ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2022, **14** (1), 135–146
doi:10.15330/cmp.14.1.135-146

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2022, Т.14, №1, С.135–146

On statistically convergent complex uncertain sequences

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In this paper, we extend the study of statistical convergence of complex uncertain sequences in a given uncertainty space. We produce the relation between convergence and statistical convergence in an uncertain environment. We also initiate statistically Cauchy complex uncertain sequence to prove that a complex uncertain sequence is statistically convergent if and only if it is statistically Cauchy. We further characterize a statistically convergent complex uncertain sequence via boundedness and density operator.

Key words and phrases: uncertainty space, uncertain measure, statistical convergence, complex uncertain sequence, statistically Cauchy sequence.

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Introduction

Convergence of sequences plays a pivotal role in the study of fundamental theory of mathematics [17, 21, 23]. B. Liu [13] first introduced and studied four types of convergences (in mean, measure, distribution and almost surely) of real uncertain sequences and C. You [25] extended this work by commencing another convergence namely convergence in uniformly almost surely. Then, X. Chen et al. [2] explored the same concepts by considering complex uncertain sequences. Convergence of a complex uncertain double sequence is initiated by D. Datta and B.C. Tripathy [8]. In current days, various notions of convergences like almost convergence, strong convergence, statistical convergence are being studied in uncertain environment. For instance, one can go through B. Das et al. [5–7], S. Saha et al. [18], B.C. Tripathy and P.K. Nath [22], D. Datta and B.C. Tripathy [9] etc. The concept of statistical convergence of a real sequences was first commenced by H. Fast [10], R.C. Buck [1], I.J. Schohenberg [20] independently in the mid of the twentieth century. But the work of T. Šalát [19] and J.A. Fridy [11] gave the real momentum of research in this direction. B.C. Tripathy [24], M. Mursaleen and H.H.O. Edely [15], F. Móricz [14] independently explored this notion considering real double sequences. Some more works on statistical convergence may be seen in [3, 12].

It is pertinent to mention here that a subset E of the set $\mathbb{N}$ of natural numbers is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n} \frac{1}{n} |\{k \leq n : \ k \in E\}|,$$

УДК 517.521:519:2
2010 *Mathematics Subject Classification:* 40A05, 40A30, 40F05, 60B10, 60E05.

where the vertical bars denote the cardinality of the enclosed set.

B.C. Tripathy and P.K. Nath [22] initiated the study of statistical convergence considering complex uncertain sequence and established the interrelationships between several types of convergences. The same work was extended by considering complex uncertain double sequence by D. Datta and B.C. Tripathy [9]. In that context, B. Das et al. [4] further characterized statistical convergence of complex uncertain double sequences. Statistical convergence is introduced by considering triple sequences of complex uncertain variable also [5].

In this paper, we analyse this concept of statistical convergence in a given uncertainty space via natural density operator δ. We introduce the notions of complex uncertain bounded sequences in mean, measure, distribution, almost surely and uniformly almost surely and characterize these sequences to some extent. Finally we initiate statistically complex uncertain Cauchy sequence and establish that a complex uncertain sequence is statistically convergent if and only if it is statistically Cauchy.

We now procure some fundamental concepts and results on uncertainty theory, those will be used throughout the paper.

1 Preliminaries

Before going to the main section we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

Definition 1 ([13]). *Let $\mathcal{L}$ be σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms.*

Axiom 1 (Normality Axiom). $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2 (Duality Axiom). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any* $\Lambda \in L$.

Axiom 3 (Subadditivity Axiom). For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$, *we have*

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined as follows.

Axiom 4 (Product Axiom). Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, 3, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{j=1}^{\infty} \Lambda_j\right\} = \bigwedge_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\},$$

where Λ_j are arbitrarily chosen events from Γ_j for $j = 1, 2, 3, \ldots$, respectively.

Axiom 5 (Monotonicity Axiom). For any two events Λ_1 and Λ_2 with $\Lambda_1 \subseteq \Lambda_2$, we have $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$.

Definition 2 ([16]). *A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e. for any Borel set B of complex numbers, the set $\{\zeta \in B\} = \{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.*

Definition 3 ([22]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be statistically convergent almost surely (a.s.) to ζ if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : ||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = 0 \quad \text{for every} \quad \gamma \in \Lambda.$$

The collection of all statistical convergent complex uncertain sequence in almost surely is denoted by $st\text{-}\Gamma_{a.s}$.

Definition 4 ([22]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be statistically convergent in measure to ζ if*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \mathcal{M}\{||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon\} \geq \delta\}| = 0 \quad \text{for every} \quad \varepsilon, \delta > 0.$$

The family of all statistical convergent complex uncertain sequence in measure is denoted by $st\text{-}\Gamma_{\mathcal{M}}$.

