ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2021, **13** (2), 326–339
doi:10.15330/cmp.13.2.326-339

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2021, Т.13, №2, С.326–339

Some spectral formulas for functions generated by differential and integral operators in Orlicz spaces

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In this paper, we investigate the behavior of the sequence of L^Φ-norm of functions, which are generated by differential and integral operators through their spectra (the support of the Fourier transform of a function f is called its spectrum and denoted by $\mathrm{sp}(f)$). With Q being a polynomial, we introduce the notion of Q-primitives, which will return to the notion of primitives if $Q(x) = x$, and study the behavior of the sequence of norm of Q-primitives of functions in Orlicz space $L^\Phi(\mathbb{R}^n)$. We have the following main result: *let Φ be an arbitrary Young function, $Q(\mathbf{x})$ be a polynomial and $(\mathcal{Q}^m f)_{m=0}^\infty \subset L^\Phi(\mathbb{R}^n)$ satisfies $\mathcal{Q}^0 f = f, Q(D)\mathcal{Q}^{m+1} f = \mathcal{Q}^m f$ for $m \in \mathbb{Z}_+$. Assume that $\mathrm{sp}(f)$ is compact and $\mathrm{sp}(\mathcal{Q}^m f) = \mathrm{sp}(f)$ for all $m \in \mathbb{Z}_+$. Then*

$$\lim_{m\to\infty} \|\mathcal{Q}^m f\|_\Phi^{1/m} = \sup_{\mathbf{x}\in \mathrm{sp}(f)} \big|1/Q(\mathbf{x})\big|.$$

The corresponding results for functions generated by differential operators and integral operators are also given.

Key words and phrases: Orlicz space, inequality in approximation, Fourier transform, generalized function.

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Introduction

Let $1 \le p \le \infty$, $\varrho > 0$, $f \in L^p(\mathbb{R})$ and $\mathrm{sp}(f) \subset [-\varrho, \varrho]$, where $\mathrm{sp}(f) := \mathrm{supp}\widehat{f}$ and $\widehat{f} = \mathcal{F}f$ is the Fourier transform of f. Then it is well-known the following Bernstein inequality (see [11,23]): $\|D^m f\|_p \le \varrho^m \|f\|_p$, $m = 1, 2, \dots$. Bernstein inequality plays an important role in function theory and has various applications. It was studied and developed by many authors, see, e.g., [16–18, 21, 25–27, 33]. The following result is an addition of the Bernstein inequality (see [4]). Let $1 \le p \le \infty$ and $D^m f \in L^p(\mathbb{R})$, $m = 0, 1, 2, \dots$, then

$$\lim_{m\to\infty} \|D^m f\|_p^{1/m} = \sup\{|\zeta| : \zeta \in \mathrm{sp}(f)\}.$$

This result has the following extensions (see [5,7]).

Let $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^n)$ and $\mathrm{sp}(f)$ be compact, then

$$\lim_{|\alpha|\to\infty} \left(\|D^\alpha f\|_p / \sup_{\zeta\in \mathrm{sp}(f)} |\zeta^\alpha| \right)^{1/|\alpha|} = 1.$$

УДК 517.9
2020 *Mathematics Subject Classification:* 41A17, 26D10, 42B10, 46E30.

Further, if $1 \le p \le \infty$, Q is a polynomial, $f \in L^p(\mathbb{R}^n)$ and $\mathrm{sp}(f)$ is compact, then

$$\lim_{m\to\infty} \|Q^m(D)f\|_p^{1/m} = \sup\{|Q(\mathbf{x})| : \mathbf{x} \in \mathrm{sp}(f)\},$$

where the differential operator $Q(D)$ is obtained from $Q(\mathbf{x})$ by substituting

$$\mathbf{x} \to (-i\partial/\partial x_1, -i\partial/\partial x_2, \dots, -i\partial/\partial x_n),$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

The novelty of these results is that the behavior of the sequence of norm of derivatives of a function f is directly investigated through its spectrum $\mathrm{sp}(f)$ but not, as usual, through a given compact K containing $\mathrm{sp}(f)$. These results were studied and developed by many authors (see [1–3, 5–7, 12–15]). It is natural to ask what will happen when we replace derivatives by integrals? For $p = 2$, V.K. Tuan proved the following result in [31].

Let $f \in L^2(\mathbb{R})$ and $\varrho := \inf\{|\zeta| : \zeta \in \mathrm{sp}(f)\} > 0$. Then there exists $I^m f$, $I^m f \in L^2(\mathbb{R})$ for all m, and

$$\lim_{m\to\infty} \|I^m f\|_2^{1/m} = \varrho^{-1}, \quad If(x) = \int_x^\infty f(y)dy,$$

the improper indefinite Riemann integral, and $I^n = (I)^n$. This result of V.K. Tuan was extended in [8] to the case $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, to the n-dimension case and Orlicz spaces in [9, 10].

The purpose of this paper is to extend above results to more general cases. With Q being a polynomial, we introduce the notion of Q-primitives, which will return to the notion of primitives used in [8] if $Q(x) = x$, and study the behavior of the sequence of norm of Q-primitives of functions in $L^\Phi(\mathbb{R}^n)$. Moreover, we also investigate the behavior of the sequence of norm of functions in Orlicz spaces which are generated by differential operators and integral operators.

