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m-quasi-$*$-Einstein contact metric manifolds

Kumara H.A.[1], Venkatesha V.[1,✉], Naik D.M.[2]

The goal of this article is to introduce and study the characterstics of m-quasi-$*$-Einstein metric on contact Riemannian manifolds. First, we prove that if a Sasakian manifold admits a gradient m-quasi-$*$-Einstein metric, then M is η-Einstein and f is constant. Next, we show that in a Sasakian manifold if g represents an m-quasi-$*$-Einstein metric with a conformal vector field V, then V is Killing and M is η-Einstein. Finally, we prove that if a non-Sasakian (κ, μ)-contact manifold admits a gradient m-quasi-$*$-Einstein metric, then it is $N(\kappa)$-contact metric manifold or a $*$-Einstein.

Key words and phrases: $*$-Ricci soliton, m-quasi-$*$-Einstein metric, Sasakian manifold, (κ, μ)-contact manifold.

[1] Kuvempu University, Shankaraghatta, 577451, Karnataka, India
[2] CHRIST (Deemed to be University), Bangalore, 560029, Karnataka, India
✉ Corresponding author
E-mail: arunmathsku@gmail.com (Kumara H.A.), vensmath@gmail.com (Venkatesha V.),
devaraja.mallesha@christuniversity.in, devarajamaths@gmail.com (Naik D.M.)

Introduction

Let M be an almost contact metric manifold. Corresponding to Ricci tensor, S. Tachibana [22] introduced the idea of $*$-Ricci tensor. In [13], T. Hamada apply these ideas to real hypersurfaces in complex spaceforms. The $*$-Ricci tensor S^* is defined by

$$S^*(X,Y) = \frac{1}{2}\,\mathrm{trace}\{\varphi \circ R(X, \varphi Y)\}$$

for all vector fields X, Y on M and φ is a $(1,1)$-tensor field. If $*$-Ricci tensor is a constant multiple of $g(X,Y)$ for all $X, Y \perp \xi$, then M is said to be a $*$-Einstein manifold. T. Hamada gave a complete classification of $*$-Einstein hypersurfaces, and further T.A. Ivey and P.J. Ryan [15] updated and refined the work of T. Hamada [13]. Further, the idea of $*$-Ricci tensor on contact Riemannian manifolds are considered in [14].

As the generalization of $*$-Einstein metric, G. Kaimakamis and K. Panagiotidou [17] introduced the so-called $*$-Ricci soliton where they essentialy modified the definition of Ricci soliton by replacing the Ricci tensor S in Ricci soliton condition with the $*$-Ricci tensor S^*.

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Definition. *A Riemannian metric g on M is called a $*$-Ricci soliton if there exist a constant λ and a vector field V such that*

$$\frac{1}{2}\pounds_V g + S^* = \lambda g \tag{1}$$

for all vector fields X, Y on M, where $\pounds_V$ denotes the Lie-derivative in the direction of V.

If the soliton constant λ in the defining condition (1) is a smooth function, then we say that it is an almost $*$-Ricci soliton. Moreover, if the vector field V is a gradient of a smooth function f, then we say that it is gradient almost $*$-Ricci soliton. Note that a $*$-Ricci soliton is trivial if the vector field V is Killing, and in this case the manifold becomes $*$-Einstein.

Einstein metrics and their generalizations are important both in mathematics and physics. A natural extension of the Ricci tensor is the m-Bakry-Emery Ricci tensor

$$S_f^m = S + \operatorname{Hess} f - \frac{1}{m} df \otimes df,$$

namely one puts $0 < m \leq \infty$, f is smooth function on M and $\operatorname{Hess} f$ stands for the Hessian form. Instead of a gradient of a smooth function f by a vector field V, m-Bakry-Emery Ricci tensor was extended by A. Barros and E. Ribeiro in [1] and M. Limoncu in [18] for an arbitrary vector field V on M as follows

$$S_f^m = S + \frac{1}{2}\pounds_V g - \frac{1}{m} V^b \otimes V^b, \tag{2}$$

where V^b is the canonical 1-form associated to the vector field V. With this setting (M, g, V, m) is called an m-quasi-Einstein metric, if there exist a vector field V and a constant λ on M such that

$$S_f^m = S + \frac{1}{2}\pounds_V g - \frac{1}{m} V^b \otimes V^b = \lambda g. \tag{3}$$

It is interesting to note that equation (3) reduces to the so-called Ricci solitons when $m = \infty$, and hence, it is considered as a direct generalization of Ricci soliton. The study of m-quasi Einstein metric in the framework of contact metric manifolds are considered when V is a gradient of a smooth function f on M, see [8–10]. Very recently, in the framework of contact metric manifolds equation (3) has been studied by A. Ghosh in [12].

