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On families of twisted power partial isometries

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We consider families of power partial isometries satisfying twisted commutation relations with deformation parameters $\lambda_{ij} \in \mathbb{C}$, $|\lambda_{ij}| = 1$. Irreducible representations of such a families are described up to the unitary equivalence. Namely any such representation corresponds, up to the unitary equivalence, to irreducible representation of certain higher-dimensional non-commutative torus.

Key words and phrases: partial isometry, centered operator, irreducible representation, non-commutative torus.

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1 Introduction

Algebras generated by families of isometries and partial isometries satisfying various deformed commutation relations form an important part in the theory of operator algebras and their representations. Among the central problems in the subject let us mention the classification of $*$-representations of corresponding algebra up to the unitary equivalence, construction of Wold type decomposition, describing the structure and dependence of isomorphism class of corresponding C^*-algebra with respect to deformation parameters, see e.g. [1,2,5–10].

In this paper we deal with the families of power partial isometries $\{s_i, s_i^*, i = 1, \ldots, d\}$ satisfying the commutation relations of the following form

$$s_i^* s_j = \lambda s_j s_i^*, \quad s_j s_i = \lambda s_i s_j, \quad 1 \le i < j \le d. \tag{1}$$

We give classification of such irreducible families up to the unitary equivalence, which can be regarded as a generalization of classical Halmos-Wallen decomposition of single power partial isometry, see [3].

Let us remind the definition of partial isometry.

Definition 1. *We call an element S of $*$-algebra $\mathcal{A}$ a partial isometry if*

$$(s^*s)^2 = s^*s, \quad \text{and} \quad (ss^*)^2 = ss^*.$$

It is easy to see from the definition above, that $s \in \mathcal{A}$ is a partial isometry if and only if s^* is so.

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Remark 1. *If S is an element of a C^*-algebra $\mathfrak{A}$, then for S to be a partial isometry is sufficient if it satisfies one of the conditions presented above. Moreover, in this case S is a partial isometry if and only if one of the following relations is satisfied*

$$S^*SS^* = S^*, \quad SS^*S = S.$$

Definition 2 ([3])**.** *An element s of a $*$-algebra $\mathcal{A}$ is called power partial isometry (centered partial isometry) if and only if for any $n \in \mathbb{N}$ elements s^n are partial isometries and the family of orthogonal projections*

$$\mathcal{T} = \{s^n(s^*)^n,\ (s^*)^m s^m,\ m, n \in \mathbb{N}\}$$

is commutative.

The key result in the description of single power partial isometry S, acting on the Hilbert space $\mathcal{H}$ is the Halmos-Wallen theorem, see [3,4].

Theorem 1. *Let $S \in B(\mathcal{H})$ be power partial isometry. Put*

- *P to be the orthogonal projection onto $\bigcap_{n=1}^{\infty} S^n\mathcal{H}$;*
- *Q to be the orthogonal projection onto $\bigcap_{n=1}^{\infty} (S^*)^n\mathcal{H}$.*

Then $PQ = QP$ and $\mathcal{H}$ can be decomposed into orthogonal sum of subspaces invariant with respect to S, S^*, namely

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \oplus \mathcal{H}_b \oplus \bigoplus_{p=1}^{\infty} \mathcal{H}_p,$$

where

$$\mathcal{H}_u = PQ\mathcal{H}, \quad \mathcal{H}_s = (\mathbf{1} - P)Q\mathcal{H}, \quad \mathcal{H}_b = (\mathbf{1} - Q)P\mathcal{H}$$

and

$$\mathcal{H}_p = \sum_{n=1}^{p} \big(S^{n-1}(S^*)^{n-1} - S^n(S^*)^n\big)\big((S^*)^{p-n}S^{p-n} - (S^*)^{p-n+1}S^{p-n+1}\big)\mathcal{H}.$$

Moreover, $\mathcal{H}_s = \mathcal{H}_b = l_2(\mathbb{Z}_+) \otimes \mathcal{F}$, for some Hilbert space $\mathcal{F}$, $\mathcal{H}_p = \mathbb{C}^p \otimes \mathcal{G}$, for some Hilbert space $\mathcal{G}$ and

- *the restriction of S onto $\mathcal{H}_u$ is unitary;*
- *the restriction of S onto $\mathcal{H}_s$ has the form $S = T \otimes \mathbf{1}_{\mathcal{F}}$, where*

  $$T\colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+)$$

  is unilateral shift;
- *the restriction of S onto $\mathcal{H}_b$ has the from $S = T^* \otimes \mathbf{1}_{\mathcal{F}}$;*
- *the restriction of S onto $\mathcal{H}_p$ has the form $S = J_p(0)$, where $J_p(0)$ denotes the nilpotent Jordan block of size p:*

  $$J_p(0)e_k = e_{k+1},\ k = 1, \dots, p-1,\ J_p(0)e_p = 0.$$

In the following we call the decomposition of power partial isometry, given by Halmosh-Wallen theorem, the HW decomposition.

