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Non-symmetric approximations of functional classes by splines on the real line

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Let $S_{h,m}$, $h > 0$, $m \in \mathbb{N}$, be the spaces of polynomial splines of order m of deficiency 1 with nodes at the points kh, $k \in \mathbb{Z}$.

We obtain exact values of the best (α,β)-approximations by spaces $S_{h,m} \cap L_1(\mathbb{R})$ in the space $L_1(\mathbb{R})$ for the classes $W^r_{1,1}(\mathbb{R})$, $r \in \mathbb{N}$, of functions, defined on the whole real line, integrable on $\mathbb{R}$ and such that their rth derivatives belong to the unit ball of $L_1(\mathbb{R})$.

These results generalize the well-known G.G. Magaril-Ilyaev's and V.M. Tikhomirov's results on the exact values of the best approximations of classes $W^r_{1,1}(\mathbb{R})$ by splines from $S_{h,m} \cap L_1(\mathbb{R})$ (case $\alpha = \beta = 1$), as well as are non-periodic analogs of the V.F. Babenko's result on the best non-symmetric approximations of classes $W^r_1(\mathbb{T})$ of 2π-periodic functions with rth derivative belonging to the unit ball of $L_1(\mathbb{T})$ by periodic polynomial splines of minimal deficiency.

As a corollary of the main result, we obtain exact values of the best one-sided approximations of classes W^r_1 by polynomial splines from $S_{h,m}(\mathbb{T})$. This result is a periodic analogue of the results of A.A. Ligun and V.G. Doronin on the best one-sided approximations of classes W^r_1 by spaces $S_{h,m}(\mathbb{T})$.

Key words and phrases: best L_1-approximation, one-sided approximation, non-symmetric approximation, polynomial spline, functional class.

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Introduction

Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{R}_+$, and $\mathbb{R}$ be the sets of integer positive, integer non-negative, integer, real non-negative, and real numbers, respectively, and $\mathbb{T}$ be the interval $[0, 2\pi]$ with identified ends.

Let $L_p(\mathbb{G})$, $1 \le p \le \infty$, $\mathbb{G} \subset \mathbb{R}$, be the spaces of all measurable on $\mathbb{G}$ functions with norms $\|\cdot\|_{L_p(\mathbb{G})}$, $C^r(\mathbb{G})$, $r \in \mathbb{Z}_+$, be the spaces of r times continuously differentiable (continuous for $r = 0$) on $\mathbb{G}$ functions, and $AC(\mathbb{G})$ be the set of all absolutely continuous (locally for $\mathbb{G} = \mathbb{R}$ and $\mathbb{G} = \mathbb{R}_+$) on $\mathbb{G}$ functions.

For $f \in L_p(\mathbb{G})$ and $\alpha, \beta > 0$ we set

$$\|f\|_{L_p(\mathbb{G});\alpha,\beta} = \|\alpha f_+ + \beta f_-\|_{L_p(\mathbb{G})},$$

where $f_\pm(t) = \max\{\pm f(t), 0\}$. The quantity

$$E(f,H)_{L_p(\mathbb{G});\alpha,\beta} := \inf_{u \in H} \|f-u\|_{L_p(\mathbb{G});\alpha,\beta} \qquad (1)$$

is called the best (α,β)-approximation of a function $f \in L_p(\mathbb{G})$ by the set $H \subset L_p(\mathbb{R})$ in the metric $L_p(\mathbb{G})$.

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The notation $E(f)_{L_p(\mathbb{G});\alpha,\beta}$ will be used for the best (α,β)-approximation of a function $f\in L_p(\mathbb{G})$ by subset of constants.