Definition 5 ([22]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be statistically convergent in mean to ζ if*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : E(||\zeta_k(\gamma) - \zeta(\gamma)||) \geq \varepsilon\}| = 0 \quad \text{for every} \quad \varepsilon > 0.$$

The collection of all statistical convergent complex uncertain sequence in mean is denoted by $st\text{-}\Gamma_E$.

Definition 6 ([22]). *Let $\Phi, \Phi_1, \Phi_2, \ldots$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, \ldots$, respectively. We say the complex uncertain sequence $\{\zeta_n\}$ statistically converges in distribution to ζ if for every $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : ||\Phi_k(c) - \Phi(c)|| \geq \varepsilon\}| = 0$$

for all c at which Φ is continuous. The collection of all statistical convergent complex uncertain sequence in distribution is denoted by $st\text{-}\Gamma_D$.

Definition 7 ([22]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be statistically convergent uniformly almost surely (u.a.s) to ζ if for every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence of events E'_k such that*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |\mathcal{M}(E'_k) - 0| \geq \varepsilon\}| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : ||\zeta_k(x) - \zeta(x)|| \geq \delta\}| = 0.$$

The collection of all statistical convergent complex uncertain sequence with respect to uniformly almost surely is denoted by $st\text{-}\Gamma_{u.a.s}$.

Definition 8 ([2]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent almost surely (a.s.) to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that $\lim\limits_{n\to\infty} ||\zeta_n(\gamma) - \zeta(\gamma)|| = 0$ for every $\gamma \in \Lambda$.*

2 Some results on statistical convergence of complex uncertain sequences

In this section, we study statistically convergent complex uncertain through density operator and also by introducing the concept of boundedness.

Theorem 1. *A complex uncertain sequence, which converges in almost surely, is also statistically convergent to the same limit.*

Proof. Let the complex uncertain sequence $\{\zeta_n\}$ converges to ζ in almost surely. Then there exists an event Λ with unit uncertain measure so that for all $\varepsilon > 0$ there must have an $n_0 \in \mathbb{N}$ such that $||\zeta_n(\gamma) - \zeta(\gamma)|| < \varepsilon$ for all $\gamma \in \Lambda$ and $n \geq n_0$. Then we have

$$\max_n |\{n : \ ||\zeta_n(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = n_0^3.$$

Consequently,

$$\lim_{p\to\infty} \frac{1}{p} |\{n : \ n \leq p, ||\zeta_n(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| \leq \lim_{p\to\infty} \frac{n_0^3}{p} = 0 \quad \text{for all } \ \gamma \in \Lambda.$$

Hence, the complex uncertain sequence $\{\zeta_n\}$ statistically converges to ζ in almost surely. □

Remark 1. *The converse of the above theorem is not true, that is, a statistically convergent complex uncertain sequence may not be convergent therein. This claim is justified in the following example.*

Example 1. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots\}, \mathcal{L} = P(\Gamma)$. Let the uncertainty measure of events be defined as follows $\mathcal{M}\{\Lambda\} = \sum_{\gamma_i \in \Lambda} 2^{-i}$, where $\Lambda \subseteq \Gamma$. Let us now consider the complex uncertain sequence $\{\zeta_n\}$, where the complex uncertain variable ζ_n is defined by*

$$\zeta_n(\gamma) = \begin{cases} n, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise,} \end{cases}$$

for all $\gamma \in \Gamma$. Let ζ be another complex uncertain variable with $\zeta(\gamma) = 1$ for all $\gamma \in \Gamma$. Now, for any given $\varepsilon > 0$ and $\gamma \in \Gamma$ the set $\{n : \ ||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}$, that is $\{K : \ ||\zeta_k(\gamma) - 1|| \geq \varepsilon\}$, has cardinality less than or equal to $\sqrt{k}$.

Consequently,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : \ k \leq n, ||\zeta_k(\gamma) - 1|| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{1}{n}\sqrt{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Hence, the complex uncertain sequence $\{\zeta_n\}$ converges statistically to ζ. But it is obvious that the complex uncertain sequence $\{\zeta_n\}$ does not converge to any finite limit in respect of almost surely.