The authors of [2,4] described finite families $\{S_i,\ i = 1, \dots, d\}$ of double-commuting power partial isometries. In particular they gave the description of such families up to the unitary equivalence. Our paper is natural continuation of this research.

2 Irreducible representations of twisted commuting partial power isometries

In this section, we give a description of families of power partial isometries $\{S_i, i = 1, \ldots, d\}$, satisfying

$$S_i^* S_j = \lambda_{ij} S_j S_i^*, \quad S_j S_i = \lambda_{ij} S_i S_j, \quad i \neq j, \tag{2}$$

where $\lambda_{ij} = \overline{\lambda}_{ji} \in \mathbb{C}$, $|\lambda_{ij}| = 1$.

Proposition 1. *Let $\{S_i,\ i = 1, \ldots, d\}$ be family of power partial isometries, satisfying (2) and acting on Hilbert space $\mathcal{H}$. Consider the HW-decomposition of S_1*

$$\mathcal{H} = \mathcal{H}_u^{(1)} \oplus \mathcal{H}_s^{(1)} \oplus \mathcal{H}_b^{(1)} \oplus \bigoplus_{p=1}^{\infty} \mathcal{H}_p^{(1)}.$$

Then any component of this decomposition is invariant with respect to S_j, S_j^*, $j = 2, \ldots, d$.

Proof. Put P_1 and Q_1 to be the orthogonal projections onto $\cap_{n=1}^{\infty} S_1^n \mathcal{H}$ and $\cap_{n=1}^{\infty} (S_1^*)^n \mathcal{H}$ respectively. Relations (2) imply that any of $S_1^n \mathcal{H}$, $(S_1^*)^n \mathcal{H}$, $n \geq 1$, is invariant with respect to S_j, S_j^*, $j = 2, \ldots, d$. Hence $\cap_{n=1}^{\infty} S_1^n \mathcal{H}$ and $\cap_{n=1}^{\infty} (S_1^*)^n \mathcal{H}$ are invariant with respect to S_j, S_j^*, $j = 2, \ldots, d$, also. Then for all $j = 2, \ldots, d$

$$P_1 S_j = S_j P_1,\ P_1 S_j^* = S_j^* P_1,\ Q_1 S_j = S_j Q_1,\ Q_1 S_j^* = S_j^* Q_1$$

implying the invariance of $\mathcal{H}_u^{(1)}$, $\mathcal{H}_s^{(1)}$ and $\mathcal{H}_b^{(1)}$.

The invariance of $\mathcal{H}_p^{(1)}$ follows from

$$S_1^n (S_1^*)^n S_j = S_j S_1^n (S_1^*)^n, \quad (S_1^*)^n S_1^n S_j = S_j (S_1^*)^n S_1^n, \quad n \geq 1,\ j = 2, \ldots, d.$$

□

Corollary 1. *Let the family $\{S_i,\ i = 1, \ldots, d\}$ of power partial isometries satisfying (2) be irreducible. Then $\mathcal{H}$ coincides with exactly one of the components of its HW-decomposition.*

Theorem 2. *Let $\{S_i,\ i = 1, \ldots, d\}$ be irreducible family of power partial isometries, satisfying (2). Consider HW-realisation of S_1.*

1. *In the case $\mathcal{H} = \mathcal{H}_s$ one has*

$$S_1 = T \otimes \mathbf{1}_{\mathcal{F}}, \quad S_j = d(\lambda_{1j}) \otimes \widetilde{S}_j, \quad j = 2, \ldots, d. \tag{3}$$

2. *In the case $\mathcal{H} = \mathcal{H}_b$ one has*

$$S_1 = T^* \otimes \mathbf{1}_{\mathcal{F}}, \quad S_j = d(\lambda_{j1}) \otimes \widetilde{S}_j, \quad j = 2, \ldots, d. \tag{4}$$

3. *In the case $\mathcal{H} = \mathcal{H}_p$ one has*

$$S_1 = J_p(0) \otimes \mathbf{1}_{\mathcal{G}}, \quad S_j = D_p(\lambda_{1j}) \otimes \widetilde{S}_j,\ j = 2, \ldots, d. \tag{5}$$

Here

$$d(\lambda)\colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+), \quad d(\lambda) e_n = \lambda^n e_n, \quad n \in \mathbb{Z}_+,$$
$$D_p(\lambda)\colon \mathbb{C}^p \to \mathbb{C}^p, \quad D_p(\lambda) e_n = \lambda^{n-1} e_n, \quad n = 1, \ldots, p,$$

and in all cases above $\{\widetilde{S}_j,\ j = 2, \ldots, d\}$ are families of irreducible power partial isometries acting on the corresponding Hilbert space and satisfying (2) with $d-1$ generators.

