Nonlocal multipoint problem for a differential equation of $2n$-th order with operator coefficients

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In the article, the spectral properties of a multipoint problem for a differential operator equation of order $2n$ are studied. The operator of the problem has an infinite number of multiple eigenvalues. Each multiple eigenvalue corresponds to a finite set of root functions. A commutative group of transmutation operators is constructed. Each element of the group corresponds to the isospectral perturbation of the problem operator with antiperiodic conditions. The conditions for the existence and uniqueness of the solution are established for the selected family of multipoint problems, and this solution is constructed too.

Key words and phrases: multipoint problem, antiperiodic boundary condition, root function, method of transmutation operators, Riesz basis.

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1 Introduction

The foundations of the theory of boundary value problems for differential-operator equations are investigated in the works of J.-L. Lions and E. Magenes [18], V.L. Gorbachuk and M.L. Gorbachuk [12], S.Ya. Yakubov [24], A.A. Dezin [9].

Nonlocal problems on a finite interval for differential equations with unbounded operator coefficients were studied by Ya.O. Baranetskij et al. [3], Yu.A. Dubinskii [10], P.I. Kalenyuk et al. [16], F.E. Lomovtsev et al. [19,20], K.S. Mamedov [21], V.K. Romanko [22], S.Ya. Yakubov et al. [23] and many others.

The basics of the theory of transmutation operators are set out in the paper of V.V. Katrakhov and S.M. Sitnik [17]. The properties of differential operators that have an infinite number of multiple eigenvalues have been studied in the papers [4,6,9,13–15].

In the works [1,2,4,5,7,15,16] the properties of operators spectrum are investigated by the methods of the theory of transmutation operators.

This paper is a continuation of the research [6,7] for the case of nonhomogeneous boundary conditions.

2   Basic designations and statement of the problem

For our investigation we will use the following notations. Let H be separable Hilbert space and $A : H \to H$ be closed unbounded linear operator with discrete spectrum

$$\sigma_p(A) = \{\mu_k : \mu_k \sim ak^\alpha, \ k \to \infty, \ a, \ \alpha > 0\}.$$

Let $V(A) = \{v_k \in H : k = 1, 2, \ldots\}$ be the system of eigenfunctions, which forms the orthonormal basis in the space H and

$$H(A^m) := \{v \in H, \ A^m v \in H, \ m \in \mathbb{R}, \ m \geq 0\}, \qquad H(A^0) = H,$$

$$\|u\|^2_{H(A^m)} := (A^m u; A^m u)_H, \qquad (u; u)_{H(A^m)} := (A^m u; A^m u)_H,$$

$$L_2((0,1), H) := \{v(t) : \ (0,1) \to H, \ \|v(\cdot)\|_H \in L_2(0,1)\},$$

$$(u(t); v(t))_{L_2((0,1),H)} := \int_0^1 (u(t), \overline{v}(t))_H \, dt.$$

Let $[(H(A^m); H(A^q)]$ be an algebra of bounded linear operators $A : H(A^m) \to H(A^q)$, $D_t : L_2((0,1), H) \to L_2((0,1), H)$ be a strong derivative in the space $L_2((0,1), H)$,

$$W_2^{2n}((0,1), A) := \Big\{ v \in L_2((0,1), H) : \ A^{2n} v, \ D_t^{2n} v \in L_2((0,1), H),$$

$$(u; v)_{W_2^{2n}((0,1),H)} := (v; u)_{L_2((0,1),H)} + (A^{2n} v; A^{2n} u)_{L_2((0,1),H)} + (A^{2n} v; A^{2n} u)_{L_2((0,1),H)},$$

$$\|y\|^2_{W_2^{2n}((0,1),A)} := \|y\|^2_{L_2((0,1),H)} + \|A^{2n} y\|^2_{L_2((0,1),H)} + \|D_t^{2n} y\|^2_{L_2((0,1),H)} \Big\}.$$

Let us consider the multipoint problem

$$L(-D_t^2, A^2)u := \sum_{r=0}^{n} (-1)^r \beta_r A^{2n-2r} D_t^{2r} u(t) = f(t), \quad t \in (0,1), \tag{1}$$

$$\begin{aligned}
\ell_j u &:= D_t^{2j-2} u(0) + D_t^{2j-2} u(1) = \varphi_j, \\
\ell_{n+j} u &:= D_t^{2j-1} u(0) + D_t^{2j-1} u(1) + \ell_{n+j}^1 u = \varphi_{n+j}, \\
\ell_{n+j}^1 u &:= \sum_{q=0}^{\chi} b_q D_t^{2j-1} u(t_q),
\end{aligned} \tag{2}$$

where $0 = t_0 < t_1 < \ldots < t_\chi = 1$, $\beta_r \in \mathbb{R}$, $b_q \in \mathbb{R}$, $q = 0, 1, \ldots, \chi$, $j = 1, 2, \ldots, n$, $r = 0, 1, \ldots, n$.
The function $u \in W_2^{2n}((0,1), H)$ is called the solution of the problem (1)–(2) if

$$\|L(-D_t^2, A^2)u - f\|_{L_2((0,1),H)} = 0, \quad \|\ell_j u - \varphi_j\|_{H(A^{m_j})} = 0, \quad \|\ell_{n+j} u - \varphi_{n+j}\|_{H(A^{m_{n+j}})} = 0,$$

with $m_j = 2n - 2j + \frac{3}{2}$, $m_{n+j} = 2n - 2j - \frac{1}{2}$, $j = 1, 2, \ldots, n$.