Remark 2. *From the above example, we can easily verify that the complex uncertain sequence $\{\zeta_n\}$ is not bounded in almost surely. Thus we can make a conclusion that an almost surely statically convergent complex uncertain triple sequence is not bounded with respect to almost surely.*

Theorem 2. *A convergent complex uncertain sequence converges statistically with preservation of limit in view of the concept uncertain measure.*

Proof. Let $\zeta = \{\zeta_n\}$ be a convergent complex uncertain sequence in measure to the limit ζ. Then for any given $\varepsilon > 0$ there exists a number $n_0 \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \mathcal{M}\{||\zeta_n - \zeta|| \geq \varepsilon\} = 0 \quad \text{for all } \ n > n_0,$$

which means that for any $\varepsilon > 0$ and $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}\{||\zeta_n - \zeta|| \geq \varepsilon\} > \delta$ for some $n \leq n_0$. This implies that the cardinality of the set $\{n : \ \mathcal{M}\{||\zeta_n - \zeta|| \geq \varepsilon\} \geq \delta\}$ is at most n_0^3. Hence,

$$\lim_{p\to\infty} \frac{1}{p}|\{n : \ n \leq p, \mathcal{M}\{||\zeta_n - \zeta|| \geq \varepsilon\} \geq \delta\}| \leq \lim_{p\to\infty} \frac{n_0^3}{p} = 0.$$

As a consequence, the complex uncertain sequence is statistically convergent to ζ in measure. □

Remark 3. *Theorem 1 is true for the cases of convergence in mean, in distribution and with respect to uniformly almost surely.*

By considering the expected value operator and complex uncertainty distribution function respectively in Theorem 2, we can easily prove that every convergent complex uncertain sequence is statistically convergent in mean and distribution, respectively.

Also, considering the sequence $\{E_x\}$ of events with uncertain measure of each of the events tending to zero and then by taking the events $\gamma \in \Gamma - E_x$ in the Theorem 1, it can be proved that every convergent complex uncertain sequence with respect to uniformly almost surely is statistically convergent therein.

Theorem 3. *A complex uncertain sequence $\{\zeta_n\}$ is statistically convergent in mean to ζ if and only if there exists a subset $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\lim\limits_{k\to\infty} E[||\zeta_k - \zeta||] = 0, k \in K$.*

Proof. Let the complex uncertain sequence $\{\zeta_n\}$ be statistically convergent in mean to ζ and $K_r = \{k \in \mathbb{N} : \ E[||\zeta_k - \zeta||] \geq 1/r\}$ with $L_r = \{k \in \mathbb{N} : \ E[||\zeta_k - \zeta||] < 1/r\}$. Then $\delta(K_r) = 0$,

$$L_1 \supset L_2 \supset \ldots \supset L_i \supset L_{i+1} \supset \ldots \tag{1}$$

and

$$\delta(L_r) = 1, \quad r = 1, 2, \ldots. \tag{2}$$

We want to show that for any $k \in L_r$ the complex uncertain sequence $\{\zeta_k\}$ is convergent in mean to ζ. If possible, let $\{\zeta_k\}$ be not convergent in mean to ζ. Therefore there is $\varepsilon > 0$ such that $E[||\zeta_k - \zeta||] \geq \varepsilon$ for infinitely many terms.

Let $L_\varepsilon = \{k : \ E[||\zeta_k - \zeta||] < \varepsilon\}$ and $r\varepsilon > 1$, $r = 1, 2, 3, \ldots$. Then $\delta(L_\varepsilon) = 0$ and by condition (1), $L_r \subset L_\varepsilon$. Hence, $\delta(L_\varepsilon) = 0$, which contradicts the condition (2). Therefore, $\{\zeta_k\}$ is convergent in mean to ζ.

Conversely, let there exists a subset $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\lim\limits_{k\to\infty} E[||\zeta_k - \zeta||] = 0$, $k \in K$, that is, there exists $n_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$ we have $E[||\zeta_k - \zeta||] < \varepsilon$ for all $k \geq n_0$. Now,

$$K_\varepsilon = \{k \in \mathbb{N} : \ E[||\zeta_k - \zeta||] \geq \varepsilon\} \subseteq \mathbb{N} - \{k_{n_0+1}, k_{n_0+2}, \ldots\},$$

that is, $\delta(K_\varepsilon) \leq 1 - 1 = 0$. Hence, $\{\zeta_k\}$ is statistically convergent in mean to ζ. □

Theorem 4. *The sequence $\{\zeta_n\}$ of complex uncertain variable is statistically convergent in distribution to ζ if and only if there exists $K \subseteq \mathbb{N}$ with density 1 and $\lim_{k\to\infty} \Phi_k(z) = \Phi(z)$, $k \in K$, where Φ, Φ_k are the distribution functions of the complex uncertain variables ζ, ζ_k, respectively, and z is the point at which Φ is continuous.*

Proof. Let $\{\zeta_n\}$ be a statistically convergent complex uncertain sequence in distribution to ζ and K_r and M_r be two sets defined as follows:

$$K_r = \{k \in \mathbb{N} : ||\Phi_k - \Phi|| \geq 1/r\}, \quad M_r = \{k \in \mathbb{N} : ||\Phi_k - \Phi|| < 1/r\}$$

uniformly for all $r \in \mathbb{N}$. Then

$$M_1 \supset M_2 \supset \ldots \supset M_i \supset M_{i+1} \supset \ldots \quad \text{and} \quad \delta(K_r) = 0. \tag{3}$$

The density of each of the sets M_i, $i = 1, 2, \ldots$, being 1.