1 Notations

Let $D = (D_1, \dots, D_n)$, $D_j = \partial/\partial x_j$ for $j = 1, 2, \dots, n$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $1/\mathbf{x} = (1/x_1, \dots, 1/x_n)$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $\mathbf{x} = (x_1, \dots, x_n)$. Let K be an arbitrary compact set in $\mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^n$ and $\varepsilon > 0$. Denote by $K_\varepsilon := \{\mathbf{x} \in \mathbb{R}^n : dist(\mathbf{x}, K) < \varepsilon\}, K_{(\varepsilon)} := \{\mathbf{x} \in \mathbb{C}^n : dist(\mathbf{x}, K) < \varepsilon\}$, $B(\mathbf{z}, \varepsilon) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{z}| < \varepsilon\}$ and $(\mathbb{R}^n, \varepsilon) := \{\mathbf{x} \in \mathbb{R}^n : \min_{1\le j\le n} |x_j| \ge \varepsilon\}$. Further, $\mathcal{S}(\mathbb{R}^n)$ stands for the Schwartz space on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of tempered distributions on $\mathbb{R}^n$. The convolution of two functions f, g is denoted by $f * g$. Let $f \in L^1(\mathbb{R}^n)$ then

$$\widehat{f}(\mathbf{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\mathbf{z}} f(\mathbf{z})d\mathbf{z},$$

where $\mathbf{x}\mathbf{z} = x_1z_1 + x_2z_2 + \dots + x_nz_n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{z} = (z_1, z_2, \dots, z_n)$. The Fourier transform of a tempered distribution f is defined via the formula $\langle \mathcal{F}f, \phi \rangle = \langle f, \mathcal{F}\phi \rangle$, $\phi \in \mathcal{S}(\mathbb{R}^n)$. Recall that (see [8]) if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}_+^n$ is the unit vector such that its j^{th} coordinate equals 1, $j = 1, 2, \dots, n$, the tempered distribution $I^{e_j} f$ is termed a x_j-primitive of f if $D^{e_j}(I^{e_j} f) = f$, that is, $\langle I^{e_j} f, D^{e_j}\varphi \rangle = -\langle f, \varphi \rangle \ \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$. Now let Q be a polynomial with n variables, $Q(\mathbf{x}) \not\equiv 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The tempered distribution $\mathcal{Q}f$ is termed a Q-primitive of f if $Q(D)\mathcal{Q}f = f$. So, each $\mathcal{Q}f$ is a solution of the differential equation $Q(D)h = f$, and this is the meaning of introducing the Q-primitive notion. Note that the notion of primitives of a generalized function in $\mathcal{D}'(a,b)$ can be found in [32]. For a polynomial Q, the differential operator $Q(D)$ (respectively, the integral operator $Q(I)$) is obtained from $Q(\mathbf{x})$ by substituting $x_j \to -i\partial/\partial x_j$ (respectively, $x_j \to iI_j$), $j = 1, \dots, n$. Then, for $Q(\mathbf{x}) = \sum_{|\alpha| \le M} a_\alpha \mathbf{x}^\alpha$, we have

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad Q(D)f = \sum_{|\alpha| \le M} a_\alpha (-i)^{|\alpha|} D^\alpha f, \quad Q(I)f = \sum_{|\alpha| \le M} a_\alpha i^{|\alpha|} I^\alpha f.$$

Let $\Phi : [0, +\infty) \to [0, +\infty]$ be an arbitrary Young function, i.e., $\Phi(0) = 0$, $\Phi(t) \ge 0$, $\Phi(t) \not\equiv 0$ and Φ is convex. Denote by $\overline{\Phi}(t) = \sup_{s \ge 0}\{ts - \Phi(s)\}$ the Young function conjugate to Φ and $L^\Phi(\mathbb{R}^n)$-the space of measurable functions u such that

$$|\langle u, v\rangle| = \Big| \int_{\mathbb{R}^n} u(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} \Big| < \infty$$

for all v with $\rho(v, \overline{\Phi}) < \infty$, where

$$\rho(v, \overline{\Phi}) = \int_{\mathbb{R}^n} \overline{\Phi}(|v(\mathbf{x})|) d\mathbf{x}.$$

Then $L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the Orlicz norm

$$\|u\|_\Phi = \sup_{\rho(v, \overline{\Phi}) \le 1} \Big| \int_{\mathbb{R}^n} u(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} \Big|,$$

which is equivalent to the Luxemburg norm

$$\|u\|_{(\Phi)} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi(|u(\mathbf{x})|/\lambda)\, d\mathbf{x} \le 1\Big\} < \infty.$$

Moreover, $\|\cdot\|_{(\Phi)} \le \|\cdot\|_\Phi \le 2\|\cdot\|_{(\Phi)}$.

We have the following results (see [28]).

Lemma 1. *Let $u \in L^\Phi(\mathbb{R}^n)$ and $v \in L^{\overline{\Phi}}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |u(\mathbf{x}) v(\mathbf{x})|\, d\mathbf{x} \le \|u\|_\Phi \|v\|_{(\overline{\Phi})}.$$

Lemma 2. *Let $u \in L^\Phi(\mathbb{R}^n)$ and $v \in L^1(\mathbb{R}^n)$. Then $\|u * v\|_\Phi \le \|u\|_\Phi \|v\|_1$.*

Note that Lebesgue spaces and their extension, Orlicz spaces, play an important role in analysis and have many applications (see [19, 20, 22, 24, 28, 29]). Recall that $\|\cdot\|_{(\Phi)} = \|\cdot\|_p$, where $\Phi(t) = t^p$ with $1 \le p < \infty$, and $\|\cdot\|_{(\Phi)} = \|\cdot\|_\infty$, where $\Phi(t) = 0$ for $0 \le t \le 1$ and $\Phi(t) = \infty$ for $t > 1$.