Let us consider the following assumptions and theorems, that are necessary for further investigation.

Assumption P_1 : $t_q = 1 - t_{\chi-q}$, $b_q = -b_{\chi-q}$, $q = 0, 1, \ldots, \chi$.

Assumption P_2 : $L(-\omega^2, 1) = \prod_{j=1}^{n}(\omega^2 - \omega_j^2)$, $\operatorname{Re}\omega_1 < \operatorname{Re}\omega_2 < \ldots < \operatorname{Re}\omega_n < 0$, $\operatorname{Re}\omega_j^2 > 0$.

Assumption P_3 : $\varphi_s \in H(A^{m_s})$, $s = 1, 2, \ldots, 2n$.

Consider the operator

$$Lu := \sum_{r=0}^{n}(-1)^r \beta_r A^{2n-2r} D_t^{2r} u, \quad u \in D(L),$$

$$D(L) := \{u \in W_2^{2n}((0,1), H) : \ \ell_m u = 0, \ m = 1, 2, \ldots, 2n\}.$$

The following statements are the main results of the paper.

Theorem 1. *Let the Assumptions P_1, P_2 hold. Then the operator L has the system of root functions $V(L)$, which is the Riesz basis in the space $L_2((0,1), H)$.*

Theorem 2. *Let the Assumptions $P_1 - P_3$ hold. Then for arbitrary $\beta_r, b_s \in \mathbb{R}$, $f \in L_2((0,1), H)$, $\varphi_j \in H(A^{m_j})$, $j = 1, 2, \ldots, 2n$, the unique solution $u \in W_2^{2n}(G)$ of the problem (1)–(2) exists.*

Our research is structured as follows. In Section 3 the spectral properties of the problem with antiperiodicity conditions are studied. In Section 4 the solution of the problem with homogeneous conditions is constructed. In Section 5 the properties of the operator for a multipoint problem are investigated and a commutative group of transmutation operators is constructed. In Section 6 the Theorem 1 is proved. In Section 7 the solution of the problem with nonhomogeneous conditions of antiperiodicity is constructed. In Section 8 the solution of the problem with nonhomogeneous multipoint conditions is constructed and the Theorem 2 is proved for this case.

3 The spectral problem with antiperiodic boundary conditions for differential-operator equation

Let us consider the partial case of the problem (1)–(2) for $\beta_r = 0$, $b_q = 0$, $r = 0, 1, \ldots, n$, $q = 0, 1, \ldots, \chi$:

$$L(-D_t^2, A^2)w_0 := \sum_{r=0}^{n}(-1)^r \beta_r A^{2n-2r} D_t^{2r} w_0(t) = f(t), \tag{3}$$

$$\ell_{0,j} w_0 := D_t^{2j-2} w_0(0) + D_t^{2j-2} w_0(1) = 0,$$
$$\ell_{0,n+j} w_0 := D_t^{2j-1} w_0(0) + D_t^{2j-1} w_0(1) = 0, \quad j = 1, 2, \ldots, n. \tag{4}$$

Consider the operator $L_0 : \ L_2(0,1) \to L_2(0,1)$ of problem (3)–(4):

$$L_0 w_0 := L(-D_t^2, A^2) w_0, \quad w_0 \in D(L_0),$$
$$D(L_0) := \{w_0 \in W_2^{2n}((0,1), H) : \ \ell_{0,s} w_0 = 0, \ s = 1, 2, \ldots, 2n\}.$$

Lemma 1. *The operator L_0 has the discrete spectrum*

$$\sigma(L_0) := \left\{ \lambda_{m,q} \in \mathbb{R} : \ \lambda_{m,q} := \sum_{r=0}^{n} \beta_r \mu_m^{2n-2r} \rho_q^{2r}, \ \rho_q = \pi(2q-1), \ m, q = 1, 2, \dots \right\}$$

and the system of the eigenfunctions $V(L_0)$, which forms the orthonormal basis in the space $L_2((0,1), H)$.

Proof. Consider the spectral problem

$$L(-D_t^2, A^2)w_0 := \sum_{r=0}^{n} (-1)^r \beta_r A^{2n-2r} D_t^{2r} w_0(t) = \lambda w_0(t), \tag{5}$$

$$\ell_{0,j} w_0 := D_t^{2j-2} w_0(0) + D_t^{2j-2} w_0(1) = 0,$$
$$\ell_{0,n+j} w_0 := D_t^{2j-1} w_0(0) + D_t^{2j-1} w_0(1) = 0, \tag{6}$$
$$\lambda \in \mathbb{C}, \ j = 1, 2, \dots, n.$$

We can find the solution of the spectral problem (5), (6) as the product

$$w_0(t) = y(t) v_m, \quad v_m \in V(A), \ m \in \mathbb{N}.$$

To determine the unknown function $y \in W_2^{2n}(0,1)$ we obtain the spectral problem

$$\sum_{r=0}^{n} (-1)^r \beta_r \mu_m^{2n-2r} y^{(2r)}(t) = \lambda y(t), \quad \lambda \in \mathbb{C},$$
$$l_{0,j} y := y^{(2j-2)}(0) + y^{(2j-2)}(1) = 0, \tag{7}$$
$$l_{0,n+j} y := y^{(2j-1)}(0) + y^{(2j-1)}(1) = 0, \quad j = 1, 2, \dots, n.$$

Let $L_{0,m}$ be an operator of the problem (7):

$$L_{0,m} y := \sum_{r=0}^{n} (-1)^r \beta_r \mu_m^{2n-2r} y^{(2r)}(t), \quad y \in D(L_{0,m}),$$
$$D(L_{0,m}) := \left\{ y \in W_2^{2n}(0,1) : \ l_{0,p} y = 0, \ p = \overline{1, 2n} \right\}.$$