For class of functions $M\subset L_p(\mathbb{G})$, the quantity

$$E(M,H)_{L_p(\mathbb{G});\alpha,\beta} := \sup_{f\in M} E(f,H)_{L_p(\mathbb{G});\alpha,\beta} \tag{2}$$

is called the best (α,β)-approximation of the class M by the set H in the space $L_p(\mathbb{G})$. If $\alpha=\beta=1$ the quantities (1) and (2) coincide with ordinary best L_p-approximation of a function f (notation $E(f,H)_{L_p(\mathbb{G})}$) and of class M (notation $E(M,H)_{L_p(\mathbb{G})}$), respectively.

Let the set $H\subset L_p(\mathbb{G})$ be fixed. We associate with the function f the subsets

$$H_f^+ = \{u(t)\colon u\in H,\ u(t)\le f(t),\ t\in\mathbb{G}\} \quad\text{and}\quad H_f^- = \{u(t)\colon u\in H,\ u(t)\ge f(t),\ t\in\mathbb{G}\}.$$

For $f\in L_p(\mathbb{G})$ and $M\subset L_p(\mathbb{G})$ we set

$$E^{\pm}(f,H)_{L_p(\mathbb{G})} = \begin{cases} \inf\{\|f-u\|_{L_p(\mathbb{G})}\colon u\in H_f^{\pm}\}, & H_f^{\pm}\ne\varnothing,\\ \infty, & H_f^{\pm}=\varnothing,\end{cases}$$

and

$$E^{\pm}(M,H)_{L_p(\mathbb{G})} = \sup_{f\in M} E^{\pm}(f,H)_{L_p(\mathbb{G})}.$$

Quantities $E^{\pm}(f,H)_{L_p(\mathbb{G})}$ and $E^{\pm}(M,H)_{L_p(\mathbb{G})}$ are called the best approximation from below $(+)$ and from above $(-)$ of a function $f\in L_p(\mathbb{G})$ and $M\subset L_p(\mathbb{G})$, respectively.

In the case when $\mathbb{G}$ is a segment, V.F. Babenko [1] (see also [7, Theorems 1.4.10 and 1.5.9]) established that if the set $H\subset L_p(\mathbb{G})$, $1\le p\le\infty$, is locally compact then for any function $f\in L_p(\mathbb{G})$, monotonously on α and β

$$\lim_{\beta\to\infty} E(f,H)_{L_p(\mathbb{G});1,\beta} = E^+(f,H)_{L_p(\mathbb{G})}, \quad \lim_{\alpha\to\infty} E(f,H)_{L_p(\mathbb{G});\alpha,1} = E^-(f,H)_{L_p(\mathbb{G})}, \tag{3}$$

for any set $M\subset L_p(\mathbb{G})$, monotonously on α and β

$$\lim_{\beta\to\infty} E(M,H)_{L_p(\mathbb{G});1,\beta} = E^+(M,H)_{L_p(\mathbb{G})}, \quad \lim_{\alpha\to\infty} E(M,H)_{L_p(\mathbb{G});\alpha,1} = E^-(M,H)_{L_p(\mathbb{G})}. \tag{4}$$

In the case $\mathbb{G}=\mathbb{R}$ one can prove (3) and (4) the same.

For $r\in\mathbb{N}$, let

$$L_p^r(\mathbb{G}) = \{f\in L_1(\mathbb{G})\colon f^{(r-1)}\in AC(\mathbb{G}), f^{(r)}\in L_p(\mathbb{G})\},$$
$$W_p^r(\mathbb{G}) = \{f\in L_p^r(\mathbb{G})\colon \|f^{(r)}\|_{L_p(\mathbb{G})}\le 1\}, \quad W_{p,q}^r(\mathbb{G}) = W_p^r(\mathbb{G})\cap L_q(\mathbb{G}).$$

For $h>0$ and $m\in\mathbb{Z}_+$, by $S_{h,m}(\mathbb{R})$ we denote the collection of functions $s\in C^{(m-1)}(\mathbb{R})$, $m\ge 1$, such that $s^{(m)}|_{(jh;(j+1)h)} = c_j = const$, $j\in\mathbb{Z}$. The set $S_{h,m}(\mathbb{R})$ is called the space of polynomial splines on $\mathbb{R}$ of order m and defect 1 with nodes at the points jh, $j\in\mathbb{Z}$.