Almost contact Riemannian manifolds can be viewed as an odd-dimensional analogue of almost Hermitian manifolds. So few authors in the earlier days called them by the name almost co-Hermitian manifolds (see, for example, [16]). Contact Riemannian manifolds, special classes of almost contact Riemannian manifolds, have recently been increasing interest in differential geometry. During the last few years, conformal vector fields, $*$-Ricci solitons, gradient almost $*$-Ricci solitons, and almost $*$-Ricci solitons are studied by several authors on almost contact Riemannian manifold. The studies of $*$-Ricci solitons on almost contact Riemannian manifolds were first initiated by A. Ghosh and D.S. Patra [11]. In the paper, the author showed that a complete Sasakian metric is an almost gradient $*$-Ricci soliton, then it is positive-Sasakian and isometric to a unit sphere $\mathbb{S}^{2n+1}$. Next, P. Majhi et al. [19] studied $*$-Ricci soliton on Sasakian 3-manifolds. Further, V. Venkatesha et al. [24] and X. Dai et al. [7] considered the almost $*$-Ricci soliton on Kenmotsu manifolds and $(\kappa, \mu)'$-almost Kenmotsu manifolds. Recently, D.M. Naik et al. [20] and X. Dai [6] studied the $*$-Ricci solitons in the backkground of cosymplectic manifolds and (κ, μ)-almost cosymplectic manifolds. Y. Wang [26] considered the $*$-Ricci solitons on contact metric 3-manifolds. In this setting, it is worth to mention that in the background of paracontact geometry, the authors in [21,25] studied $*$-Ricci solitons on paraSasakian manifolds and para Kenmotsu manifolds respectively.

Motivated by the above cited works about $*$-Ricci solitons, in this work we essentially modified the m-Bakry-Emery Ricci tensor by replacing the Ricci tensor S in the fundamental equation (2) with the $*$-Ricci tensor S^*, called m-Bakry-Emery $*$-Ricci tensor

$$S^{*m}_{f} = S^* + \frac{1}{2}\pounds_V g - \frac{1}{m} V^b \otimes V^b.$$

In this setting, (M, g) is called an m-quasi $*$-Einstein metric, if there exist a vector V, real constant λ and m, $0 < m \leq \infty$, such that

$$S^* + \frac{1}{2}\pounds_V g - \frac{1}{m} V^b \otimes V^b = \lambda g. \tag{4}$$

The above equation is very much interesting when $m = \infty$. In this case, it is exactly the $*$-Ricci soliton and hence, it is considered as a direct generalization of $*$-Ricci solitons. If the vector field V is a gradient of a smooth function f, then we say that it is gradient m-quasi $*$-Einstein metric and in such a case (4) becomes

$$S^* + \text{Hess} f - \frac{1}{m} df \otimes df = \lambda g. \tag{5}$$

This paper focuses on the study of contact metric manifolds, which admits a m-quasi $*$-Einstein metric. The paper is organized as follows. In Section 2, preliminaries relations and basic results for contact metric manifolds are presented. In Section 3, we show that if there is a gradient m-quasi $*$-Einstein structure (g, f, m) associated with the Sasakian metric g, then M is η-Einstein and f is constant. Next, we proved that in a Sasakian manifold if g represents a m-quasi $*$-Einstein metric with V conformal, then V is Killing and M is η-Einstein. We study gradient m-quasi $*$-Einstein metric on non-Sasakian (κ, μ)-contact manifold and prove that either it is $N(\kappa)$-contact metric manifold or it is $*$-Einstein.

