною похідною $S^r_{p,\theta}B(\mathbb{R}^d)$. Одержано точні за порядком оцінки наближення функцій з даних класів $B^r_{p,\theta}(\mathbb{R}^d)$ у метриці простору Лебега $L_q(\mathbb{R}^d)$ за допомогою цілих функцій експоненціального типу з певними обмеженнями на їхній спектр у випадку $1 \leqslant p \leqslant q \leqslant \infty$, $(p,q) \neq \{(1,1),(\infty,\infty)\}$, $d \geq 1$. У випадку $2 < p = q < \infty$, $d = 1$, встановлена оцінка є новою й для класів $S^r_{p,\theta}B(\mathbb{R})$.

Ключові слова і фрази: ізотропні класи Нікольського-Бєсова, ціла функція експоненціального типу, носій функції, перетворення Фур'є.