Let us consider boundary value problem

$$-z^{(2)}(t) = h(t), \quad t \in (0,1),$$
$$z^{(r)}(0) + z^{(r)}(1) = 0, \quad r = 0, 1. \tag{8}$$

Let B_0 be an operator of problem (8):

$$B_0 z(t) := -z^{(2)}(t), \quad z(t) \in D(B_0),$$
$$D(B_0) := \{ z \in W_2^2(0,1) : \ z^{(r)}(0) + z^{(r)}(1) = 0, \quad r = 0, 1 \}.$$

It is well known (see [6, 21]), that the operator B_0 has the point spectrum $\sigma(B_0) := \{ \phi_q \in \mathbb{R} : \phi_q = \pi^2(2q-1)^2, \ q \in \mathbb{N} \}$ and the system of eigenfunctions

$$T := \left\{ \tau_{s,q}(t) \in L_2(0,1) : \tau_{0,q}(t) = \sqrt{2} \sin(2q-1)\pi t, \tau_{1,q}(t) = \sqrt{2} \cos(2q-1)\pi t, \ q = 1, 2, \dots \right\}.$$

By direct substitution we obtain $L_{0,m} = \sum\limits_{r=0}^{n} \beta_r \mu_m^{2n-2r} B_0^r$. Therefore, the operator $L_{0,m}$ has the point spectrum

$$\sigma(L_{0,m}) := \left\{ \lambda_{m,q} := \sum_{r=0}^{n} \beta_r \mu_m^{2n-2r} \phi_q^r, \ m = \overline{1, \infty} \right\} \tag{9}$$

and the system of eigenfunctions T.

Consider the orthonormal basis of space $L_2((0,1), H)$:

$$V(L_0) := \{ v_{s,q,m}(t, L_0) \in L_2((0,1), H) : \ v_{s,q,m}(t, L_0) := \tau_{s,q}(t) v_m, \ s = 0, 1, \ q, \ m = 1, 2, \ldots \}.$$

From the definition of operator L_0 we have equality

$$L_0 v_{s,q,m}(t, L_0) = \lambda_{q,m} v_{s,q,m}(t, L_0), \quad s = 0, 1, \ q, m = 1, 2, \ldots.$$

The space $L_2((0,1), H)$ is isomorphic to the Hilbert tensor product of spaces $L_2(0,1)$ and H. Then $D(L_0) = \{ w_0 \in D(A^{2n}) \otimes D(B_0^n) \} \subset L_2((0,1), H)$. Therefore, the operator L_0 can be represented as a polynomial $L_0 = \sum\limits_{r=0}^{n} \beta_r A^{2n-2r} B_0^r$ from the operators A and B_0.

Taking into account the orthonormality of the system $V(L_0)$ in space $L_2((0,1), H)$, we obtain the statement of the lemma. $\qquad \square$

4   The boundary value problem with homogeneous conditions

Let us consider the problem (3)–(4).

Theorem 3. *Let the Assumption P_2 holds. Then for arbitrary function $f \in L_2((0,1), H)$ the unique solution $w_0 \in W_2^{2n}((0,1), H)$ of the problem (3)–(4) exists.*

Proof. We expand the functions $w_0(t)$, $f(t)$ into the series with respect to the system $V(L_0)$:

$$w_0 = \sum_{m,q=1}^{\infty} \sum_{s=0}^{1} w_{0,s,q,m} v_{s,q,m}(t, L_0), \qquad f = \sum_{m,q=1}^{\infty} \sum_{s=0}^{1} f_{s,q,m} v_{s,q,m}(t, L_0),$$

$$f_{s,q,m} = (f; v_{s,q,m}(t, L_0))_{L_2((0,1),H)}, \quad s = 0, 1, \ q, m = 1, 2, \ldots,$$

where $V(L_0) := \{ v_{s,q,m}(t, L_0) \in L_2((0,1), H), \ s = 0, 1, \ q, \ m = 1, 2, \ldots \}$ is the system of eigenfunctions of the operator, which is orthonormal basis of the space $L_2((0,1), H)$.

Substituting into the equation (3) and taking into account the completeness of the system $V(L_0)$, we obtain the equality $w_{0,s,q,m} = \lambda_{m,q}^{-1} f_{s,q,m}$, $s = 0, 1, q, m = 1, 2, \ldots$.

From Assumption P_2 we have the following inequality

$$|w_{0,s,q,m}| \leq C_1 |f_{s,q,m}|, \quad 0 < C_1 < \infty, \ s = 0, 1, \ q, \ m = 1, 2 \ldots.$$

Therefore

$$\|w_0\|_{L_2((0,1),H)} \leq C_1 \|f\|_{L_2((0,1),H)}.$$

Show that $w_0(t) \in W_2^{2n}((0,1), H)$. Taking into account the theorem on intermediate derivatives [6, 14, 17], we should show that $A^{2n} w_0(t) \in L_2((0,1), H)$ and $D_t^{2n} w_0(t) \in L_2((0,1), H)$.