We now show that the sequence $\{\zeta_k\}$ is convergent in distribution to ζ. If possible, let $\{\zeta_k\}$ be not convergent in distribution to ζ. Therefore there is $\varepsilon > 0$ such that $||\Phi_k(z) - \Phi(z)|| \geq \varepsilon$ for infinitely many terms.

Let $L_\varepsilon = \{k : ||\Phi_k(z) - \Phi(z)|| \geq \varepsilon\}$ and $r\varepsilon > 1$, $r = 1, 2, 3, \ldots$. Then $\delta(L_\varepsilon) = 0$ and by equation (3), $L_r \subset M_\varepsilon$. Hence, $\delta(L_\varepsilon) = 0$, which is a contradiction. Therefore $\{\zeta_k\}$ is convergent in distribution to ζ.

Conversely, let there exists $K \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\lim_{k\to\infty} \Phi_k(z) = \Phi(z)$, $k \in K$, where Φ is continuous at the point z, which implies there exists $n_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$ $||\Phi_k(z) - \Phi(z)|| < \varepsilon$ for all $k \geq n_0$.

Now,

$$T_\varepsilon = \{k \in \mathbb{N} : ||\Phi_k(z) - \Phi(z)|| \geq \varepsilon\} \subseteq \mathbb{N} - \{k_{n_0+1}, k_{n_0+2}, \ldots\},$$

that is, $\delta(T_\varepsilon) \leq 1 - 1 = 0$. Hence, $\{\zeta_k\}$ is statistically convergent in distribution to ζ. □

Now we state the following results without proofs as these can be established using the same techniques as above.

Theorem 5. *A complex uncertain sequence $\{\zeta_n\}$ is statistically convergent in measure to ζ if and only if there exists a set $K \subseteq \mathbb{N}$ with natural density unity such that*

$$\lim_{k\to\infty} \mathcal{M}\{||\zeta_k - \zeta|| \geq \varepsilon\} = 0, \quad k \in K.$$

Theorem 6. *The sequence $\{\zeta_n\}$ of complex uncertain variables converges statistically to ζ with respect to almost surely if there exists a subset K of $\mathbb{N}$ with unit density and event Λ with unit uncertain measure such that*

$$\lim_{k\to\infty} \zeta_k(\gamma) = \zeta(\gamma), \quad k \in K, \quad \gamma \in \Lambda.$$

Theorem 7. *The sequence $\{\zeta_n\}$ of complex uncertain variables converges statistically to ζ with respect to uniformly almost surely if there exists a subset K of $\mathbb{N}$ with $\delta(K) = 1$ and a sequence of events $\{E_k\}$ having uncertain measure zero of each events such that*

$$\lim_{m\to\infty} \zeta_m(\gamma) = \zeta(\gamma), \quad m \in K, \quad \gamma \in \Gamma - E_k.$$

Definition 9. *A complex uncertain sequence $\{\zeta_n\}$ is said to be bounded in measure if there exists a positive number δ such that $\mathcal{M}(||\zeta_n|| \geq \delta) = 0$. The collection of all bounded complex uncertain sequences in measure is denoted by $\ell_\infty(\Gamma_{\mathcal{M}})$.*

The remaining types of boundedness may be defined as follows.

Definition 10. *A complex uncertain sequence $\{\zeta_n\}$ is said to be bounded in mean if there exists a positive number δ such that $\sup_n E[||\zeta_n||]$ is a finite value. The collection of such sequences is denoted by $\ell_\infty(\Gamma_E)$.*

Definition 11. *Let Φ_n, Φ be the distribution functions for the complex uncertain variables ζ_n and ζ, respectively.*

Then the sequence $\{\zeta_n\}$ is said to be bounded in distribution if $\sup_n ||\Phi_n(z)|| < \infty$, where z is the point at which Φ is continuous. The family of all sequences of such type is denoted by $\ell_\infty(\Gamma_{\mathcal{D}})$.

Definition 12. *A complex uncertain sequence $\{\zeta_n\}$ is said to be bounded in almost surely if for every $\varepsilon > 0$ there exists some event Λ with unit uncertain measure such that*

$$\sup_k ||\zeta_k(\gamma)|| < \infty, \quad \gamma \in \Lambda.$$

The class of all bounded complex uncertain sequences is denoted by $\ell_\infty(\Gamma_{a.s})$.