1 Preliminaries

First, we look into the basic definitions and formulas of contact metric manifolds. A (2n+1)-dimensional smooth manifold M is said to be contact if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. This 1-form is called a contact 1-form. For a contact 1-form η, there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact sub-bundle D (defined by $\eta = 0$), we obtain a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$d\eta(X, Y) = \Psi(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{6}$$

for all $X, Y \in TM$. From these equations one can also deduce that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$.

The structure (φ, ξ, η, g) on M is known as a contact metric structure and the metric g is called an associated metric. A Riemannian manifold M together with the structure (φ, ξ, η, g)

is said to be a contact metric manifold and we denote it by $(M, \varphi, \xi, \eta, g)$. On a contact metric manifold, the following identities are known

$$\nabla_X \xi = -\varphi X - \varphi h X, \quad h\varphi + \varphi h = 0 \tag{7}$$

for any vector fields X, Y on M and ∇ denotes the operator of covariant differentiation of g. If the vector field ξ is Killing (equivalently, $h = 0$) with respect to g, then the contact metric manifold M is said to be K-contact. On a K-contact (Sasakian) manifold the following formulas are known (see [2]):

$$\nabla_X \xi = -\varphi X, \tag{8}$$

$$Q\xi = 2n\xi, \tag{9}$$

$$(\nabla_X \varphi)Y = R(\xi, X)Y,$$

where Q and R denote the Ricci operator and the Riemann curvature tensor of g, respectively. A contact metric manifold is said to be Sasakian if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X. \tag{10}$$

On a Sasakian manifold the curvature tensor satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{11}$$

Also, the contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the metric cone $(M \times R^+, r^2 g + dr^2)$ over M is Kaehler [2]. A Sasakian manifold is K-contact but the converse is true only in dimension 3. For more details see [2] and [5].

2 m-quasi $*$-Einstein metric and Sasakian manifolds

In this section, we consider Sasakian metric satisfying m-quasi $*$-Einstein metric. To prove our results we require the following lemma.

Lemma 1. *Let (M, g, m, λ) be a gradient m-quasi $*$-Einstein manifold. If g represents a Sasakian metric, then*

$$\begin{aligned} R(X, Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y &+ \frac{\lambda + (2n-1)}{m}\{(Yf)X - (Xf)Y\} \\ &+ \frac{1}{m}\{(Xf)QY - (Yf)QX + (Yf)\eta(X)\xi - (Xf)\eta(Y)\xi\} \\ &+ 2g(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y \end{aligned} \tag{12}$$

and

$$\frac{m-1}{m} S(Y, Df) = \frac{1}{2}(Yr) + \frac{2n(\lambda + (2n-1)) + 1 - r}{m}(Yf) - \frac{1}{m}(\xi f)\eta(Y). \tag{13}$$

Proof. In [11], A. Ghosh and D.S. Patra find the expression of $*$-Ricci tensor, which is of the form

$$S^*(X, Y) = S(X, Y) - (2n-1)g(X, Y) - \eta(X)\eta(Y). \tag{14}$$

Making use of the above equation in (5), we obtain

$$S + \text{Hess}\, f - \frac{1}{m} df \otimes df = (\lambda + (2n-1))g + \eta \otimes \eta. \tag{15}$$

Here, we note that equation (15) may be exhibited as

$$\nabla_Y Df + QY - \frac{1}{m} g(Y, Df) = (\lambda + (2n-1))Y + \eta(Y)\xi. \tag{16}$$

By straightforward computations, using the well-known expression of the curvature tensor, we obtain $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, and the repeated use of equation (16) gives the equation (12). Next, contracting (12) over X we get the equation (13). □

Remark. *The relation* (15) *is the m-quasi $*$-Einstein conditon on Sasakian manifolds. We observe that if f is constant then Sasakian manifolds becomes η-Einstein.*

Theorem 1. *Let (M, g, m, λ) be a gradient m-quasi $*$-Einstein manifold. If g represents a Sasakian metric and $m \neq 1$, then M is η-Einstein and f is constant.*

Proof. Taking covariant differentiation of (9) and then making use of (8) we obtain