2 Some spectral formulas for Q-primitives of a function

Theorem 1. *Let Φ be an arbitrary Young function, $f \in L^\Phi(\mathbb{R}^n)$ and Q be a polynomial. Assume that $\operatorname{sp}(f)$ is compact and $Q(\mathbf{x}) \ne 0$ for all $\mathbf{x} \in \operatorname{sp}(f)$. Then there exists exactly one sequence of functions $(\mathcal{Q}^m f)_{m=0}^\infty \subset L^\Phi(\mathbb{R}^n)$ satisfying $\mathcal{Q}^0 f = f$, $Q(D)\mathcal{Q}^{m+1} f = \mathcal{Q}^m f$, $\operatorname{sp}(f) = \operatorname{sp}(\mathcal{Q}^m f)$ for $m \in \mathbb{Z}_+$.*

Proof. We consider a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\varphi(\mathbf{x}) = 1$ for $\mathbf{x}$ in a neighborhood of $\mathrm{sp}(f)$. Put $\mathcal{Q}^m f = (2\pi)^{-n/2} f * \mathcal{F}^{-1}(\varphi(\mathbf{x})/Q^m(\mathbf{x}))$. Clearly, $\mathcal{Q}^0 f = f$, $\mathcal{Q}^m f \in L^\Phi(\mathbb{R}^n)$, $\mathrm{sp}(f) = \mathrm{sp}(\mathcal{Q}^m f)$ and

$$Q(D)\mathcal{Q}^{m+1} f = (2\pi)^{-n/2} f * (Q(D)(\mathcal{F}^{-1}(\varphi(\mathbf{x}))/Q^{m+1}(\mathbf{x}))) = \mathcal{Q}^m f$$

for all $m = 0, 1, \ldots$. Moreover, $\mathcal{Q}_1 f$ and $\mathcal{Q}_2 f$ are two Q-primitives in $L^\Phi(\mathbb{R}^n)$ of f satisfying $\mathrm{sp}(f) = \mathrm{sp}(\mathcal{Q}_1 f) = \mathrm{sp}(\mathcal{Q}_2 f)$ then

$$\begin{aligned}\langle f, \mathcal{F}(\varphi(\mathbf{x})\psi(\mathbf{x})/Q(\mathbf{x}))\rangle &= \langle Q(D)\mathcal{Q}_j f, \mathcal{F}(\varphi(\mathbf{x})\psi(\mathbf{x})/Q(\mathbf{x}))\rangle \\ &= \langle \mathcal{Q}_j f, Q(-D)\mathcal{F}(\varphi(\mathbf{x})\psi(\mathbf{x})/Q(\mathbf{x}))\rangle = \langle \mathcal{Q}_j f, \mathcal{F}(\varphi(\mathbf{x})\psi(\mathbf{x})\rangle \\ &= \langle \mathcal{F}(\mathcal{Q}_j f), \varphi\psi\rangle = \langle \mathcal{F}(\mathcal{Q}_j f), \psi\rangle = \langle \mathcal{Q}_j f, \mathcal{F}\psi\rangle\end{aligned}$$

for $j = 1, 2$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Hence, $\langle \mathcal{Q}_1 f - \mathcal{Q}_2 f, \mathcal{F}\psi\rangle = 0$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$, and then $\mathcal{Q}_1 f = \mathcal{Q}_2 f$ because of $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n)$, so the uniqueness of the sequence $(\mathcal{Q}^m f)_{m=0}^\infty$ is proved. ☐

Remark 1. *Let $f \in L^\Phi(\mathbb{R}^n)$ and $\mathrm{sp}(f)$ be compact. It should be noticed that the assumption $Q(\mathbf{x}) \neq 0\, \forall\, \mathbf{x} \in \mathrm{sp}(f)$ is essential for the existence of Q-primitives in $L^\Phi(\mathbb{R}^n)$ of f. For example, if $f(\mathbf{x}) = 1 + \cos\mathbf{x}$, $Q(\mathbf{x}) = \mathbf{x}$, $n = 1$, $\Phi(\mathbf{x}) = 0$ for $0 \le \mathbf{x} \le 1$ and $\Phi(\mathbf{x}) = \infty$ for $\mathbf{x} > 1$, then $\mathrm{sp}(f) = \{-1, 0, 1\}$ and each Q-primitive of f has the form $\mathbf{x} + \sin\mathbf{x} + c$, $c \in \mathbb{C}$, which does not belong to $L^\Phi(\mathbb{R})(= L^\infty(\mathbb{R}))$.*

Theorem 2. *Let Φ be an arbitrary Young function, Q be a polynomial and $(\mathcal{Q}^m f)_{m=0}^\infty \subset L^\Phi(\mathbb{R}^n)$ satisfies $\mathcal{Q}^0 f = f$, $Q(D)\mathcal{Q}^{m+1} f = \mathcal{Q}^m f$ for $m \in \mathbb{Z}_+$. Then*

$$\liminf_{m\to\infty} \|\mathcal{Q}^m f\|_\Phi^{1/m} \ge \sup_{\mathbf{x}\in\mathrm{sp}(f)} |1/Q(\mathbf{x})|. \tag{1}$$

Before giving the proof of above theorem, we recall the following result in [7].