Definition 13. *A complex uncertain sequence $\{\zeta_n\}$ is said to be bounded with respect to uniformly almost surely if for any $\varepsilon > 0$ there exist events E_k with an approximate zero uncertain measure and $\sup_n ||\zeta_n(\gamma)|| < \infty$ for all $\gamma \in \Gamma - E_k$. The set of all such types of sequences is denoted by $\ell_\infty(\Gamma_{u.a.s})$.*

Theorem 8. *The set of all statistically convergent and bounded complex uncertain sequences in measure is a closed linear subspace of the bounded complex uncertain sequence space $\ell_\infty(\Gamma_{\mathcal{M}})$.*

Proof. Let $\zeta^m = \{\zeta_i^m\} \in st\text{-}(\Gamma_{\mathcal{M}}) \cap \ell_\infty(\Gamma_{\mathcal{M}})$ and $\zeta^m \to \zeta \in \ell_\infty(\Gamma_{\mathcal{M}})$. Since $\zeta^m \in st\text{-}(\Gamma_{\mathcal{M}}) \cap \ell_\infty(\Gamma_{\mathcal{M}})$, there exists complex number z_m such that $st\text{-}(\Gamma_{\mathcal{M}})\text{-}\lim_{k\to\infty} \zeta_k^m = z_m$, $m = 1, 2, \ldots$. Since $\zeta^m \to \zeta$ in measure, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}\{||\zeta^p - \zeta^m|| \geq \delta\} \leq \mathcal{M}\{||\zeta^p - \zeta|| \geq \delta'\} + \mathcal{M}\{||\zeta^m - \zeta|| \geq \delta'\}$$

for some

$$\delta' \leq \frac{\delta}{2} < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3} \quad \text{for all} \quad p \geq m \geq n_0.$$

By the Theorem 5, there exists $K_1, K_2 \subset \mathbb{N}$ with $\delta(K_1) = \delta(K_2) = 1$ and $\lim_{k\to\infty} \zeta_k^m = z_m$, $k \in K_1$, $\lim_{k\to\infty} \zeta_k^p = z_p$, $k \in K_2$ with $\delta(K_1 \cap K_2) = 1$.

Now, let $k_1, k_2 \in K_1 \cap K_2$. Then from the above we have

$$\mathcal{M}\{||\zeta_{k_1k_2}^m - z_m|| \geq \delta\} \leq \frac{\varepsilon}{3}, \quad \mathcal{M}\{||\zeta_{k_1k_2}^p - z_p|| \geq \delta\} < \frac{\varepsilon}{3}.$$

Then for each $p \geq m \geq n_0$ and $\delta > 0$ there exists $\delta'' \leq \delta/2$ such that

$$\begin{aligned}\mathcal{M}\{||z_p - z_m|| \geq \delta\} \leq \mathcal{M}\{||z_p - \zeta_{k_1k_2}^p|| \geq \delta''\} + \mathcal{M}\{||\zeta_{k_1k_2}^p - \zeta_{k_1k_2}^m|| \geq \delta''\} \\ + \mathcal{M}\{||\zeta_{k_1k_2}^m - z_m|| \geq \delta''\} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

for some $\delta'' \leq \delta/3$. Therefore, the complex sequence $\{z_m\}$ is a Cauchy sequence in measure and hence convergent, since $\mathbb{C}$, the set of complex numbers is complete.

Let $\lim_{m\to\infty} z_m = z$, that is, for every $\varepsilon, \delta > 0$ there exists a number $n_1(\varepsilon) \in \mathbb{N}$ such that $\mathcal{M}\{||z_k - z|| \geq \delta\} < \varepsilon/3$ for all $k \geq n_1$. Now, we have to show that $\{\zeta_k\}$ is statistically convergent in measure to z. Since $\{\zeta_k^m\}$ is convergent in measure to ζ_k, for each $\varepsilon, \delta > 0$ there exists an $n_2 \in \mathbb{N}$ such that $\mathcal{M}\{||\zeta_k^m - \zeta_k|| \geq \delta\} < \varepsilon/3$ for all $k \geq n_2$. Also, $\{\zeta_k^m\}$ is statistically convergent in measure to z_m. Then there exists $K = \{k \in \mathbb{N}\}$ such that $\delta(K) = 1$ and for every $\varepsilon, \delta > 0$ there exists an $n_3 \in \mathbb{N}$ such that $\mathcal{M}\{||\zeta_k^m - z_k|| \geq \delta\} < \varepsilon/3$ for all $j \geq n_3$.