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \tag{17}$$

We know that in a Sasakian manifold the Ricci operator Q and φ commute each other, i.e. $Q\varphi = \varphi Q$. Thus, taking inner product of (12) with ξ and then using (9) and (17) yields

$$g(R(X,Y)Df, \xi) = 2g(X, Q\varphi Y) - 2(2n-1)g(X, \varphi Y) + \frac{\lambda}{m}\{(Yf)\eta(X) - (Xf)\eta(Y)\}.$$

Replacing Y by ξ in the foregoing equation and recalling (11), we obtain

$$\left\{\frac{\lambda}{m} - 1\right\}[(Xf) - (\xi f)\eta(X)] = 0.$$

Since, m and λ are constant, we have either $\lambda/m \neq 1$, or $\lambda/m = 1$. We now discuss the two cases separately.

Case (i). When $\lambda/m \neq 1$, we have $Df = (\xi f)\xi$. Differentiating this and making use of (8), we obtain $\nabla_X Df = X(\xi f)\xi - (\xi f)\varphi X$. Applying Poincare lemma ($d^2 = 0$) we see that $(X(\xi f))\eta(Y) - (Y(\xi f))\eta(X) + 2(\xi f)d\eta(X,Y) = 0$. Choosing $X, Y \perp \xi$ and noting that $d\eta$ is non-vanishing for any Sasakian manifold, we find $\xi f = 0$. This shows that f is constant.

Case (ii). When $\lambda/m = 1$, we remember that for a Sasakian manifold ξ is Killing, and hence $\pounds_\xi Q = 0$. In view of (8) and (9), this is equivalent to $\nabla_\xi Q = Q\varphi - \varphi Q$. For a Sasakian manifold, Q and φ commute each other (see [2]) and hence $\nabla_\xi Q = 0$. Now replacing Y by ξ in (12), recalling the last equation, (9), (11) and (17) we find

$$Q\varphi X - (2n-1)\varphi X + \left\{\frac{\lambda}{m} - 1\right\}(Xf)\xi + \frac{(\xi f)}{m}\{QX - (\lambda + (2n-1) - m)X - \eta(X)\xi\} = 0.$$

Taking inner product of this equation with Y and by virtue of $\lambda/m = 1$, one immediately finds

$$g(Q\varphi X - (2n-1)\varphi X, Y) + \frac{(\xi f)}{m} g(QX - (2n-1)X - \eta(X)\xi, Y) = 0.$$

Anti-symmetrizing the foregoing equation yields $(Q\varphi + \varphi Q)X = 2(2n-1)\varphi X$ for all vector fields X. As $Q\varphi = \varphi Q$ we have $Q\varphi X = (2n-1)\varphi X$. Replacing X by φX and making use of last equation of (6), it follows that

$$QX = (2n-1)X + \eta(X)\xi. \tag{18}$$

This shows that M is η-Einstein with constant scalar curvature $r = 4n^2$. Since M is η-Einstein and $\lambda/m = 1$, making use of these in (13) yeilds $m\{Df - (\xi f)\xi\} = 0$. Since $m \neq 0$, we have $Df - (\xi f)\xi = 0$. We conclude that f is constant. □

By virtue of (18) and (14), it follows that M is $*$-Ricci flat. From this we state the following corollary.

Corollary. *Let (M, g, m, λ) be a gradient m-quasi $*$-Einstein manifold. If g represents a Sasakian metric and $m \neq 1$, then $*$-Ricci tensor vanishes and f is constant.*

Now we consider a Sasakian manifold with conformal m-quasi-$*$-Einstein metric and we prove the following theorem.

Theorem 2. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. If g represents an m-quasi $*$-Einstein metric with V as a conformal vector field, then V is Killing and M is η-Einstein.*

Proof. In view of (14), it follows from (4) that

$$S(X,Y) + \frac{1}{2}(\pounds_V g)(X,Y) - \frac{1}{m}V^b(X)V^b(Y) = \{\lambda + (2n-1)\}g(X,Y) + \eta(X)\eta(Y). \tag{19}$$

Since V is conformal and hence there exists a smooth function σ such that $(\pounds_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = 2\sigma g(X,Y)$. Therefore, equation (19) reduces to

$$S(X,Y) = \{\lambda + (2n-1) - \sigma\}g(X,Y) + \eta(X)\eta(Y) + \frac{1}{m}V^b(X)V^b(Y). \tag{20}$$