Lemma 3. *Let Φ be an arbitrary Young function, Q be a polynomial, $f \in L^\Phi(\mathbb{R}^n)$ and sp(f) be compact. Then*

$$\lim_{m\to\infty} \|Q^m(D)f\|_\Phi^{1/m} = \sup\{|Q(\mathbf{x})| : \mathbf{x} \in \mathrm{sp}(f)\}. \tag{2}$$

Proof of Theorem 2. Given $\varrho \in \mathrm{sp}(f)$. Then for any $\varepsilon > 0$ there exists $\psi \in C_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}\,\psi \subset B(\varrho, \varepsilon)$ such that $\langle \widehat{f}, \psi\rangle \neq 0$. Put $\mathcal{H}_m := \mathcal{F}(\psi(\mathbf{x})Q^m(\mathbf{x}))$. Clearly, $\mathcal{H}_m \in \mathcal{S}(\mathbb{R}^n)$, and

$$\langle \widehat{f}, \psi\rangle = \langle f, \mathcal{F}\psi\rangle = \langle Q^m(D)\mathcal{Q}^m f, \mathcal{F}\psi\rangle = \langle \mathcal{Q}^m f, Q^m(-D)\mathcal{F}\psi\rangle = \langle \mathcal{Q}^m f, \mathcal{H}_m\rangle.$$

Using Lemma 1, we get

$$\|\mathcal{Q}^m f\|_\Phi \|\mathcal{H}_m\|_{(\overline{\Phi})} \ge |\langle \mathcal{Q}^m f, \mathcal{H}_m\rangle| = |\langle \widehat{f}, \psi\rangle| > 0.$$

Hence,

$$\liminf_{m\to\infty} \|\mathcal{Q}^m f\|_\Phi^{1/m} \ge 1/\limsup_{m\to\infty} \|\mathcal{H}_m\|_{(\overline{\Phi})}^{1/m}. \tag{3}$$

From Lemma 3, we deduce

$$\limsup_{m\to\infty} \|\mathcal{H}_m\|_{(\Phi)}^{1/m} \le \sup_{\mathbf{x}\in B(\varrho,\varepsilon)} |Q(\mathbf{x})|.$$

Therefore, since (3),

$$\liminf_{m\to\infty} \|\mathcal{Q}^m f\|_{\Phi}^{1/m} \ge 1/\sup_{\mathbf{x}\in B(\varrho,\varepsilon)} |Q(\mathbf{x})|.$$

Letting $\varepsilon \to 0$, we get

$$\liminf_{m\to\infty} \|\mathcal{Q}^m f\|_{\Phi}^{1/m} \ge 1/|Q(\varrho)|. \tag{4}$$

Because (4) holds for any $\varrho \in \mathrm{sp}(f)$, we confirm (1), which completes the proof. □

We have the following theorem for entire functions of exponential type.

Theorem 3. *Let Φ be an arbitrary Young function, Q be a polynomial and $(\mathcal{Q}^m f)_{m=0}^{\infty} \subset L^{\Phi}(\mathbb{R}^n)$ satisfies $\mathcal{Q}^0 f = f$, $Q(D)\mathcal{Q}^{m+1}f = \mathcal{Q}^m f$ for $m \in \mathbb{Z}_+$. Assume that $\mathrm{sp}(f)$ is compact and $\mathrm{sp}(\mathcal{Q}^m f) = \mathrm{sp}(f)$ for all $m \in \mathbb{Z}_+$. Then*

$$\lim_{m\to\infty} \|\mathcal{Q}^m f\|_{\Phi}^{1/m} = \sup_{\mathbf{x}\in \mathrm{sp}(f)} |1/Q(\mathbf{x})|.$$

Proof. We divide the proof into two cases.

Case 1 ($Q(\mathbf{x}) \neq 0\ \forall\, \mathbf{x} \in \mathrm{sp}(f)$). Put $K := \mathrm{sp}(f)$. Now we prove

$$\limsup_{m\to\infty} \|\mathcal{Q}^m f\|_{\Phi}^{1/m} \le \sup_{\mathbf{x}\in K} |1/Q(\mathbf{x})|. \tag{5}$$

Indeed, by virtue of $Q(\mathbf{x}) \neq 0\ \forall\, \mathbf{x} \in K$ there exists a small number $\varepsilon > 0$ such that $Q(\mathbf{x}) \neq 0$ $\forall\, \mathbf{x} \in K_\varepsilon$. We choose a function $\mathcal{J} \in C_0^\infty(\mathbb{R}^n)$: $\mathcal{J}(\mathbf{x}) = 1$ if $\mathbf{x} \in K_{\varepsilon/2}$ and $\mathcal{J}(\mathbf{x}) = 0$ if $\mathbf{x} \notin K_\varepsilon$. Then $\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$. Because of $Q^m(D)\mathcal{Q}^m f = f$ and $\mathrm{sp}(\mathcal{Q}^m f) = \mathrm{sp}(f)$, we have $\widehat{f}\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}) = \widehat{\mathcal{Q}^m f}$. Consequently, $\mathcal{Q}^m f = (2\pi)^{-n/2} f * \mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))$.