The subspace of 2π-periodic polynomial splines of order m and defect 1 with nodes at the points $j\pi/n$, $j\in\mathbb{Z}$, we denote as $S_{\pi/n,m}(\mathbb{T})$.

For $\alpha,\beta>0$, $\lambda>0$, $m\in\mathbb{N}$, by $\varphi_{\lambda,m}(\alpha,\beta;t)$ we denote $(2\pi/\lambda)$-periodic integral of order m with zero mean over the period from even $(2\pi/\lambda)$-periodic function $\varphi_{\lambda,0}(\alpha,\beta;t)$, which for $t\in[0,\pi/n)$ is defined as follows

$$\varphi_{\lambda,0}(\alpha,\beta;t)=\begin{cases}\alpha, & 0\le t\le \pi\beta/(\lambda(\alpha+\beta)),\\ -\beta, & \pi\beta/(\lambda(\alpha+\beta))<t<\pi/n.\end{cases}$$

For $\alpha=\beta=1$ instead of $\varphi_{\lambda,m}(\alpha,\beta;t)$ we will write $\varphi_{\lambda,m}(t)$.

By

$$B_{\lambda,m}(t)=-2\lambda^{-m}\sum_{k=1}^{n}\frac{cos(k\lambda t-\pi m/2)}{k^m},\quad m\in\mathbb{N},$$

we denote the Bernoulli kernel of order m (see [7, p. 107]).

Note (see [7, p.109]) that for $m\ge 2$ and $\beta\to\infty$ one has

$$\|\varphi_{1,m}(1,\beta;\cdot)-B_{\lambda,m}\|_{L_\infty(\mathbb{T})}\to 0. \tag{5}$$

Moreover, for all $m\in\mathbb{N}$ and $\beta\to\infty$,

$$E(\varphi_{1,m}(1,\beta;\cdot))_{L_\infty(\mathbb{T})}\to E(B_{1,m})_{L_\infty(\mathbb{T})}. \tag{6}$$

1 Preliminary information and main results

Let $p=1,\infty$. It is well known, that for $n,r,m\in\mathbb{N}$, $m\ge r-1$,

$$E(W_p^r(\mathbb{T}),S_{\pi/n,m}(\mathbb{T}))_{L_p(\mathbb{T})}=\frac{\|\varphi_{1,r}\|_{L_\infty(\mathbb{T})}}{n^r}. \tag{7}$$

In the case $p=\infty$ and $m=r-1$, the equality (7) was established by V.M. Tikhomirov [14] and in the other cases by A.A. Ligun [8].

Similar results for one-sided approximations of classes $W_p^r(\mathbb{T})$ by splines were obtained by V.G. Doronin and A.A. Ligun [6]. They proved that for $n,m,r\in\mathbb{N}$, $m\ge r$,

$$E^{\pm}(W_1^r(\mathbb{T}),S_{\pi/n,m}(\mathbb{T}))_{L_1(\mathbb{T})}=\frac{E(B_{1,r})_{L_\infty(\mathbb{T})}}{n^r}. \tag{8}$$

Later V.F. Babenko [1] established, that for $n,m,r\in\mathbb{N}$, $m\ge r$, and $\alpha,\beta>0$,

$$E^{\pm}(W_1^r(\mathbb{T}),S_{\pi/n,m}(\mathbb{T}))_{L_1(\mathbb{T});\alpha,\beta}=\frac{E(\varphi_{1,r}(\alpha,\beta;\cdot))_{L_\infty(\mathbb{T})}}{n^r}. \tag{9}$$

Note that considering (5), (6), the result (8) can be obtained from (9) by passing to the limit.