Replacing X by φX in the above equation, we get

$$S(\varphi X, Y) = \{\lambda + (2n-1) - \sigma\}g(\varphi X, Y) + \frac{1}{m}V^b(\varphi X)V^b(Y).$$

Again replacing Y by φY in (20), we obtain

$$S(X, \varphi Y) = \{\lambda + (2n-1) - \sigma\}g(X, \varphi Y) + \frac{1}{m}V^b(X)V^b(\varphi Y).$$

Adding the last two equations and keeping in mind that in Sasakian manifold Q and φ commute, we find that

$$\frac{1}{m}\{V^b(\varphi X)V^b(Y) + V^b(X)V^b(\varphi Y)\} = 0.$$

Substituting $X = \varphi V$ and $Y = \varphi V$, the foregoing equation entails that $\varphi V = 0$. Operating this by φ shows that $V = \eta(V)\xi = \rho\xi$, where $\rho = \eta(V)$. Differentiating this and making use of (8) yields $\nabla_X V = (X\rho)\xi - \rho\varphi X$. We know that V is conformal, and therefore

$$(\pounds_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = (X\rho)\eta(Y) + (Y\rho)\eta(Y). \tag{21}$$

Choosing X, Y orthogonal to ξ, the foregoing equation gives $\pounds_V g = 0$. This shows that V is Killing. Thus, (21) yields $(X\rho)\eta(Y) + (Y\rho)\eta(Y) = 0$. Putting $Y = \xi$ in this equation and proceeding as in case (i) of Theorem 1 we easily conclude that $\rho = \eta(V)$ is constant. Making use of this in (20) takes the form $S(X,Y) = \{\lambda + (2n-1)\}g(X,Y) + \{\rho^2/m + 1\}\eta(X)\eta(Y)$. Finally, replacing X by $\varphi^2 X$ in the last equation and recalling (9) implies the equation $S(X,Y) = \{\lambda + (2n-1)\}g(X,Y) + (1-\lambda)\eta(X)\eta(Y)$ and hence $\rho^2 = -\lambda m$. □

If S^* satisfies the relation $S^*(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$ for $\alpha, \beta \in \mathbb{R}$, then M is said to be a $*$-η-Einstein almost contact metric manifold. If we set $X = Y = \xi$, we find $\alpha + \beta = 0$ so that $\alpha = -\beta$.

Let M be a $*$-η-Einstein K-contact manifold with $V = \rho\xi$. Differentiating this along X and using (8), we obtain $\nabla_X V = (X\rho)\xi - \rho\varphi X$. This together with $*$-η-Einstein imply

$$S^*(X,Y) + \frac{1}{2}g(X,Y) - \frac{1}{m}V^b(X)V^b(Y)$$
$$= \alpha g(X,Y) - \alpha\eta(X)\eta(Y) + \frac{1}{2}\{(X\rho)\eta(Y) + (Y\rho)\eta(X)\} - \frac{\rho^2}{m}\eta(X)\eta(Y).$$

If we choose $\rho^2 = -m\alpha$, where $\alpha > 0$, then it is easily see that M admits an m-quasi $*$-Einstein metric with $\lambda = \alpha$. Thus, we say that any $*$-η-Einstein K-contact manifold satisfies the m-quasi $*$-Einstein condition with $V = \rho\xi$, where $\rho^2 = -m\alpha$ and $\alpha > 0$. Next, suppose that M admits an m-quasi $*$-Einstein metric with $V = \rho\xi$, where ρ is a smooth function. Then (4) reduces to

$$S^*(X,Y) + \frac{1}{2}\{(X\rho)\eta(Y) + (Y\rho)\eta(X)\} - \frac{\rho^2}{m}\eta(X)\eta(Y) = \lambda g(X,Y).$$

Replacing $Y = \xi$ in the last equation and making use of $S^*(X,\xi) = 0$, we obtain

$$\frac{1}{2}\{(X\rho) + (\xi\rho)\eta(X)\} = \Big\{\frac{\rho^2}{m} + \lambda\Big\}\eta(X).$$

Again, taking $X = \xi$ the foregoing equation gives $(\xi\rho) = \rho^2/m + \lambda$. Hence, we have $D\rho = (\xi\rho)\xi$. Proceeding as in the case (i) of Theorem 1 we easily see that $\xi\rho = 0$. This shows that ρ is constant and $\rho^2 = -\lambda m$. Consequently, $S^*(X,Y) = \lambda g(\varphi X, \varphi Y)$. Thus, we have proved the following proposition.