Hence, using Lemma 2, we obtain

$$\|\mathcal{Q}^m f\|_\Phi \le (2\pi)^{-n/2}\|\mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))\|_1 \|f\|_\Phi = (2\pi)^{-n/2}\|\mathcal{F}(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))\|_1 |f\|_\Phi.$$

So,

$$\limsup_{m\to\infty} \|\mathcal{Q}^m f\|_{\Phi}^{1/m} \le \limsup_{m\to\infty} \|\mathcal{F}(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))\|_1^{1/m}. \tag{6}$$

We define the function $\mathcal{G}_m$ as follows $\mathcal{G}_m = \mathcal{F}(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))$. Then for $\sigma \in \mathbb{Z}_+^n, \sigma \le (2, 2, \ldots, 2)$ we have the following estimate

$$\begin{aligned}
\sup_{\mathbf{y}\in\mathbb{R}^n} |\mathbf{y}^\sigma \mathcal{G}_m(\mathbf{y})| &\le (2\pi)^{-n/2} \sup_{\mathbf{y}\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-i\mathbf{y}\mathbf{x}} D^\sigma(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))\, d\mathbf{x} \right| \\
&= (2\pi)^{-n/2} \sup_{\mathbf{y}\in\mathbb{R}^n} \left| \int_{K_\varepsilon} e^{-i\mathbf{y}\mathbf{x}} D^\sigma(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))\, d\mathbf{x} \right| \\
&\le (2\pi)^{-n/2} \int_{K_\varepsilon} |D^\sigma(\mathcal{J}(\mathbf{x})/Q^m(\mathbf{x}))| d\mathbf{x},
\end{aligned}$$

which together with Leibniz's rule imply

$$
\begin{aligned}
\sup_{\mathbf{y}\in\mathbb{R}^n} |\mathbf{y}^\sigma \mathcal{G}_m(\mathbf{y})| &\le (2\pi)^{-n/2} \int_{K_\varepsilon} \left| \sum_{\gamma\le\sigma} \frac{\sigma!}{\gamma!(\sigma-\gamma)!} D^\gamma \mathcal{J}(\mathbf{x}) D^{\sigma-\gamma}(1/Q^m(\mathbf{x})) \right| d\mathbf{x} \\
&\le (2\pi)^{-n/2} \sum_{\gamma\le\sigma} \left(\frac{\sigma!}{\gamma!(\sigma-\gamma)!} \sup_{\mathbf{y}\in K_\varepsilon} |D^{\sigma-\gamma}(1/Q^m(\mathbf{y}))| \int_{K_\varepsilon} |D^\gamma \mathcal{J}(\mathbf{x})|\, d\mathbf{x} \right) \\
&\le (2\pi)^{-n/2} \max_{\tau\le(2,2,\dots,2)} \sup_{\mathbf{y}\in K_\varepsilon} |D^\tau(1/Q^m(\mathbf{y}))| \sum_{\gamma\le\sigma} \left(\frac{\sigma!}{\gamma!(\sigma-\gamma)!} \int_{K_\varepsilon} |D^\gamma \mathcal{J}(\mathbf{x})|\, d\mathbf{x} \right).
\end{aligned}
\tag{7}
$$

From

$$
D^\tau\big(1/Q^m(\mathbf{y})\big) = \sum_{\substack{\kappa\in\mathbb{Z}_+,(\kappa_j)_{j=1}^n\subset\mathbb{Z}_+^n,\\ \kappa\le 2n,\kappa_j\le(2,2,\dots,2)}} c_{\kappa,(\kappa_j)_{j=1}^n} (m-1+\kappa)! \Big(\prod_{j=1}^n D^{\kappa_j} Q(\mathbf{y}) \Big) / \big((m-1)! Q^{m+\kappa}(\mathbf{y})\big)
$$

and $\inf\{|Q(\mathbf{y})| : \mathbf{y}\in K_\varepsilon\} > 0$, there is a constant $C<\infty$ independent of m such that

$$
\sup_{\mathbf{y}\in K_\varepsilon} |D^\tau(1/Q^m(\mathbf{y}))| \le C m^{2n} \sup_{\mathbf{y}\in K_\varepsilon} |1/Q^m(\mathbf{y})|, \quad \forall\, \tau\in\mathbb{Z}_+^n, \quad \tau\le(2,2,\dots,2). \tag{8}
$$

Combining (7) and (8), we have

$$
\begin{aligned}
\sup_{\mathbf{y}\in\mathbb{R}^n} |\mathbf{y}^\sigma \mathcal{G}_m(\mathbf{y})| &\le (2\pi)^{-n/2} C m^{2n} \sup_{\mathbf{y}\in K_\varepsilon} |1/Q^m(\mathbf{y})| \sum_{\gamma\le\sigma} \left(\frac{\sigma!}{\gamma!(\sigma-\gamma)!} \int_{K_\varepsilon} |D^\gamma \mathcal{J}(\mathbf{x})|\, d\mathbf{x} \right) \\
&= C_1 m^{2n} \sup_{\mathbf{y}\in K_\varepsilon} |1/Q(\mathbf{y})|^m,
\end{aligned}
\tag{9}
$$

where

$$
C_1 := C(2\pi)^{-n/2} \sum_{\gamma\le\sigma} \left(\frac{\sigma!}{\gamma!(\sigma-\gamma)!} \int_{K_\varepsilon} |D^\gamma \mathcal{J}(\mathbf{x})|\, d\mathbf{x} \right).
$$

Clearly, C_1 is independent of m. Then it follows from (9) and

$$
\begin{aligned}
\|\mathcal{G}_m\|_1 &\le \left(\sup_{\mathbf{y}\in\mathbb{R}^n} |(1+y_1^2)(1+y_2^2)\dots(1+y_n^2)\mathcal{G}_m(\mathbf{y})| \right) \left(\int_{\mathbb{R}^n} \frac{d\mathbf{y}}{(1+y_1^2)(1+y_2^2)\dots(1+y_n^2)} \right) \\
&= \pi^n \sup_{\mathbf{y}\in\mathbb{R}^n} |(1+y_1^2)(1+y_2^2)\dots(1+y_n^2)\mathcal{G}_m(\mathbf{y})|
\end{aligned}
$$

that

$$
\limsup_{m\to\infty} \|\mathcal{G}_m\|_1^{1/m} \le \sup_{\mathbf{y}\in K_\varepsilon} |1/Q(\mathbf{y})|. \tag{10}
$$

From (6) and (10), we obtain $\limsup_{m\to\infty} \|\mathcal{Q}^m f\|_\Phi^{1/m} \le \sup_{\mathbf{y}\in K_\varepsilon} |1/Q(\mathbf{y})|$. Letting $\varepsilon\to 0$, we confirm (5). By Theorem 2 and (5), we get $\lim_{m\to\infty} \|\mathcal{Q}^m f\|_\Phi^{1/m} = \sup_{\mathbf{y}\in \mathrm{sp}(f)} |1/Q(\mathbf{y})|$.