Let $y_1(t) = A^{2n}L_0^{-1}f(t)$, $y_2(t) = B_0^n L_0^{-1}f(t)$. The operators $A^{2n}L_0^{-1}$, $B_0^n L_0^{-1}$ have the system $V(L_0)$ of eigenfunctions and corresponding eigenvalues $\chi_{1,m,q} = \mu_q^{2n}\lambda_{m,q}^{-1}$, $\chi_{2,m,q} = \phi_q^n \lambda_{m,q}^{-1}$, $m, q = 1, 2, \ldots$. Taking into account the Assumption P_2, we obtain the estimates

$$|\chi_{s,m,q}| \le C_2 < \infty, \quad s = 1, 2, \ m, q = 1, 2, \ldots.$$

Therefore, $y_s(t) \in L_2((0,1), H)$, $s = 1, 2$. Then from the definition of the norm in space $W_2^{2n}((0,1), H)$ we obtain

$$w_0(t) \in W_2^{2n}((0,1), H) \quad \text{and} \quad \|w_0\|_{W_2^{2n}((0,1),H)} \le C_3\|f\|_{L_2((0,1),H)}, \quad 0 < C_3 < \infty.$$

The theorem is proved. □

Remark 1. *From the statement of the Theorem 3 we have the following relation*

$$L_0^{-1} \in [L_2((0,1), H); W_2^{2n}((0,1), H)].$$

5   The spectral boundary problem for a differential-operator equation and transmutation operators

Let us consider boundary value problem

$$\begin{aligned}
-z^{(2)}(t) &= g(t), \quad t \in (0,1), \\
l^0 z &:= z(0) + z(1) = 0, \\
l^1 z &:= z^{(1)}(0) + z^{(1)}(1) + h(z^{(1)}(0) - z^{(1)}(1)) = 0, \quad h \in \mathbb{R}, \ r = 0, 1.
\end{aligned} \tag{10}$$

Let B be an operator of problem (10):

$$\begin{aligned}
Bz(t) &:= -z^{(2)}(t), \quad z(t) \in D(B), \\
D(B) &:= \{z \in W_2^2(0,1) : \ l^r z = 0, 1\}.
\end{aligned}$$

In the paper [5] it is proved that the operator B has eigenvalues μ_m and the system of root functions

$$V(B) := \{v_{1,q}(t, B) := \tau_{1,q}(t), \ v_{0,q}(t, B) = (1 - h(2t - 1))\tau_{0,q}(t), \ q = 1, 2, \ldots\},$$

for which the following relations hold

$$Bv_{1,q}(t, B) = \phi_q v_{1,q}(t, B), \qquad Bv_{0,m}(t, B) = \phi_q v_{0,q}(t, B) + 4\rho_q h v_{1,q}(t, B), \quad q = 1, 2, \ldots.$$

Let B^* be an operator of the conjugate problem:

$$\begin{aligned}
-y^{(2)}(t) &= g(t), \quad t \in (0,1), \\
y(0) + y(1) + h(y(1) - y(0)) &= 0, \\
y^{(1)}(0) + y^{(1)}(1) &= 0, \quad r = 0, 1.
\end{aligned}$$

The system of root functions of this operator is defined by the following relations

$$\begin{aligned}
V(B^*) := \{v_{s,q}(t, B^*) \in L_2(0,1), \quad v_{s,q}(t, B^*) : \ v_{0,q}(t, B^*) := \tau_{0,q}(t), \\
v_{1,q}(t, B^*) = (1 + h(2t - 1))\tau_{1,q}(t), \quad q = 1, 2, \ldots\},
\end{aligned}$$

where

$$T := \{\tau_{s,q}(t) \in L_2(0,1) : \ \tau_{0,q}(t) = \sqrt{2}\sin(2q - 1)\pi t, \ \tau_{1,q}(t) = \sqrt{2}\cos(2q - 1)\pi t, \ q = 1, 2, \ldots\},$$

and it is biorthogonal to the system $V(B)$.

Lemma 2. *For any $h \in \mathbb{R}$ the operator B has the system of root functions $V(B)$, which is the Riesz basis of the space $L_2(0,1)$.*

Proof. The system $V(B)$ has the unique biorthogonal system B^*. Therefore, $V(B)$ is complete and minimal in the space $L_2(0,1)$. For any $g \in L_2(0,1)$ we obtain the inequality

$$\sum_{s=0}^{1} \sum_{q=1}^{\infty} (g; v_{s,q}(t,B))_{L_2(0,1)}^2 \leq (1+h^2)\|g\|_{L_2(0,1)}^2 < \infty.$$

Therefore, the system of functions $V(B)$ is the Riesz basis of the space $L_2(0,1)$ (see [10]). $\square$

For any $\theta_q \in \mathbb{R}$, $q = 1, 2, \ldots$, we define the operator $B_\theta : L_2(0,1) \rightarrow L_2(0,1)$ as the operator, for which the root functions are defined by the following relations

$$\begin{aligned}
v_{1,q}(t, B_\theta) &:= \tau_{1,q}(t), \\
v_{0,q}(t, B_\theta) &= (1 + \theta_q(2t-1))\tau_{0,q}(t), \quad q = 1, 2, \ldots.
\end{aligned} \tag{11}$$

Taking into account (11), we obtain

$$B_\theta v_{1,q}(t, B_\theta) = \phi_q v_{1,q}(t, B_\theta), \qquad B_\theta v_{0,q}(t, B_\theta) = \phi_q v_{0,m}(t, B_\theta) + 4\rho_q \theta_q v_{1,m}(t, B_\theta), \quad q = 1, 2, \ldots.$$

We denote by $\Phi_0(B) := \{B_\theta\}$ the set of operators B_θ having the root functions (11) and

$$\Phi_{0,c}(B) := \{B_\theta \in \Phi_0(B), \ \theta_q \in \mathbb{R}, \ |\theta_q| \leq C < \infty, \ q = 1, 2, \ldots\}.$$

The operator, which maps the system $V(B)$ into the system $V(B_\theta)$ of the root functions for the operator $B_\theta \in \Phi_0(B)$, we define by the equalities

$$R(B_\theta)\tau_{s,q}(t) := v_{s,q}(t, B_\theta), \quad s = 0, 1, \ q = 1, 2, \ldots,$$

and let $R(B_\theta) := E + S(B_\theta)$.