Proposition. *Let $(M, \varphi, \xi, \eta, g)$ be a K-contact manifold. Then M satisfies the m-quasi $*$-Einstein condition if it is $*$-η-Einstein and $V = \rho\xi$, where $\rho^2 = -m\alpha$ and $\alpha > 0$. Moreover, M is $*$-η-Einstein if it satisfies the m-quasi $*$-Einstein condition with $V = \rho\xi$ for some smooth function ρ.*

Example. *Define (ξ, η, φ, g) on the Euclidean space $M = \mathbb{R}^3$ in the following way:*

$$\xi = \frac{\partial}{\partial z}, \quad \eta = ydx - xdy + dz, \quad \varphi\Big(\frac{\partial}{\partial x}\Big) = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},$$
$$\varphi\Big(\frac{\partial}{\partial y}\Big) = y\frac{\partial}{\partial z} - \frac{\partial}{\partial x}, \quad \varphi\Big(\frac{\partial}{\partial z}\Big) = 0, \quad (g_{ij}) = \begin{pmatrix} y^2+1 & -xy & y \\ -xy & x^2+1 & -x \\ y & -x & 1 \end{pmatrix}.$$

It is not difficult to verify that the structure (ξ, η, φ, g) is an almost contact Riemannian structure. Recalling $\Psi = g(\cdot, \varphi\cdot)$, we find $\Psi = -2dx \wedge dy$. Thus, it follows that $\Psi = d\eta$ and so M is a contact Riemannian manifold. Now, we employ Koszul's formula in order to deduce Levi-Civita connection ∇ as given below:

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -2y\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial z}, && \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = -2x\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial z}, \\ &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (x^2 - y^2)\frac{\partial}{\partial z}, && \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \\ &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} = \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial x} = -\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, && \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = 0. \end{aligned} \tag{22}$$

We use the equation (22) *to check that* (10) *holds and so the defined structure is Sasakian. Further use of the equation* (22) *gives:*

$$R\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)\frac{\partial}{\partial z}=-x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y},\qquad R\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial y}=-x\frac{\partial}{\partial x}-xy\frac{\partial}{\partial z},$$

$$R\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial x}=y\frac{\partial}{\partial x}-(y^2+1)\frac{\partial}{\partial z},\qquad R\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial y}=-x\frac{\partial}{\partial y}-(x^2+1)\frac{\partial}{\partial z},$$

$$R\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)\frac{\partial}{\partial x}=-xy\frac{\partial}{\partial x}-(y^2-3)\frac{\partial}{\partial y}+4x\frac{\partial}{\partial z},\qquad R\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial z}=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},$$

$$R\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\qquad R\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial x}=y\frac{\partial}{\partial y}+xy\frac{\partial}{\partial z},$$

$$R\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)\frac{\partial}{\partial y}=(x^2-3)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+4y\frac{\partial}{\partial z}.$$

We use the preceding expression of curvature tensor to find the Ricci tensor as given below:

$$(S_{ij})=\begin{pmatrix}2y^2-2 & -2xy & 2y\\ -2xy & 2x^2-2 & -2x\\ 2y & -2x & 2\end{pmatrix}.$$

Now it is not hard to verify that $S=-2g+4\eta\otimes\eta$. *By virtue of this,* (14) *and definition of* $*$*-Ricci tensor, one can easily find* $S^*=-3g+3\eta\otimes\eta$. *Hence, if we take* $V=3\sqrt{m/3}\xi$, *then* M *admits an* m*-quasi-*$*$*-Einstein metric with* $\lambda=-3$.