Case 2 ($Q(\mathbf{x}) = 0$ for some $\mathbf{x}\in\mathrm{sp}(f)$). Then it follows from Theorem 2 that

$$
\liminf_{m\to\infty} \|\mathcal{Q}^m f\|_\Phi^{1/m} = \infty.
$$

□

Remark 2. *Note that due to Theorem 1, the assumption* $\mathrm{sp}(\mathcal{Q}^m f) = \mathrm{sp}(f)$ *for all* $m \in \mathbb{Z}_+$ *may be replaced by the following stricter condition:* $Q(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in \mathrm{sp}(f)$.

Remark 3. *In general, it is impossible to calculate all* $\|\mathcal{Q}^m f\|_\Phi$, $m = 1, 2, \ldots$, *while Theorems 2 and 3 give us asymptotic estimations of them by using the following rather easier calculation of* $\sup_{\mathbf{x}\in\mathrm{sp}(f)} |1/Q(\mathbf{x})|$, *which is especially effect if* $f \in \mathcal{F}^{-1}(\mathcal{M})$, *where* $\mathcal{M}$ *is the set of all measurable functions belonging to* $\mathcal{S}'(\mathbb{R}^n)$. *The similar situation occurs with our theorems obtained in the following sections. This shows that our results have the potential to apply to computational science.*

3 Some spectral formulas for functions generated by integral operators

Let Φ be an arbitrary Young function and $f \in L^\Phi(\mathbb{R}^n)$. If $\mathrm{sp}(f) \subset (\mathbb{R}^n, v)$ for some $v > 0$, then it was shown in [10] that there exists exactly one sequence $(I^\alpha f)_{\alpha\in\mathbb{Z}^n_+} \subset L^\Phi(\mathbb{R}^n)$ satisfying $D^\alpha I^{\alpha+\ell} f = I^\ell f$ for all $\alpha, \ell \in \mathbb{Z}^n_+$.

Theorem 4. *Let* Φ *be an arbitrary Young function,* Q *be a polynomial and* $(I^\alpha f)_{\alpha\in\mathbb{Z}^n_+} \subset L^\Phi(\mathbb{R}^n)$ *satisfying* $D^\alpha I^{\alpha+\ell} f = I^\ell f$ *for all* $\alpha, \ell \in \mathbb{Z}^n_+$. *Assume that* $\mathrm{sp}(f) \subset (\mathbb{R}^n, v)$ *for some* $v > 0$. *Then* $\mathrm{sp}(Q^m(I)f) \subset \mathrm{sp}(f)\ \forall\, m \in \mathbb{Z}_+$, *and*

$$\liminf_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} \geq \sup_{\mathbf{x}\in\mathrm{sp}(f)} |Q(1/\mathbf{x})|.$$

Proof. It was shown in [10] that $\mathrm{sp}(I^\alpha f) = \mathrm{sp}(f)\ \forall\, \alpha \in \mathbb{Z}^n_+$. Hence, $\mathrm{sp}(Q^m(I)f) \subset \mathrm{sp}(f)$ $\forall\, m \in \mathbb{Z}_+$. Next, we show that

$$\liminf_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} \geq |Q(1/\varrho)|, \tag{11}$$

where ϱ is an arbitrary element in $\mathrm{sp}(f)$. Indeed, if $Q(1/\varrho) = 0$, then (11) is obvious. If $Q(1/\varrho) \neq 0$, then for a small enough number $\varepsilon \in (0, v/2)$ we have $\prod_{j=1}^n |x_j| > 0$ and $Q(1/\mathbf{x}) \neq 0\ \forall\, \mathbf{x} \in B(\varrho, \varepsilon)$. From $\varrho \in \mathrm{sp}(f)$, there is $\mathcal{J} \in C_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}\mathcal{J} \subset B(\varrho, \varepsilon)$ such that $\langle \widehat{f}, \mathcal{J}\rangle \neq 0$. Put $\mathcal{H}_m = \mathcal{F}(\mathcal{J}(\mathbf{x})/Q^m(1/\mathbf{x}))$. Clearly, $\mathcal{H}_m$ is well defined, $\mathcal{H}_m \in \mathcal{S}(\mathbb{R}^n)$ and it follows from $f = D^\alpha I^\alpha f$ that

$$\begin{aligned}\langle f, \mathcal{F}(\mathcal{J}(\mathbf{x})/((i\mathbf{x})^\alpha Q^m(1/\mathbf{x})))\rangle &= \langle D^\alpha I^\alpha f, \mathcal{F}(\mathcal{J}(\mathbf{x})/((i\mathbf{x})^\alpha Q^m(1/\mathbf{x})))\rangle \\ &= (-1)^{|\alpha|}\langle I^\alpha f, D^\alpha \mathcal{F}(\mathcal{J}(\mathbf{x})/((i\mathbf{x})^\alpha Q^m(1/\mathbf{x})))\rangle = \langle I^\alpha f, \mathcal{H}_m\rangle\end{aligned}$$