For other results on best approximations of classes $W_1^r(\mathbb{T})$ by splines in a periodic case see [2–5, 11, 12] and references therein.

In was proved in [9] that for all $r\in\mathbb{N}$, $m\in\mathbb{Z}_+$, $m\ge r-1$, and $h>0$,

$$E(W_{1,1}^r(\mathbb{R}),S_{h,m}(\mathbb{R})\cap L_1(\mathbb{R}))_{L_1(\mathbb{R})}=\frac{\|\varphi_{1,r}\|_{L_\infty(\mathbb{T})}\cdot h^r}{\pi^r}. \tag{10}$$

Moreover, it is shown in [10] that this result can be obtained from its periodic analogue (7).

In this paper, we obtain non-symmetric analogs of equality (10), based on the result of V.F. Babenko (9).

Theorem. *Let $\alpha,\beta>0$, $r,n,m\in\mathbb{N}$, $m\ge r$, $h>0$. Then*

$$E(W_{1,1}^r(\mathbb{R}),S_{h,m}(\mathbb{R})\cap L_1(\mathbb{R}))_{L_1(\mathbb{R});\alpha,\beta}=\frac{E(\varphi_{1,r}(\alpha,\beta;\cdot))_{L_\infty(\mathbb{T})}\cdot h^r}{\pi^r}.$$

Corollary. *Let $\alpha,\beta>0$, $r,n,m\in\mathbb{N}$, $m\ge r$, $h>0$. Then*

$$E^{\pm}(W_{1,1}^r(\mathbb{R}),S_{h,m}(\mathbb{R})\cap L_1(\mathbb{R}))_{L_1(\mathbb{R})}=\frac{E(B_{1,r})_{L_\infty(\mathbb{T})}h^r}{\pi^r}.$$

2 Proof of the main result

We give a proof of our theorem.

Proof. We obtain an upper bound first. In doing so, we will use ideas and methods from [10]. Let $n \in \mathbb{N}$ and $h > 0$. Let us consider the class $W_p^r(\frac{nh}{\pi}\mathbb{T})$ of $2nh$-periodic functions. Note that $f \in W_p^r(\frac{nh}{\pi}\mathbb{T})$ if and only if $g(t) = f(\frac{nh}{\pi}t) \in (\frac{nh}{\pi})^r \cdot W_p^r(\mathbb{T})$. Similarly, $s \in S_{h,m}(\frac{nh}{\pi}\mathbb{T})$ if and only if $\sigma(t) = s(\frac{nh}{\pi}t) \in S_{h,m}(\mathbb{T})$. Then from (9) we obtain

$$\begin{aligned} E\Big(W_p^r\Big(\frac{nh}{\pi}\mathbb{T}\Big), S_{h,m}\Big(\frac{nh}{\pi}\mathbb{T}\Big)\Big)_{L_p(\frac{nh}{\pi}\mathbb{T});\alpha,\beta} &= \Big(\frac{nh}{\pi}\Big)^r E(W_p^r(\mathbb{T}), S_{h,m}(\mathbb{T}))_{L_p(\mathbb{T});\alpha,\beta} \\ &= \frac{E(\varphi_{1,r}(\alpha,\beta;\cdot))_{L_\infty(\mathbb{T})} \cdot h^r}{\pi^r}. \end{aligned} \tag{11}$$

Let now $f \in W_{1,1}^r(\mathbb{R})$ and $\eta(\cdot)$ infinitely differentiable on $\mathbb{R}$ function such that $0 \le \eta(t) \le 1$, $t \in \mathbb{R}$, $\operatorname{supp}\eta(\cdot) \subset [-1,1]$ and $\eta(t) = 1$ subject to $t \in [-1/2, 1/2]$. For any $l > 0$ put $\eta_l(t) := \eta(t/l)$ and $f_l(\cdot) = f(\cdot)\eta_l(\cdot)$. By Leibniz's formula we have

$$f_l^{(r)}(t) = \sum_{j=0}^{r} \binom{r}{j} f^{(j)}(t) l^{-(r-j)} \eta_l^{(r-j)}(t). \tag{12}$$