Let $n_4 = \max\{n_1, n_2, n_3\}$. Then for any preassigned $\varepsilon, \delta > 0$ and $k \geq n_4, k \in K$, we have

$$\begin{aligned}\mathcal{M}\{||\zeta_k - z_k|| > \delta\} &= \mathcal{M}\{||(\zeta_k - \zeta_k^m) + (\zeta_k^m - z_k) + (z_k - z)|| > \delta\} \\ &\leq \mathcal{M}\{||\zeta_k - \zeta_k^m|| > \delta'''\} + \mathcal{M}\{||\zeta_k^m - z_k|| > \delta'''\} + \mathcal{M}\{||z_k - z|| > \delta'''\},\end{aligned}$$

for some

$$\delta''' \leq \frac{\delta}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, $\{\zeta_k\}$ statistically converges in measure to z and hence the space $st\text{-}(\Gamma_{\mathcal{M}}) \cap \ell_\infty(\Gamma_{\mathcal{M}})$ is a closed linear subspace. □

Once again we put the following results without proofs as those can be showed by similar technique adopted in above.

Theorem 9. *The space* $st\text{-}(\Gamma_{a.s}) \cap \ell_\infty(\Gamma_{a.s})$ *is a closed linear subspace of the bounded complex uncertain sequence space* $\ell_\infty(\Gamma_{a.s})$.

Theorem 10. *The set of all statistically convergent and bounded complex uncertain sequence in distribution is a closed linear subspace of the bounded complex uncertain sequence space* $\ell_\infty(\Gamma_{\mathcal{D}})$.

Theorem 11. *The space* $st\text{-}(\Gamma_{u.a.s}) \cap \ell_\infty(\Gamma_{u.a.s})$ *is a closed linear subspace of the bounded complex uncertain sequence space* $\ell_\infty(\Gamma_{u.a.s})$.

Theorem 12. *The set of all statistically convergent and bounded complex uncertain sequence in mean is a closed linear subspace of the bounded complex uncertain sequence space* $\ell_\infty(\Gamma_E)$.

3 Statistically complex uncertain Cauchy sequences

In this section, we present the notion of statistically complex uncertain Cauchy sequence and establish the interrelationship with statistically convergent complex uncertain sequence.

Definition 14. *The complex uncertain sequence* $\{\zeta_n\}$ *is said to be statistically Cauchy in measure if for every* $\varepsilon, \delta > 0$ *there exists* $n_1 \in \mathbb{N}$ *such that for all* $k, p \geq n_1$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \ \mathcal{M}\{||\zeta_k - \zeta_p|| \geq \delta\} < \varepsilon\}| = 0.$$

Definition 15. *The complex uncertain sequence* $\{\zeta_n\}$ *is said to be statistically Cauchy in mean if for every* $\varepsilon > 0$ *there exists* $n_1 \in \mathbb{N}$ *such that for all* $k, p \geq n_1$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \ E[||\zeta_k - \zeta_p||] \geq \varepsilon\}| = 0.$$

Definition 16. *The complex uncertain sequence $\{\zeta_n\}$ is called statistically Cauchy in distribution if for every positive ε there exists $n_1 \in \mathbb{N}$ such that for all $k, p \geq n_1$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: \; ||\Phi_k(z) - \Phi_p(z)|| \geq \varepsilon\}| = 0,$$

where z is the point at which Φ is continuous and Φ, Φ_k are uncertain distribution functions for ζ, ζ_k, respectively.

Definition 17. *The complex uncertain sequence $\{\zeta_n\}$ is said to be statistically Cauchy with respect to almost surely if for any positive $\varepsilon > 0$ there exist event Λ with unit uncertain measure and $n_1 \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: \; ||\zeta_k(\gamma) - \zeta_p(\gamma)|| \geq \varepsilon\}| = 0, \quad k, p \geq n_1,$$

for all $\gamma \in \Lambda$.

Definition 18. *A complex uncertain sequence $\{\zeta_n\}$ is called statistically Cauchy with respect to uniformly almost surely if for any positive $\varepsilon > 0$ there exist sequence of events $\{E_t\}$ approaching to uncertain measure zero and natural numbers n_1 with $k, p \geq n_1$ such that*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: \; ||\zeta_k(\gamma) - \zeta_p(\gamma)|| \geq \varepsilon\}| = 0$$

for all $\gamma \in \Gamma - E_t$.