for all $\alpha \in \mathbb{Z}^n_+$. Then, for $Q^m(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$, we get

$$\begin{aligned}\langle Q^m(I)f, \mathcal{H}_m\rangle &= \sum_\alpha c_\alpha i^{|\alpha|}\langle I^\alpha f, \mathcal{H}_m\rangle = \sum_\alpha c_\alpha i^{|\alpha|}\langle f, \mathcal{F}(\mathcal{J}(\mathbf{x})/((i\mathbf{x})^\alpha Q^m(1/\mathbf{x})))\rangle \\ &= \langle f, \mathcal{F}(\mathcal{J}(\mathbf{x})\sum_\alpha c_\alpha (1/\mathbf{x})^\alpha/(Q^m(1/\mathbf{x})))\rangle = \langle f, \widehat{\mathcal{J}}\rangle = \langle \widehat{f}, \mathcal{J}\rangle.\end{aligned}$$

So, using Lemma 1, we have

$$0 < |\langle \widehat{f}, \mathcal{J}\rangle| = |\langle Q^m(I)f, \mathcal{H}_m\rangle| \leq \|Q^m(I)f\|_\Phi\, \|\mathcal{H}_m\|_{(\overline{\Phi})}.$$

It follows that

$$\liminf_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} \geq 1/\limsup_{m\to\infty} \|\mathcal{H}_m\|_{(\overline{\Phi})}^{1/m}. \tag{12}$$

Arguing as in the proof of Theorem 3 and taking account of $B(\varrho, \varepsilon) \subset (\mathbb{R}^n, v/2)$, we have a constant $A < \infty$ independent of m, σ such that

$$\sup_{\mathbf{z}\in\mathbb{R}^n} |\mathbf{z}^\sigma \mathcal{H}_m(\mathbf{z})| \leq Am^{2n} \sup_{\mathbf{z}\in B(\varrho,\varepsilon)} |1/Q^m(1/\mathbf{z})|,$$

for all $\sigma \in \mathbb{Z}^n_+$, $\sigma \leq (2, 2, \ldots, 2)$. Consequently,

$$\sup_{\mathbf{z}\in\mathbb{R}^n} |(1+z_1^2)(1+z_2^2)\ldots(1+z_n^2)\mathcal{H}_m(\mathbf{z})| \leq A2^n m^{2n} \sup_{\mathbf{z}\in B(\varrho,\varepsilon)} |1/Q^m(1/\mathbf{z})|. \tag{13}$$

Moreover,

$$\|\mathcal{H}_m\|_{(\overline{\Phi})} \leq \|\mathrm{Y}\|_{(\overline{\Phi})} \sup_{\mathbf{z}\in\mathbb{R}^n} |(1+z_1^2)(1+z_2^2)\ldots(1+z_n^2)\mathcal{H}_m(\mathbf{z})|, \tag{14}$$

where

$$\mathrm{Y}(\mathbf{z}) = \frac{1}{(1+z_1^2)(1+z_2^2)\ldots(1+z_n^2)}.$$

We choose $\kappa > 0$ such that $\overline{\Phi}(\kappa) < \infty$. Since $\overline{\Phi}$ is a Young function, $\overline{\Phi}(x)/x$ is increasing on $[0, +\infty)$. Then $\overline{\Phi}(x) \leq \kappa_1 x$ for all $x \in [0, \kappa]$, where $\kappa_1 = \overline{\Phi}(\kappa)/\kappa$. Hence,

$$\int_{\mathbb{R}^n} \overline{\Phi}\Big(\frac{1}{\lambda(\prod_{j=1}^n (1+z_j^2))}\Big)\, d\mathbf{z} \leq \int_{\mathbb{R}^n} \frac{\kappa_1}{\lambda(\prod_{j=1}^n (1+z_j^2))}\, d\mathbf{z} = \frac{\kappa_1 \pi^n}{\lambda} < 1$$

for all $\lambda > \max\{1/\kappa, \kappa_1\pi^n\}$. Consequently, $\|\mathrm{Y}\|_{(\overline{\Phi})} \leq \max\{1/\kappa, \kappa_1\pi^n\} < \infty$, which together with (13) and (14) imply

$$\limsup_{m\to\infty} \|\mathcal{H}_m\|^{1/m}_{(\overline{\Phi})} \leq \sup_{\mathbf{z}\in B(\varrho,\varepsilon)} 1/|Q(1/\mathbf{z})|.$$

Therefore, since (12), $\liminf_{m\to\infty} \|Q^m(I)f\|^{1/m}_\Phi \geq \inf_{\mathbf{z}\in B(\varrho,\varepsilon)} |Q(1/\mathbf{z})|$. Letting $\varepsilon \to 0$, we confirm (11). Because (11) holds for any $\varrho \in \mathrm{sp}(f)$, we obtain

$$\liminf_{m\to\infty} \|Q^m(I)f\|^{1/m}_\Phi \geq \sup_{\mathbf{x}\in\mathrm{sp}(f)} |Q(1/\mathbf{x})|.$$

□

We have the following theorem for entire functions of exponential type.