Stain's inequality [13] implies the boundedness of the derivatives $x^{(j)}$, $j = 1, 2, \dots, r-1$, in $L_1(\mathbb{R})$. From here and from (12) we obtain

$$\|f_l^{(r)}\|_{L_1(\mathbb{R})} \le \rho(l) + \|f^{(r)}\|_{L_1(\mathbb{R})} \le \rho(l) + 1,$$

where $\rho(l) \to 0$ as $l \to \infty$.

By $\tilde{f}_h$ denote $4nh$-periodic continuation of function $(\rho(2nh)+1)^{-1} f_{2nh}(\cdot) \subset W^r_{[-2nh,2nh]}$ and $\|\tilde{f}_h^{(r)}\|_{L_p(\frac{nh}{\pi}\mathbb{T})} \le 1$. Thus $\tilde{f}_h \in W_p^r(\frac{nh}{\pi}\mathbb{T})$. Since $S_{h,m}(\frac{nh}{\pi}\mathbb{T})$ is a finite-dimensional space, there exists a spline s_n^* from this space such that

$$\|\tilde{f}_h - s_n^*\|_{L_p(\frac{nh}{\pi}\mathbb{T});\alpha,\beta} = E\Big(\tilde{f}_h, S_{h,m}\Big(\frac{nh}{\pi}\mathbb{T}\Big)\Big)_{L_p(\frac{nh}{\pi}\mathbb{T});\alpha,\beta}.$$

Since

$$E\Big(\tilde{f}_h, S_{h,m}\Big(\frac{nh}{\pi}\mathbb{T}\Big)\Big)_{L_1(\frac{nh}{\pi}\mathbb{T});\alpha,\beta} \le \|\tilde{f}_h\|_{L_1(\frac{nh}{\pi}\mathbb{T});\alpha,\beta},$$

using the inequality

$$\min\{\alpha,\beta\}\|\cdot\|_{L_p(\mathbb{G})} \le \|\cdot\|_{L_p(\mathbb{G});\alpha,\beta} \le \max\{\alpha,\beta\}\|\cdot\|_{L_p(\mathbb{G})},$$

we get

$$\min\{\alpha,\beta\}\|s_n^*\|_{L_1(\frac{nh}{\pi}\mathbb{T})} \le \|s_n^* - \tilde{f}_h + \tilde{f}_h\|_{L_1(\frac{nh}{\pi}\mathbb{T});\alpha,\beta} \le 2\|\tilde{f}_h\|_{L_1(\frac{nh}{\pi}\mathbb{T});\alpha,\beta} \le 2\max\{\alpha,\beta\}\|\tilde{f}_h\|_{L_1(\frac{nh}{\pi}\mathbb{T})},$$

whence

$$\|s_n^*\|_{L_1(\frac{nh}{\pi}\mathbb{T})} \le \frac{2\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}\|\tilde{f}_h\|_{L_1(\frac{nh}{\pi}\mathbb{T})}.$$

Thus the conditions of Proposition 1 in [10] are satisfied. According to them there exists a spline $\xi \in S_{h,m}(\mathbb{R}) \cap L_1(\mathbb{R})$ and a sequence $\{s_{n_j}^*\}_{j\in\mathbb{N}}$ such that $\|\xi - s_{n_j}^*\|_{L_\infty(I)} \to 0$, $j \to \infty$, for any finite segment $I \subset \mathbb{R}$.