4. *If $\mathcal{H} = \mathcal{H}_u$, then*

$$S_1 = U_1, \quad U_1^* S_j = \lambda_{1j} S_j U_1^*, \quad S_j U_1 = \lambda_{1j} U_1 S_j, \quad j = 2, \ldots, d, \tag{6}$$

where U_1 is unitary, operators S_j, $j = 2, \ldots, d$, satisfy (2) *and the family $\{U_1, S_j, j = 2, \ldots, d\}$ is irreducible.*

Families, corresponding to different cases are non-equivalent, families corresponding to $\{\widetilde{S}_j^{(1)}\}$ and $\{\widetilde{S}_j^{(2)}\}$ inside the same case are equivalent if and only if the latter families are equivalent.

Proof. *1.* Suppose $\mathcal{H} = \mathcal{H}_s = l_2(\mathbb{Z}_+) \otimes \mathcal{F}$ and $S_1 = T \otimes \mathbf{1}_{\mathcal{F}}$. Let e_n, $n \in \mathbb{Z}_+$ be standard basis of $l_2(\mathbb{Z}_+)$. Present $l_2(\mathbb{Z}_+) \otimes \mathcal{F}$ as

$$l_2(\mathbb{Z}_+) \otimes \mathcal{F} = \bigoplus_{i=0}^{\infty} e_i \otimes \mathcal{F}$$

and put $\mathcal{H}_i = e_i \otimes \mathcal{F}$, $i \in \mathbb{Z}_+$. Then operators

$$P_i = S_1^i (S_1^*)^i - S_1^{i+1} (S_1^*)^{i-1}$$

are orthogonal projections onto $\mathcal{H}_i$. It is easy to see that

$$S_j P_i = P_i S_j, \quad S_j^* P_i = P_i S_j^*, \quad i \in \mathbb{Z}_+, \ j = 2, \ldots, d,$$

implying that operators S_j, S_j^*, $j = 2, \ldots, d$, leave any of $\mathcal{H}_i$ invariant. Denote by $S_j^{(i)}$ the restriction of S_j onto $\mathcal{H}_i$. Evidently, we can identify $S_j^{(i)}$ with operator on $\mathcal{F}$ denoted by the same symbol. Then for any $x \in \mathcal{F}$ one has

$$S_j S_1 (e_i \otimes x) = e_{i+1} \otimes S_j^{i+1}(x), \quad S_1 S_j (e_i \otimes x) = e_{i+1} \otimes S_j^{(i)} x.$$

Then the relation $S_j S_1 = \lambda_{1j} S_1 S_j$ gives $S_j^{(i)} = \lambda_{1j}^i S_j^{(0)}$, $i \in \mathbb{Z}_+$, $j = 2, \ldots, d$. Finally, put $\widetilde{S}_j = S_j^{(0)}$. Obviously

$$S_j = d(\lambda_{1j}) \otimes \widetilde{S}_j, \quad j = 2, \ldots, d.$$

To deal with irreducibility we study a structure of operator C commuting with S_i, S_i^*, $i = 1, \ldots, d$. Namely, if

$$CS_1 = S_1 C, \ CS_1^* = S_1^* C, \text{ with } S_1 = T \otimes \mathbf{1}_{\mathcal{F}},$$

one has

$$C = \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \widetilde{C},$$

then $CS_j = S_j C$, $CS_j^* = S_j^* C$, $j = 2, \ldots, d$, iff

$$C\widetilde{S}_j = \widetilde{S}_j C, \quad C\widetilde{S}_j^* = \widetilde{S}_j^* C, \quad j = 2, \ldots, d.$$

The Schur's lemma implies that $\{S_i,\ i = 1, \ldots, d\}$ is irreducible iff $\{\widetilde{S}_j,\ j = 2, \ldots, d\}$ is irreducible. By the similar way one can show that two families $\{S_i^{(\varepsilon)},\ i = 1, \ldots, d\}$, $\varepsilon = 1, 2$, defined by (3) are unitarily equivalent iff the corresponding families $\{\widetilde{S}_j^{\varepsilon},\ j = 2, \ldots, d\}$, $\varepsilon = 1, 2$, are unitarily equivalent.