Define $\Gamma_0(B) := \{R(B_\theta) : L_2(0,1) \rightarrow L_2(0,1) : B_\theta \in \Phi_0(B)\}$, $\Gamma_{0,c}(B) := \Gamma_0(B) \cap [L_2(0,1)]$. For any $B_\theta \in \Phi_0(B)$ we define the biorthogonal system

$$V(B_\theta^*) := \left\{ v_{s,q}(t, B_\theta^*) \in L_2(0,1) : v_{0,q}(t, B_\theta^*) = \tau_{0,q}(t), \right.$$

$$\left. v_{1,q}(t, B_\theta^*) = \tau_{1,q}(t) - \sum_{s=1}^{\infty} (\tau_{1,s}(t); v_{0,q}(t, B_\theta))_{L_2(0,1)} \tau_{0,q}(t), \ q = 1, 2, \ldots \right\},$$

where

$$T := \{\tau_{s,q}(t) \in L_2(0,1) : \tau_{0,q}(t) = \sqrt{2}\sin(2q-1)\pi t, \ \tau_{1,q}(t) = \sqrt{2}\cos(2q-1)\pi t, \ q = 1, 2, \ldots\}.$$

Therefore the following Lemma 3 holds.

Lemma 3. *For any sequence $\{\theta_q\}_{q=1}^{\infty} \subset \mathbb{R}$ the system of functions $V(B_\theta)$ is complete and minimal in $L_2(0,1)$.*

Lemma 4. *The system of functions $V(B_\theta)$ is the Riesz basis in the space $L_2(0,1)$ if and only if the sequences $\{\theta_q\}_{q=1}^{\infty}$ are bounded.*

Proof. Necessity. If the system of functions $V(B_\theta)$ is a Riesz basis in the space $L_2(0,1),$ then it is almost normalized. Taking into account (11), we obtain the inequality

$$0 < 1 \le \|v_{0,q}(t, B_\theta)\|^2_{L_2(0,1)} = 1 + |\theta_q|^2 \le C_4 < \infty, \quad q = 1, 2, \ldots, \ C_4 = 1 + \max \theta_q^2.$$

Sufficiency. Let us consider the relations

$$v_{1,q}(t, B_\theta) = R(B_\theta)\tau_{1,q}(t) = (1 - \theta_q)\tau_{1,q}(t) + \theta_q B\tau_{1,q}(t),$$

$$(h; v_{1,q}(t, B_\theta))_{L_2(0,1)} = (1 - \theta_q)(h; \tau_{1,q})_{L_2(0,1)} + \theta_q(h; B\tau_{1,q})_{L_2(0,1)},$$

$$(h; v_{1,q}(t, B_\theta))^2_{L_2(0,1)} \le 4(1 + \theta_q^2)(h; \tau_{1,q})^2_{L_2(0,1)} + 2\theta_q^2(h; v_{1,q}(t, B))^2_{L_2(0,1)},$$

$$(h; v_{0,q}(t, B_\theta))^2_{L_2(0,1)} = (h; \tau_{0,q}))^2_{L_2(0,1)},$$

$$\sum_{r=0}^{1}\sum_{q=1}^{\infty}((R(B_\theta^*)h; \tau_{r,q})^2_{L_2(0,1)} = \sum_{r=0}^{1}\sum_{q=1}^{\infty}(h; v_{r,q}(t, B_\theta))^2_{L_2(0,1)} < C_5\|h\|^2_{L_2(0,1)},$$

$$C_5 = 4(1 + \|R(B)\|^2_{[L_2(0,1)]}).$$

Therefore,

$$R(B_\theta^*) = E + S(B_\theta^*) \in [L_2(0,1)], \quad R(B_\theta) = E + S(B_\theta) \in [L_2(0,1)].$$

Taking into account the Bary theorem [8, 10], we obtain the statement of the lemma. $\square$

We choose two arbitrary sequences $\{\theta_q^s\}_{q=1}^{\infty}, s = 1, 2,$ and consider the product of transmutation operators $R(B_\theta^q) = E + S(B_\theta^q), \ q = 1, 2,$ and inverse operator on the set $\Gamma_0(B):$

$$R(B_\theta^1)R(B_\theta^2) = E + S(B_\theta^1) + S(B_\theta^2), \quad R(B_\theta)^{-1} = E - R(B_\theta).$$

Remark 2. *The set $\Gamma_0(B)$ is an Abelian group.*

Let B_1 be an operator of the problem

$$-z^{(2)}(t) = h(t), \quad t \in (0,1),$$

$$l_1^0 z := z(0) + z(1) = 0, \quad l_1^1 z := z^{(1)}(0) + z^{(1)}(1) + \sum_{r=0}^{\chi} b_r z^{(1)}(t_r),$$

$$B_1 z(t) := -z^{(2)}(t), \quad z(t) \in D(B_1), \quad D(B_1) := \{z \in W_2^2(0,1) : l_1^{(r)} z = 0, 1\}.$$

We define the root functions of the operator B_1 by the relations

$$v_{1,q}(t, B_1) := \tau_{1,q}(t),$$

$$v_{0,q}(t, B_1) := \left(1 + \theta_q^1(2t - 1)\right)\tau_{0,q}(t), \quad \theta_q^1 = \sum_{r=0}^{\chi} b_r \tau_{0,q}(t_r), \ q = 1, 2, \ldots. \tag{12}$$

From the relations (12) we obtain the inequality

$$\|v_{1,q}(t, B_1)\|_{L_2(0,1)} < C_6, \quad s = 0, 1, \ q = 1, 2, \ldots.$$

Therefore, taking into account the statement of the Lemma 4, we obtain the following assertion.