Theorem 13. *A complex uncertain sequence $\{\zeta_n\}$ is statistically convergent in measure if and only if $\{\zeta_n\}$ is statistically Cauchy in measure.*

Proof. Let $\{\zeta_n\}$ be statistically convergent in measure to ζ. Then for each $\varepsilon, \delta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: \; \mathcal{M}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon\}| = 0, \quad k, p \geq n_0.$$

Let us choose a natural number n_1 such that $\mathcal{M}\{||\zeta_{n_1} - \zeta|| \geq \delta\} \geq \varepsilon$. We define three sets $A_\varepsilon, B_\varepsilon$ and C_ε as follows:

$$\begin{aligned}
A_\varepsilon &= \{k \leq n: \; \mathcal{M}\{||\zeta_k - \zeta_{n_1}|| \geq \delta\} \geq \varepsilon\}, \\
B_\varepsilon &= \{k \leq n: \; \mathcal{M}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon\}, \\
C_\varepsilon &= \{k = n_1 \leq n: \mathcal{M}\{||\zeta_{n_1} - \zeta|| \geq \delta\} \geq \varepsilon\}.
\end{aligned}$$

Obviously, $A_\varepsilon \subseteq B_\varepsilon \cup C_\varepsilon$. Therefore $\delta(A_\varepsilon) \leq \delta(B_\varepsilon) + \delta(C_\varepsilon) = 0$, since $\{\zeta_n\}$ is statistically convergent in measure to ζ. Hence, $\{\zeta_n\}$ is statistically Cauchy in measure.

Conversely, let the complex uncertain sequence $\{\zeta_n\}$ be statistically Cauchy in measure. Then $\delta(A_\varepsilon) = 0$. Hence, for the set $E_\varepsilon = \{k \leq n: \; \mathcal{M}\{||\zeta_k - \zeta_{n_1}|| \geq \delta\} < \varepsilon\}$ we have $\delta(E_\varepsilon) = 1$. Now, for each $\delta > 0$ there exists some $0 < \delta' \leq \delta/2$ such that

$$\mathcal{M}\{||\zeta_k - \zeta_{n_1}|| \geq \delta\} \leq 2\mathcal{M}\{||\zeta_k - \zeta|| \geq \delta'\} < \varepsilon. \tag{4}$$

Now, if $\{\zeta_n\}$ is not statistically convergent in measure, then $\delta(B_\varepsilon) = 1$. Hence, for the set $F_\varepsilon = \{k \leq n: \; \mathcal{M}\{||\zeta_k - \zeta|| \geq \delta\} < \varepsilon\}$ we have $\delta(F_\varepsilon) = 0$.

Thus, from the equation (4), for the set $G_k = \{k \leq n: \; \mathcal{M}\{||\zeta_k - \zeta_{n_1}|| \geq \delta\} < \varepsilon\}$ we have $\delta(G_k) = 0$, which implies that $\delta(A_\varepsilon) = 1$ and thus it arises a contradiction that $\{\zeta_n\}$ is a statistically Cauchy sequence in measure. Hence, the complex uncertain sequence $\{\zeta_n\}$ is statistically convergent in measure to ζ. □

Theorem 14. *A complex uncertain sequence $\{\zeta_n\}$ is statistically convergent almost surely if and only if $\{\zeta_n\}$ is statistically Cauchy with respect to almost surely.*

Proof. Let $\{\zeta_n\}$ be a statistically convergent sequence with respect to almost surely to ζ. Therefore for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ with $k, p \geq n_0$ and event Λ with unit uncertain measure such that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \ ||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = 0 \quad \text{for all} \quad \gamma \in \Lambda.$$

Take $n_1 \in \mathbb{N}$ such that $||\zeta_{n_1}(\gamma) - \zeta(\gamma)|| \geq \varepsilon$, $\gamma \in \Lambda$, and consider three sets

$$\begin{aligned} A_\varepsilon &= \{k \leq n : \ ||\zeta_k(\gamma) - \zeta_{n_1}(\gamma)|| \geq \varepsilon\}, \\ B_\varepsilon &= \{k \leq n : \ ||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}, \\ C_\varepsilon &= \{k = n_1 \leq n : \ ||\zeta_{n_1}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}, \end{aligned}$$

where $\gamma \in \Lambda$. Here $A_\varepsilon \subseteq B_\varepsilon \cup C_\varepsilon$ and hence $\delta(A_\varepsilon) \leq \delta(B_\varepsilon) + \delta(C_\varepsilon) = 0$, since $\{\zeta_n\}$ is statistically convergent with respect to almost surely. Hence, the sequence $\{\zeta_n\}$ is a statistically Cauchy sequence with respect to almost surely.