The remained cases can be considered analogously. □

Now we are ready to formulate our classification result. Consider arbitrary decomposition

$$\{1,2,\dots,d\} = \Phi_s \cup \Phi_b \cup \Psi \cup \Phi_u,$$

where components are disjoint sets and

$$\Psi = \bigcup_{p=1}^{\infty} \Phi_p$$

with finite number of non-empty components. Given such a decomposition, construct the following family of operators acting on

$$\mathcal{H} = \bigotimes_{i\in\Phi_s} l_2(\mathbb{Z}_+) \otimes \bigotimes_{j\in\Phi_b} l_2(\mathbb{Z}_+) \otimes \bigotimes_{p,\,\Phi_p\neq\varnothing} \mathbb{C}^p \bigotimes \mathcal{H}_u,$$

namely

$$\begin{aligned}
S_j &= \bigotimes_{i<j,\, i\in\Phi_s} d(\lambda_{ij}) \otimes T \bigotimes_{i>j,\, i\in\Phi_s} \mathbf{1}_{l_2(\mathbb{Z}_+)} \bigotimes_{i\in\Phi_b} \mathbf{1}_{l_2(\mathbb{Z}_+)} \bigotimes_{i\in\Phi_q,\,\Phi_q\neq\varnothing} \mathbf{1}_{\mathbb{C}^q} \otimes \mathbf{1}_{\mathcal{H}_u}, \quad j\in\Phi_s,\\
S_j &= \bigotimes_{i\in\Phi_s} d(\lambda_{ij}) \bigotimes_{i<j,\, i\in\Phi_b} d(\lambda_{ji}) \otimes T^* \bigotimes_{i>j,\, i\in\Phi_b} \mathbf{1}_{l_2(\mathbb{Z}_+)} \bigotimes_{i\in\Phi_q,\,\Phi_q\neq\varnothing} \mathbf{1}_{\mathbb{C}^q} \otimes \mathbf{1}_{\mathcal{H}_u}, \quad j\in\Phi_b,\\
S_j &= \bigotimes_{i\in\Phi_s} d(\lambda_{ij}) \bigotimes_{i\in\Phi_b} d(\lambda_{ji}) \bigotimes_{\substack{i\in\Phi_q,\\ \Phi_q\neq\varnothing,\, i<j}} D_q(\lambda_{ij}) \otimes J_p(0) \bigotimes_{\substack{i\in\Phi_q,\\ \Phi_q\neq\varnothing,\, i>j}} \mathbf{1}_{\mathbb{C}^q} \otimes \mathbf{1}_{\mathcal{H}_u}, \quad j\in\Phi_p\neq\varnothing,\\
S_j &= \bigotimes_{i\in\Phi_s} d(\lambda_{ij}) \bigotimes_{i\in\Phi_b} d(\lambda_{ji}) \bigotimes_{i\in\Phi_q,\,\Phi_q\neq\varnothing} D_q(\lambda_{ij}) \otimes U_j, \quad j\in\Phi_u,
\end{aligned} \tag{7}$$

where $\{U_j,\ j\in\Phi_u\}$ is an irreducible family of unitary operators on $\mathcal{H}_u$ satisfying

$$U_i^* U_j = \lambda_{ij}\, U_j U_i^*, \quad i\neq j,\ i,j\in\Phi_u.$$

Theorem 3. *Any irreducible family of power partial isometries $\{S_i,\ i=1,\dots,d\}$ satisfying* (2) *is unitarily equivalent to the family described above corresponding to certain decomposition*

$$\{1,2,\dots,d\} = \Phi_s \cup \Phi_b \cup \bigcup_{p=1}^{\infty} \Phi_p \cup \Phi_u.$$

Families, corresponding to different decompositions are non-equivalent. Families corresponding to the same decomposition are equivalent iff the related families $\{U_i,\ i\in\Phi_u\}$ are equivalent.

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Ми вивчаємо набори центрованих часткових ізометрій, що задовольняють деформовані комутаційні співвідношення, що відповідають параметрам деформації $\lambda_{ij} \in \mathbb{C}$, $|\lambda_{ij}| = 1$. Надано опис відповідних незвідних наборів, з точністю до унітарної еквівалентності. А саме, показано, що кожне незвідне зображення відповідає, з точністю до унітарної еквівалентності, незвідному зображенню певного багатовимірного некомутативного тору.

Ключові слова і фрази: часткова ізометрія, центрований оператор, незвідне зображення, некомутативний тор.