We set $a_n = \rho(2nh) + 1$. Taking into account that $\tilde{f}_h(\cdot)|_{[-nh,nh]} = a_n^{-1} f(\cdot)$ for all $n \in \mathbb{N}$ and for all $j \in \mathbb{N}$ such that $n_j \geq n$, based on (11) we will have

$$\begin{aligned} \|f - s_{n_j}^*\|_{L_1([-nh,nh]);\alpha,\beta} &\leq \|a_{n_j}\tilde{f}_{n_j} - s_{n_j}^*\|_{L_1([-n_jh,n_jh]);\alpha,\beta} \\ &\leq a_{n_j} E\Big(W_1^r\Big(\frac{nh}{\pi}\mathbb{T}\Big), S_{h,m}\Big(\frac{nh}{\pi}\mathbb{T}\Big)\Big)_{L_1(\frac{nh}{\pi}\mathbb{T});\alpha,\beta} \\ &= a_{n_j}\frac{E(\varphi_{1,r}(\alpha,\beta;\cdot))_{L_\infty(\mathbb{R})}h^r}{\pi^r}. \end{aligned} \tag{13}$$

Passing in (13) to the limit as $j \to \infty$, according to [10, Proposition 1] we obtain

$$\|\tilde{f} - \xi\|_{L_1([-nh,nh]);\alpha,\beta} \leq \frac{E(\varphi_{1,r}(\alpha,\beta;\cdot))_{L_\infty(\mathbb{R})}h^r}{\pi^r}.$$

Whence, taking into account Fatou's lemma, we have

$$\|\tilde{f} - \xi\|_{L_1(\mathbb{R});\alpha,\beta} \leq \frac{E(\varphi_{1,r}(\alpha,\beta;\cdot))_{L_\infty(\mathbb{R})}h^r}{\pi^r}.$$

The upper bound is obtained.

Let c_0 be the constant of the best approximation of function $\varphi_{\pi/h,r}(\alpha,\beta;t)$. For $\varepsilon > 0$ we consider the set

$$e = \{t \in [-h,h] : |\varphi_{\pi/h,r}(\alpha,\beta;t) - c_0| > E(\varphi_{\pi/h,r}(\alpha,\beta;\cdot))_{L_\infty(\mathbb{T})} - \varepsilon\}.$$

Let $f_\varepsilon(t)$ be the $2h$-periodic function such that

$$f_\varepsilon(t) = \begin{cases} (\operatorname{mes} e)^{-1}\operatorname{sign}(\varphi_{\pi/h,r}(\alpha,\beta;t) - c_0), & t \in e, \\ 0, & t \in ([-h,h] \setminus e), \end{cases}$$

and $f_{\varepsilon,r}$ be the rth antiderivative of $f_\varepsilon(t)$.

By $\tilde{f}_{\varepsilon,r}$ we denote the product $f_{\varepsilon,r}(t)\eta(t)$, where $\eta(t)$ is infinitely differentiable on $\mathbb{R}$ function such that $0 \leq \eta(t) \leq 1$, $t \in \mathbb{R}$ and

$$\eta(t) = \begin{cases} 1, & t \in [-\varepsilon - h, h + \varepsilon], \\ 0, & t \in (\mathbb{R} \setminus [-3\varepsilon - h, h + 3\varepsilon]), \end{cases}$$

(see [15, p. 77]). It is clear that $\tilde{f}_{\varepsilon,r} \in L_1(\mathbb{R})$, $\operatorname{supp}\tilde{f}_{\varepsilon,r} = [-3\varepsilon - h, h + 3\varepsilon]$, and

$$\tilde{f}_{\varepsilon,r}^{(r)}(t) = \sum_{j=0}^{r}\binom{r}{j} f_{\varepsilon,r}^{(j)}(t)\eta^{(r-j)}(t).$$

Since for $j = 0, 1, \ldots, r-1$

$$\operatorname{supp}\eta^{(r-j)} = [-h - 3\varepsilon, -h - \varepsilon] \cup [h + \varepsilon, h + 3\varepsilon],$$