Lemma 5. *Let Assumption P_1 holds. Then the functions $V(B_1)$ is the Riesz basis in the space $L_2(0,1).$*

6   The spectral problem with nonlocal conditions for differential-operator equation

Consider the spectral problem

$$L(-D_t^2, A^2)w := \sum_{r=0}^{n}(-1)^r \beta_r A^{2n-2r} D_t^{2r} w(t) = \lambda w(t), \quad t \in (0,1), \ \lambda \in \mathbb{C}. \tag{13}$$

$$\begin{aligned}
\ell_j w &:= D_t^{2j-2} w(0) + D_t^{2j-2} w(1) = 0, \\
\ell_{n+j} w &:= D_t^{2j-1} w(0) + D_t^{2j-1} w(1) + \ell_{n+j}^1 w = 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{14}$$

Let us prove the Theorem 1. We find the solution of spectral problem (13), (14) as the product

$$w(t) = y(t)v_m, \quad v_k \in V(A), \ m = 1, 2, \ldots.$$

To determine the unknown function $y \in W_2^{2n}(0,1)$ we obtain the spectral problem

$$\begin{aligned}
&\sum_{r=0}^{n}(-1)^r \beta_r \mu_m^{2n-2r} y^{(2r)}(t) = \lambda y(t), \ \lambda \in \mathbb{C}, \\
&l_j y := y^{(2j-2)}(0) + y^{(2j-2)}(1) = 0, \\
&l_{n+j} y := y^{(2j-1)} y(0) + y^{(2j-1)}(1) + \sum_{s=0}^{\chi} b_s y^{(2j-1)}(t_s) = 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{15}$$

Let L_m be an operator of the problem (15):

$$L_m y := \sum_{r=0}^{n}(-1)^r \beta_r \mu_m^{2n-2r} y^{(2r)}(t), \quad D(L_m) := \big\{ y \in W_2^{2n}(0,1) : \quad l_j y = 0, \ j = 1, 2, \ldots, n \big\}.$$

The operator L_m has a point spectrum (9) and the system of root functions $V(B_1)$ of this operator is defined by relations

$$L_m v_{1,q}(t, B_1) = \lambda_{m,q} v_{1,q}(t, B_1), \qquad L_m v_{0,q}(t, B_1) = \lambda_{m,q} v_{0,q}(t, B_1) + \xi_{q,m} v_{1,q}(t, B_1),$$

$$\xi_{q,m} := \sum_{r=1}^{n} \beta_r 2r \mu_m^{2n-2r-1} \rho_q^{2r-1}, \quad q = 1, 2, \ldots.$$

Therefore the operator L has the system of root functions

$$V := \big\{ v_{s,q,m}(t) \in L_2((0,1), H) : \ v_{s,q,m}(t) := v_{s,q}(t, B_1)v_m, \ s = 0, 1, \ q, m = 1, 2, \ldots \big\},$$

and is represented as a polynomial $L = \sum_{r=0}^{n} \beta_r A^{2n-2r} B_1^r$ of the operators A and B_1.

Taking into account that the system V is the Riesz basis in space $L_2((0,1), H)$, we obtain the statement of the Theorem 1.

Let us consider the problem (1), (14).

Theorem 4. *Let the Assumptions P_1, P_2 hold. Then for arbitrary function $f \in L_2((0,1), H)$ a unique solution $w \in W_2^{2n}((0,1), H)$ of problem (1), (14) exists.*

Proof. We expand the functions $w(t)$, $f(t)$ into a series according to the system $V(L)$:

$$w = \sum_{m,q=1}^{\infty} \sum_{s=0}^{1} w_{s,q,m} v_{s,q,m}(t, L), \qquad f = \sum_{m,q=1}^{\infty} \sum_{s=0}^{1} f_{s,q,k} v_{s,q,m}(t, L).$$

Substituting this series into the equation (1) and taking into account the completeness of the system $V(L)$ we obtain equalities

$$w_{0,q,m} = \lambda_{m,q}^{-1} f_{0,q,m}, \qquad w_{1,q,m} = \lambda_{m,q}^{-1} f_{1,q,m} - \xi_{q,m} \lambda_{m,q}^{-2} f_{1,q,m}, \quad q,m = 1,2,\dots.$$

From Assumption P_2 we have the inequality $|w_{s,q,m}| \leq C_7 |f_{s,q,m}|$, $C_7 < \infty$, $s = 0,1$, $q,m = 1,2,\dots$. Therefore,

$$\|w\|_{L_2((0,1),H)} \leq C_8 \|f\|_{L_2((0,1),H)}, \quad 0 < C_8 < \infty.$$

We show that $w(t) \in W_2^{2n}((0,1), H)$. Taking into account the theorem on intermediate derivatives [6, 14, 17], we should show that

$$w^1(t) := A^{2n} w(t) \in L_2((0,1), H), \qquad w^2(t) := D_t^{2n} w(t) \in L_2((0,1), H).$$

The operators $A^{2n} L^{-1}$ and $D_t^{2n} L^{-1}$ have the system $V(L)$ of root functions and eigenvalues $\chi_{1,m,q} = \mu_q^{2n} \lambda_{m,q}^{-1}$, $\chi_{2,m,q} = \nu_m^n \lambda_{m,q}^{-1}$.