3 m-quasi $*$-Einstein metric and (κ,μ)-contact manifolds

In [3], D.E. Blair et al. introduced and studied a new type of contact metric manifold known as a (κ,μ)-contact manifold. Later on, E. Boeckx [4] classified these manifolds completely. A contact metric manifold (M,φ,ξ,η,g) is said to be (κ,μ)-space if the curvature tensor satisfies

$$R(X,Y)\xi=\kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\}\tag{23}$$

for all vector fields X,Y on M and for some real numbers (κ,μ). This type of space arises through a D-homothetic deformation (see [23]) to a contact metric manifold which satisfies $R(X,Y)\xi=0$. The class of (κ,μ)-spaces covers Sasakian manifolds (for $\kappa=1$) and the trivial sphere bundle $E^{n+1}\times S^n(4)$ (for $\kappa=\mu=0$). There exist examples of non-Sasakian (κ,μ)-contact metric manifolds. For instance, the unit tangent bundles of Riemannian manifolds of constant curvature $\kappa\neq 1$. Since a D-homothetic deformation preserves (κ,μ)-contact structures, one can construct lot of examples of (κ,μ)-contact structures (see [3]). The following formulas are also valid for a non-Sasakian (κ,μ)-contact manifolds [3]:

$$QX=[2(n-1)-n\mu]X+[2(n-1)+\mu]hX+[2(1-n)+n(2\kappa+\mu)]\eta(X)\xi,\quad Q\xi=2n\kappa\xi,$$
$$h^2=(\kappa-1)\varphi^2,\quad \kappa<1,\tag{24}$$

equality holds when $\kappa=1$ (equivalently, $h=0$), i.e. M is Sasakian. For the non-Sasakian case, i.e. $\kappa<1$, the (κ,μ)-nullity condition determines the curvature of M completely. In view of this, E. Boeckx [4] proved that a non-Sasakian (κ,μ)-contact manifold is locally homogeneous and hence analytic. Moreover, the constant scalar curvature r of such structures is given by

$$r=2n(2(n-1)+\kappa-n\mu),$$

which is constant.

Now we prepare the following result for later use.

Lemma 2. *Let* $(M, \varphi, \xi, \eta, g)$ *be a non-Sasakian* (κ, μ)*-contact manifold. If* g *represents an* m*-quasi* $*$*-Einstein metric, then*

$$\begin{aligned} R(X,Y)Df = &\frac{\lambda + (n\mu + \kappa)}{m}\{(Yf)X - (Xf)Y\} + \frac{1}{m}\{(Xf)\eta(Y)\xi - (Yf)\eta(X)\xi\} \\ &- 2g(X, \varphi Y)\xi + \eta(Y)(\varphi X + \varphi hX) - \eta(X)(\varphi Y + \varphi hY). \end{aligned} \tag{25}$$

Proof. In [11], A. Ghosh and D.S. Patra obtain the expression of $*$-Ricci tensor in non-Sasakian (κ, μ)-contact manifolds, which is of the form $S^*(X,Y) = (n\mu + \kappa)\{-g(X,Y) + \eta(X)\eta(Y)\}$. By virtue of this, equation (5) takes the form

$$\mathrm{Hess}\, f - \frac{1}{m} df \otimes df = (\lambda + (n\mu + \kappa))g - \eta \otimes \eta.$$

The above equation can be exhibited as

$$\nabla_Y Df = \frac{1}{m}(Yf)Df + (\lambda + (n\mu + \kappa))Y - \eta(Y)\xi. \tag{26}$$

By a straightforward computations, using the well-known expression of the curvature tensor

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

and the repeated use of equation (26) gives equation (25). □

Theorem 3. *Let* $(M, \varphi, \xi, \eta, g)$ *be a non-Sasakian* (κ, μ)*-contact manifold. Suppose there exists a gradient* m*-quasi* $*$*-Einstein structure* (g, f, m) *associated with the metric* g*. Then either it is* $N(\kappa)$*-contact metric manifold or it is* $*$*-Einstein.*

Proof. Taking scalar product of (25) with ξ and then replacing Y by ξ, we obtain

$$g(R(X,\xi)Df, \xi) = \frac{\lambda + n\mu + \kappa - 1}{m}\{(\xi f)\eta(X) - (Xf)\}. \tag{27}$$

Recalling (23), it follows that

$$g(R(X,\xi)\xi, Df) = -g(R(X,\xi)Df, \xi) = \kappa\{(\xi f)\eta(X) - (Xf)\} - \mu g(hX, Df).$$