$\eta^{(r-j)}$ and $f_{\varepsilon,r}^{(j)}(t)$ are continuous on both segments $[-h - 3\varepsilon, -h - \varepsilon]$ and $[h + \varepsilon, h + 3\varepsilon]$, we can represent the function $\tilde{f}_{\varepsilon,r}^{(r)}(t)$ as $\tilde{f}_{\varepsilon,r}^{(r)}(t) = f_\varepsilon(t)\eta(t) + \rho(t)$, where

$$\rho(t) = \begin{cases} O(1), & t \in [-h - 3\varepsilon, -h - \varepsilon] \cup [h + \varepsilon, h + 3\varepsilon], \\ 0, & t \in (\mathbb{R} \setminus ([-h - 3\varepsilon, -h - \varepsilon] \cup [h + \varepsilon, h + 3\varepsilon])), \end{cases}$$

in particular, there exists a constant C such that $|\rho(t)| \le C$.

Now, we establish the upper estimate for the norm $\|\tilde{f}_{\varepsilon,r}^{(r)}\|_{L_1(\mathbb{R})}$:

$$\|\tilde{f}_{\varepsilon,r}^{(r)}\|_{L_1(\mathbb{R})} = \int_{-h-3\varepsilon}^{h+3\varepsilon} |f_\varepsilon(t)\eta(t) + \rho(t)|\, dt \le \int_{-h-3\varepsilon}^{h+3\varepsilon} |f_\varepsilon(t)\eta(t)|\, dt + \int_{[-h-3\varepsilon,-h-\varepsilon]\cup[h+\varepsilon,h+3\varepsilon]} |\rho(t)|\, dt$$
$$\le \|f_\varepsilon\|_{L_1([-h,h])} + \int_{[-h-3\varepsilon,-h]\cup[h,h+3\varepsilon]} |f_\varepsilon(t)|\, dt + 4C\varepsilon \le 1 + C_1\varepsilon, \quad C_1 > 0.$$

Thus the function $\tilde{f}_{\varepsilon,r}/(1+C_1\varepsilon) \in W_{1,1}^r(\mathbb{R})$. Let us obtain a lower estimate for the quantity $E := E(W_{1,1}^r(\mathbb{R}), S_{h,m}(\mathbb{R}) \cap L_1(\mathbb{R}))_{L_1(\mathbb{R});\alpha,\beta}$.

Using the duality theorem (see, for example, [7, Theorem 1.4.9]) we can write

$$E = \sup_{f\in W_{1,1}^r(\mathbb{R})} \sup_{\substack{\|g\|_{\infty;\alpha^{-1},\beta^{-1}} \\ g\perp S_{h,m}(\mathbb{R})\cap L_1(\mathbb{R})}} \int_{\mathbb{R}} f(t)g(t)\, dt.$$

Based on [9, Lemmas 1.4 and 1.1] after r times integration by parts we have

$$E \ge \sup_{f\in W_{1,1}^r(\mathbb{R})} \sup_{\substack{g\in W^{m+1}_{\infty;\alpha^{-1},\beta^{-1}}(\mathbb{R}) \\ g(kh)=0,k\in\mathbb{Z}}} \int_{\mathbb{R}} f(t)g^{(m+1)}(t)\, dt = \sup_{f\in W_{1,1}^r(\mathbb{R})} \sup_{\substack{g\in W^{m+1}_{\infty;\alpha^{-1},\beta^{-1}}(\mathbb{R}) \\ g(kh)=0,k\in\mathbb{Z}}} \int_{\mathbb{R}} f^{(r)}(t)g^{(m-r+1)}(t)\, dt.$$