Taking into account the Assumption P_2 we obtain estimates

$$|\chi_{s,m,q}| \leq C_7 < \infty, \quad s = 1,2, \ m,q = 1,2,\dots.$$

Therefore $w^s(t) \in L_2((0,1), H)$, $s = 1,2$.

From the definition of the norm in space $W_2^{2n}((0,1), H)$ we obtain

$$\|w\|_{W_2^{2n}((0,1),H)} \leq C_8 \|f\|_{L_2((0,1),H)}, \quad 0 < C_8 < \infty.$$

The theorem is proved. $\square$

Remark 3. *From the statement of the Theorem 4 we obtain the following relation*

$$L^{-1} \in [L_2((0,1), H); W_2^{2n}((0,1), H)].$$

7   The antiperiodic problem with nonhomogeneous conditions

For the equation

$$L(-D_t^2, A) v_0 := \sum_{r=0}^{n} (-1)^r \beta_r A^{2n-2r} D_t^{2r} v_0(t) = 0 \tag{16}$$

consider the problem with nonhomogeneous antiperiodic boundary conditions

$$\begin{aligned}
\ell_{0,j} v_0 &:= D_t^{2j-2} v_0(0) + D_t^{2j-2} v_0(1) = \varphi_j, \\
\ell_{0,n+j} v_0 &:= D_t^{2j-1} v_0(0) + D_t^{2j-1} v_0(1) = \varphi_{n+j}, \quad j = 1,2,\dots, n.
\end{aligned} \tag{17}$$

Remark 4. *It is well known [11, 14, 17], in the case $\operatorname{Re} \omega < 0$ we have*

$$U(\omega, A, t) = A^{\frac{1}{2}} \exp \omega A t \in [H, L_2((0,1), H)],$$

and the following relations hold

$$D_t^s \in \left[W_2^{2n}((0,1), H); H(A^{2n-s-\frac{1}{2}})\right], \qquad D_t^s v \in L_2\left((0,1), H(A^{2n-s-\frac{1}{2}})\right),$$

$$t \in [0,1], \ s = 1, 2, \dots, 2n-1,$$

for any $v(t) \in W_2^{2n}((0,1), H)$.

Let

$$
\begin{aligned}
Z_p(A, t) &= A^{-2n+\frac{1}{2}}(E + \exp \omega_p A)^{-1}(\exp \omega_p A t + \exp \omega_p A(1 - t)), \\
Z_p(A, t) &\in \left[L_2((0,1), H); W_2^{2n}((0,1), H)\right], \\
Z_{n+p}(A, t) &= A^{-2n+\frac{1}{2}}(E + \exp \omega_p A)^{-1}(\exp \omega_p A t - \exp \omega_p A(1 - t)), \\
Z_{n+p}(A, t) &\in \left[L_2((0,1), H); W_2^{2n}((0,1), H)\right], \\
p &= 1, 2, \dots, n, \quad \operatorname{Re} \omega_1 < \operatorname{Re} \omega_2 < \dots < \operatorname{Re} \omega_n < 0.
\end{aligned}
\tag{18}
$$

Theorem 5. *Let the Assumptions P_2, P_3 hold. Then the function*

$$v_0(t) := \sum_{s=1}^{2n} Z_s(A, t) h_s \tag{19}$$

is the solution of the equation (16) for any $h_s \in L_2((0,1), H)$, $s = 1, 2, \dots, 2n$.

Proof. Consider the following relations for the functions $v_s(t) := Z_s(A, t) h_s$:

$$A^{2n-2r} D_t^{2r} v_p(t) = \omega_p^{2r} A^{\frac{1}{2}}(E + \exp \omega_p A)^{-1}(\exp \omega_p A t + \exp \omega_p A(1 - t)) \in L_2((0,1), H),$$

$$A^{2n-2r} D_t^{2r} v_{n+p}(t) = \omega_p^{2r} A^{\frac{1}{2}}(E + \exp \omega_p A)^{-1}(\exp \omega_p A t - \exp \omega_p A(1 - t)) \in L_2((0,1), H),$$

$$p, r = 1, 2, \dots, n, \ s = 1, 2, \dots, 2n.$$

Therefore,

$$v_s(t) \in W_2^{2n}((0,1), H), \quad s = 1, 2, \dots, 2n.$$

Substituting the function (19) into the equation (16), we obtain

$$L(-D_2^2, A^2) v_0(t) = \sum_{p=1}^{n} L(-\omega_p^2, 1) A^{2n} (Z_p(A, t) h_p + Z_{n+p}(A, t) h_{n+p}) = 0.$$

Substituting the function (19) into boundary conditions (17), we obtain two systems to determine the unknowns functions h_s, $s = 1, 2, \dots, 2n$:

$$
\begin{aligned}
\sum_{p=1}^{n} \omega_p^{2j-2} h_p &= \gamma_j, & \gamma_j &= \frac{1}{2} A^{m_j} \varphi_j \in H, \\
\sum_{p=1}^{n} \omega_p^{2j-1} h_p &= \gamma_{n+j}, & \gamma_{n+j} &= \frac{1}{2} A^{m_{n+j}} \varphi_{n+j} \in H, \\
m_j &= 2n - 2j + \frac{3}{2}, & m_{n+j} &= 2n - 2j - \frac{1}{2}, \quad j = 1, 2, \dots, n.
\end{aligned}
\tag{20}
$$

The determinant Δ_0 of coefficient matrix for the first system coincides with the Vandermonde determinant, which is built on numbers $\omega_r^2 \neq 0$, $r = 1, 2, \ldots, n$.