Making use of last equation in (27) we have

$$hDf = \sigma\{Df - (\xi f)\xi\}, \tag{28}$$

where $\sigma = (\lambda + n\mu + \kappa - 1 - m\kappa)/(m\mu)$ and clearly this is constant. Differentiating (28) along an arbitrary vector field X and taking into account (7), (26) and (28), we find

$$\begin{aligned} &(\nabla_X h)Df - \frac{\sigma}{m}(Xf)(\xi f)\xi + (\lambda + n\mu + \kappa)hX \\ &= \sigma\{(\lambda + n\mu + \kappa)X - \eta(X)\xi - (X(\xi f))\xi + (\xi f)(\varphi X + \varphi hX)\} \end{aligned} \tag{29}$$

From (26) it follows that

$$g(\nabla_\xi Df, \xi) = \frac{(\xi f)^2}{m} + (\lambda + n\mu + \kappa - 1).$$

By definition, we have $g(\xi, Df) = \xi f$. Taking covariant differentiation of this along ξ and noting that $\nabla_\xi \xi = 0$ (follows from (7)), we obtain $g(\nabla_\xi Df, \xi) = \xi(\xi f)$. Hence,

$$\xi(\xi f) = \frac{(\xi f)^2}{m} + (\lambda + n\mu + \kappa - 1). \tag{30}$$

On the other hand, we note that for any contact metric manifolds we have [2]

$$\nabla_\xi h = \varphi - \varphi h^2 - \varphi l, \tag{31}$$

where $l = R(\cdot, \xi)\xi$. From (23) it follows that $l = -\kappa\varphi^2 + \mu h$. Making use of this and (24) in (31) we at once obtain $\nabla_\xi h = \mu h\varphi$. Replacing X by ξ in (29), using the last equation and (30), one immediately finds $\mu h Df = 0$. This implies either (i) $\mu = 0$, or (ii) $\mu \neq 0$.

Case (i). If $\mu = 0$, then from (23) it follows that $R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\}$, i.e. ξ belongs to κ-nullity distribution. This shows M is a $N(\kappa)$-contact metric manifold.

Case (ii). Suppose $\mu \neq 0$. In this case, we have $hDf = 0$. Operating the preceding equation by h, recalling (24), it follows that $(\kappa - 1)\varphi^2 Df = 0$. Since M is non-Sasakian, we have $Df = (\xi f)\xi$. Differentiating this along an arbitrary vector field X together with (7) entails that $\nabla_X Df = X(\xi f)\xi - (\xi f)(\varphi X + \varphi h X)$. Since $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$, the last equation shows that $(X(\xi f))\eta(X) - (Y(\xi f))\eta(Y) + (\xi f)d\eta(X,Y) = 0$. Replacing X by φX and Y by φY and since $d\eta$ is non-zero for any contact metric structure it follows that $(\xi f) = 0$. Hence $Df = 0$, i.e. f is constant and consequently (5) implies that M is $*$-Einstein. □

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Кумара Х.А., Венкатеша В., Найк Д.М. *m-квазі-$*$-Айнштайнівські контактні метричні многовиди* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 61–71.

Метою даної статті є введення та дослідження характеристик m-квазі-$*$-Айнштайнівської метрики на контактних рімановиx многовидах. Спершу ми доводимо, що якщо Сасакяновий многовид має градієнт m-квазі-$*$-Айнштайнівської метрики, то M є η-Айнштайнівським, а f є константою. Далі ми показуємо, що у Сасакяновому многовиді, якщо g представляє m-квазі-$*$-Айнштайнівську метрику з конформним векторним полем V, то V є Кіллінговим, а M — η-Айнштайнівським. Нарешті, ми доводимо, що якщо не-Сасакяновий (κ, μ)-контактний многовид допускає градієнт m-квазі-$*$-Айнштайнівської метрики, то він є $N(\kappa)$-контактний метричний або $*$-Айнштайнівський многовид.

Ключові слова і фрази: $*$-Річчі солітон, m-квазі-$*$-Айнштайнівська метрика, Сасакяновий многовид, (κ, μ)-контактний многовид.