Since (with the corresponding shift) the function $\varphi_{\pi/h,m+1}(\alpha,\beta;t) \in W^{m+1}_{\infty;\alpha^{-1},\beta^{-1}}(\mathbb{R})$ satisfies the conditions $\varphi_{\pi/h,m+1}(\alpha,\beta;kh) = 0$, $k\in\mathbb{Z}$, we obtain

$$\begin{aligned}
E \ge \sup_{f\in W_{1,1}^r(\mathbb{R})} \int_{\mathbb{R}} f^{(r)}(t)\varphi_{\pi/h,r}(\alpha,\beta;t)\, dt &\ge \frac{1}{1+C_1\varepsilon}\int_{\mathbb{R}} \tilde{f}_{\varepsilon,r}^{(r)}(t)\varphi_{\pi/h,r}(\alpha,\beta;t)\, dt \\
&= \frac{1}{1+C_1\varepsilon}\Big(\int_{-h-3\varepsilon}^{h+3\varepsilon} (f_\varepsilon(t)\eta(t)+\rho(t))\varphi_{\pi/h,r}(\alpha,\beta;t)\, dt\Big) \\
&= \frac{1}{1+C_1\varepsilon}\Big(\int_{-h}^{h} f_\varepsilon(t)(\varphi_{\pi/h,r}(\alpha,\beta;t) - c_0)\, dt \\
&\quad + \int_{[-h-3\varepsilon,-h]\cup[h,h+3\varepsilon]} (f_\varepsilon\eta(t)+\rho(t))\varphi_{\pi/h,r}(\alpha,\beta;t)\, dt\Big) \\
&\ge \frac{1}{1+C_1\varepsilon}\Big(\frac{1}{\operatorname{mes} e}\int_e |\varphi_{\pi/h,r}(\alpha,\beta;t) - c_0|\, dt + C_2\varepsilon\Big) \\
&> \frac{1}{1+C_1\varepsilon}\big(E(\varphi_{\pi/h,r}(\alpha,\beta;\cdot)) + (C_2-1)\varepsilon\big).
\end{aligned}$$

Since ε is arbitrary, we obtain the required lower bound. Thus, the theorem is proved. □

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Парфінович Н.В. *Несиметричні наближення функціональних класів сплайнами на дійсній осі* // Карпатські матем. публ. — 2021. — Т.13, №3. — C. 831–837.

Нехай $S_{h,m}$, $h > 0$, $m \in \mathbb{N}$ — простори поліноміальних сплайнів порядку m дефекту 1 з вузлами в точках kh, $k \in \mathbb{Z}$.

Отримано точні значення найкращих (α, β)-наближень просторами $S_{h,m} \cap L_1(\mathbb{R})$ у просторі $L_1(\mathbb{R})$ для класів $W^r_{1,1}(\mathbb{R})$, $r \in \mathbb{N}$, функцій, визначених на всій дійсній прямій, інтегрованих на $\mathbb{R}$ і таких, що r-ті похідні належать одиничній кулі $L_1(\mathbb{R})$.

Ці результати узагальнюють відомі результати Г.Г. Магарила-Ілляєва та В.М. Тихомирова щодо точних значень найкращих наближень класів $W^r_{1,1}(\mathbb{R})$ сплайнами з $S_{h,m} \cap L_1(\mathbb{R})$ (випадок $\alpha = \beta = 1$), а також є неперіодичними аналогами В.Ф. Бабенка щодо найкращих несиметричних наближень класів $W^r_1(\mathbb{T})$ 2π-періодичних функцій з r-тою похідною, що належить до одиничної кулі простору $L_1(\mathbb{T})$ періодичними поліноміальніми сплайнами мінімального дефекту.

Як наслідок основного результату, ми отримуємо точні значення найкращих односторонніх наближень класів W^r_1 поліноміальними сплайнами з $S_{h,m}(\mathbb{T})$. Цей результат є періодичним аналогом результатів А.А. Лігуна і В.Г. Дороніна про найкращі односторонні наближення класів W^r_1 просторами $S_{h,m}(\mathbb{T})$.

Ключові слова і фрази: найкраще L_1-наближення, одностороннє наближення, несиметричне наближення, поліноміальний сплайн, функціональний клас.