Conversely, let $\{\zeta_n\}$ be statistically Cauchy with respect to almost surely. Then $\delta(A_\varepsilon) = 0$. Therefore, for the set $E_\varepsilon = \{k \leq n : \ ||\zeta_k(\gamma) - \zeta_{n_1}(\gamma)|| < \varepsilon\}$, $\gamma \in \Lambda$, for some event Λ such that $\mathcal{M}\{\Lambda\} = 1$, we have $\delta(E_\varepsilon) = 1$. In particular, we can write

$$||\zeta_k(\gamma) - \zeta_{n_1}(\gamma)|| \leq 2||\zeta_k(\gamma) - \zeta(\gamma)|| < \varepsilon \quad \text{if} \quad ||\zeta_k(\gamma) - \zeta(\gamma)|| < \frac{\varepsilon}{2}. \tag{5}$$

If possible, let $\{\zeta_n\}$ be not statistically convergent sequence with respect to almost surely. Then $\delta(B_\varepsilon) = 1$. Hence, for the set $F_\varepsilon = \{k \leq n : \ ||\zeta_k(\gamma) - \zeta(\gamma)|| < \varepsilon\}$ we have $\delta(F_\varepsilon) = 0$. Hence, from condition (5), for the set $G_k = \{k \leq n : \ ||\zeta_k(\gamma) - \zeta_{n_1}(\gamma)|| < \varepsilon\}$ we have $\delta(G_k) = 0$, which implies that $\delta(A_\varepsilon) = 1$ and so it is not a statistically Cauchy sequence with respect to almost surely. This is a contradiction to our assumption. Hence, $\{\zeta_n\}$ is statistically convergent with respect to almost surely to ζ. □

The above results are true for mean, distribution and uniformly almost surely also. We claim these results below.

Theorem 15. *A complex uncertain sequence $\{\zeta_n\}$ is statistically convergent in mean if and only if $\{\zeta_n\}$ is statistically Cauchy in mean.*

Theorem 16. *A complex uncertain sequence $\{\zeta_n\}$ is statistically convergent with respect to uniformly almost surely if and only if $\{\zeta_n\}$ is statistically Cauchy with respect to uniformly almost surely.*

Theorem 17. *A complex uncertain sequence $\{\zeta_n\}$ is statistically convergent in distribution if and only if $\{\zeta_n\}$ is statistically Cauchy in distribution.*

Theorem 18. *Let $\{\zeta_k\}$ be a sequence of complex uncertain variables. Then $\{\zeta_k\}$ is statistically Cauchy in measure if and only if there exists a subsequence $\{\xi_p\}$ of $\{\zeta_k\}$ such that*

$$\lim_{n\to\infty} \frac{1}{n}|\{p \leq n : \ \mathcal{M}\{||\xi_p - \zeta|| \geq \varepsilon\} \geq \delta\}| = 0 \quad \textit{for every} \quad \varepsilon, \delta > 0,$$

where ζ is the limit to which the sequence $\{\zeta_k\}$ statistically converges in measure.

Proof. Combining the Theorem 5 and Theorem 13, the proof can be established. □

Theorem 19. *A complex uncertain sequence $\{\zeta_k\}$ is statistically Cauchy in mean if and only if there exists a subsequence $\{\xi_p\}$ of $\{\zeta_k\}$ converging statistically in mean to the same limit ζ that of $\{\zeta_k\}$, that is*

$$\lim_{n\to\infty}\frac{1}{n}|\{p \leq n : \ E[||\xi_p - \zeta||] \geq \varepsilon\}| = 0 \quad \textit{for every} \quad \varepsilon > 0.$$

Proof. Straightforward from the Theorem 3 and the Theorem 15 and hence omitted. □

Combining the respective theorems from the Section 2 and the Section 3, we obtain the following results very easily.

Theorem 20. *The sequence $\{\zeta_k\}$ of complex uncertain variables is statistically Cauchy in distribution if and only if there exists a subsequence $\{\xi_p\}$ of $\{\zeta_k\}$ such that*

$$\lim_{n\to\infty}\frac{1}{n}|\{k \leq n : \ ||\Phi_p(z) - \Phi(z)|| \geq \varepsilon\}| = 0,$$

where z is the point at which Φ is continuous.

Theorem 21. *A complex uncertain sequence $\{\zeta_k\}$ is statistically Cauchy with respect to almost surely if and only if there exists a subsequence $\{\xi_p\}$ of $\{\zeta_k\}$ converging statistically with respect to almost surely to the same limit as that of $\{\zeta_k\}$.*

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Received 29.07.2021
Revised 10.09.2021

Дас Б., Дебнат П., Тріпаті Б.Ч. *Про статистично збіжні комплексні невизначені послідовності* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 135–146.

У роботі розширено дослідження статистичної збіжності комплексних невизначених послідовностей у заданому просторі невизначеності. Встановлено зв'язок між збіжністю і статистичною збіжністю у невизначеному середовищі, а також ініційовано статистично комплексну невизначену послідовність Коші, щоб довести, що комплексна невизначена послідовність статистично збігається тоді і тільки тоді, коли вона статистично Коші. Охарактеризовано статистично збіжну комплексну невизначену послідовність за допомогою оператора обмеженості і щільності.

Ключові слова і фрази: простір невизначеності, невизначена міра, статистична збіжність, комплексна невизначена послідовність, статистично послідовність Коші.