The determinant Δ_1 of coefficient matrix for the second system is determined by the formula $\Delta_1 = \Delta_0 \prod_{r=1}^{n} \omega_r^2$.

Taking into account $\omega_r \neq 0$, $r = 1, 2, \ldots, n$, we obtain that $\Delta_0 \neq 0$ and $\Delta_1 \neq 0$. Thus each of the systems (20) has a unique solution. Therefore the statement of theorem is proved. □

Theorem 6. *Let the Assumptions P_2, P_3 hold. Then for any $\varphi_s \in H(A^{m_s})$, $s = 1, 2, \ldots, 2n$, a unique solution $v_0 \in W_2^{2n}((0,1), H)$ for the problem (16), (17) exists, it is determined by the ratios (18)–(20).*

8 Existence and uniqueness of the solution of the multipoint problem with nonhomogeneous conditions

Let us consider the problem for equation

$$L(-D_t^2, A)v := \sum_{r=0}^{n} (-1)^r \beta_r A^{2n-2r} D_t^{2r} v(t) = 0 \tag{21}$$

with nonhomogeneous multipoint conditions

$$\begin{aligned}
\ell_j v &:= D_t^{2j-2} v(0) + D_t^{2j-2} v(1) = \varphi_j, \\
\ell_{n+j} v &:= D_t^{2j-1} v(0) + D_t^{2j-1} v(1) + \ell_{n+j}^1 v = \varphi_{n+j}, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{22}$$

Theorem 7. *Let the Assumptions $P_1 - P_3$ hold. Then for any $\varphi_j \in H(A^{m_j})$, $j = 1, 2, \ldots, 2n$, a unique solution $v \in W_2^{2n}((0,1), H)$ for the problem (21), (22) exists.*

Proof. The solution of the problem (21), (22) is determined by the following relations

$$\begin{aligned}
v(t) &:= v^0(t) + v^1(t), \\
v^0(t) &:= \sum_{p=1}^{n} Z_p(A,t)g_p, \qquad v^1(t) := \sum_{p=1}^{n} Z_{n+p}(A,t)g_{n+p}.
\end{aligned} \tag{23}$$

To determine the unknown functions g_s, $s = 1, 2, \ldots, 2n$, we substitute the functions (23) into multipoint conditions (22).

We obtain two systems that are solved sequentially

$$\begin{aligned}
\sum_{p=1}^{n} \omega_p^{2j-2} g_p &= \gamma_j, \qquad \gamma_j = \frac{1}{2} A^{m_j} \varphi_j \in H, \\
\sum_{p=1}^{n} \omega_p^{2j-1} g_{n+p} &= \gamma_{n+j} - A^{1-2j} \ell_{n+j}^1 v^0, \qquad \gamma_{n+j} = \frac{1}{2} A^{m_{n+j}} \varphi_{n+j} \in H, \\
m_j &= 2n - 2j + \frac{3}{2}, \quad m_{n+j} = 2n - 2j - \frac{1}{2}, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{24}$$

The solutions of the systems (24) coincide. Therefore, $v^0(t) \in W_2^{2n}((0,1), H)$ and we obtain

$$A^{1-2j} \ell_{n+j}^1 v^0 = \sum_{p=1}^{n} (E + \exp \omega_p A)^{-1} \sum_{r=0}^{\chi} b_r (E + \exp \omega_p A t_r) g_p \in H.$$

Therefore, the conditions of the Theorem 6 are satisfied. Taking into account the statement of the Theorem 6, we obtain $v(t) \in W_2^{2n}((0,1), H)$. □

Let us define the solution $u(t)$ of the problem (1)–(2) as the sum $w(t) + v(t)$, where $w(t)$ is the solution of the problem (3), (14) and $v(t)$ is the solution of the problem (21), (22), respectively. Taking into account the statement of the Theorem 4 and Theorem 6, we obtain the statement of the Theorem 2.

Let $u_0(t)$ be the solution of the problem (1), (2) for the case $b_s = 0$, $s = 0, 1, \ldots, \chi$. Consider the operator

$$G : \ W_2^{2n}((0,1), H) \to W_2^{2n}((0,1), H), \ \ Gu_0(t) := u(t).$$

Remark 5. *Taking into account the Theorems 2, 3 and 6, we obtain*

$$G^{-1}, G \in [W_2^{2n}((0,1), H)].$$

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Баранецький Я.О., Демків І.І., Соломко А.В., Сусь О.М. *Нелокальна багатоточкова задача для диференціального рівняння порядку 2n з операторними коефіцієнтами* // Карпатські матем. публ. — 2021. — Т.13, №2. — С. 501–514.

У статті вивчаються спектральні властивості багатоточкової задачі для диференціального операторного рівняння порядку 2n. Оператор задачі має нескінченну кількість кратних власних значень. Кожному кратному його власному значенню відповідає скінченний набір кореневих функцій. Побудована комутативна група операторів перетворення. Кожному елементу групи відповідає ізоспектральне збурення оператора задачі з антиперіодичними умовами. Для вибраної сім'ї багатоточкових задач встановлено умови існування та єдиності розв'язку, а також цей розв'язок побудовано.

Ключові слова і фрази: багатоточкова задача, антиперіодична крайова умова, коренева функція, метод операторів перетворення, базис Рісса.