Theorem 5. *Let Φ be an arbitrary Young function, Q be a polynomial and $(I^\alpha f)_{\alpha\in\mathbb{Z}^n_+} \subset L^\Phi(\mathbb{R}^n)$ satisfying $D^\alpha I^{\alpha+\sigma} f = I^\sigma f$ for all $\alpha, \sigma \in \mathbb{Z}^n_+$. Assume that $\mathrm{sp}(f)$ is compact and $\mathrm{sp}(f) \subset (\mathbb{R}^n, v)$ for some $v > 0$. Then*

$$\lim_{m\to\infty} \|Q^m(I)f\|^{1/m}_\Phi = \sup_{\mathbf{x}\in\mathrm{sp}(f)} |Q(1/\mathbf{x})|.$$

Proof. We put $K = \mathrm{sp}(f)$ and consider $\varepsilon \in (0, v)$. Then there exists $\mathcal{J} \in C_0^\infty(\mathbb{R}^n)$ such that $\mathcal{J}(\mathbf{x}) = 1$ if $\mathbf{x} \in K_{\varepsilon/2}$ and $\mathcal{J}(\mathbf{x}) = 0$ if $\mathbf{x} \notin K_\varepsilon$. So, $\mathrm{supp}\mathcal{J} \subset (\mathbb{R}^n, v-\varepsilon)$. From $\widehat{f} = (i\mathbf{x})^\alpha \widehat{I^\alpha f}$, we get $\mathcal{J}(\mathbf{x})\widehat{f} = (i\mathbf{x})^\alpha \widehat{I^\alpha f}$ and then it follows from $\mathrm{sp}(I^\alpha f) = \mathrm{sp}(f)$ that $\widehat{f}\mathcal{J}(\mathbf{x})/(i\mathbf{x})^\alpha = \widehat{I^\alpha f}$ for all $\alpha \in \mathbb{Z}^n_+$. Therefore, $\widehat{Q^m(I)f} = \widehat{f}\mathcal{J}(\mathbf{x})Q^m(1/\mathbf{x})$ for all $\forall m \in \mathbb{Z}_+$. So,

$$Q^m(I)f = (2\pi)^{-n/2} f * \mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})Q^m(1/\mathbf{x})) \quad \forall m \in \mathbb{Z}_+.$$

Then, using Lemma 2, we get

$$\|Q^m(I)f\|_\Phi \le (2\pi)^{-n/2}\|f\|_\Phi \|\mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})Q^m(1/\mathbf{x}))\|_1 \quad \forall m \in \mathbb{Z}_+.$$

Thus, $\limsup_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} \le \limsup_{m\to\infty} \|\mathcal{G}_m\|_1^{1/m}$, where $\mathcal{G}_m := \mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})Q^m(1/\mathbf{x}))$. Then it follows from $\|\mathcal{G}_m\|_1 \le \pi^n \sup_{\mathbf{z}\in\mathbb{R}^n} |(1+z_1^2)(1+z_2^2)\dots(1+z_n^2)\mathcal{G}_m(\mathbf{z})|$ that

$$\limsup_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} \le \limsup_{m\to\infty} \sup_{\mathbf{z}\in\mathbb{R}^n} |(1+z_1^2)(1+z_2^2)\dots(1+z_n^2)\mathcal{G}_m(\mathbf{z})|^{1/m}. \tag{15}$$

Arguing as in the proof of Theorem 3, we have a constant $B < \infty$ independent of m, σ such that

$$\sup_{\mathbf{z}\in\mathbb{R}^n} |\mathbf{z}^\sigma \mathcal{G}_m(\mathbf{z})| \le Bm^{2n} \sup_{\mathbf{z}\in K_\varepsilon} |Q^m(1/\mathbf{z})|, \tag{16}$$

for all $\sigma \in \mathbb{Z}_+^n$, $\sigma \le (2,2,\dots,2)$. Combining (15) and (16), we obtain

$$\limsup_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} \le \sup_{\mathbf{z}\in K_\varepsilon} |Q(1/\mathbf{z})|.$$

Letting $\varepsilon \to 0$, we deduce

$$\limsup_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} \le \sup_{\mathbf{x}\in K} |Q(1/\mathbf{x})|. \tag{17}$$

Combining (17) and Theorem 4, we arrive

$$\lim_{m\to\infty} \|Q^m(I)f\|_\Phi^{1/m} = \sup_{\mathbf{x}\in \mathrm{sp}(f)} |Q(1/\mathbf{x})|.$$

□

4 Some spectral formulas for functions generated by differential operators

If $f \in \mathcal{S}'(\mathbb{R}^n)$ has compact spectrum, then f is the Fourier transform of $v := \mathcal{F}^{-1}f$. The Fourier-Laplace transform of v (see [32]), still denoted by the same symbol f, is known as follows $f(\zeta) = (2\pi)^{-n/2}\langle v(\cdot), e^{-i\zeta\cdot}\rangle$, $\zeta \in \mathbb{C}^n$. This is an entire function on $\mathbb{C}^n$. Hence, for $f \in L^\Phi(\mathbb{R}^n)$ and $\mathbf{a} \in \mathbb{R}^n$ (or $f \in \mathcal{F}(\mathcal{E}'(\mathbb{R}^n))$ and $\mathbf{a} \in \mathbb{C}^n$) we can define a function $f_\mathbf{a} \in \mathcal{S}'(\mathbb{R}^n)$ as follows $f_\mathbf{a}(\mathbf{x}) = f(\mathbf{x}+\mathbf{a})$, $\mathbf{x} \in \mathbb{R}^n$. Let P, Q be polynomials. Denote by $\top(P,Q,\mathbf{a})f(\mathbf{x}) = P(D)f(\mathbf{x}) + Q(D)f_\mathbf{a}$ and $\top^{m+1}(P,Q,\mathbf{a})f(\mathbf{x}) = \top(P,Q,\mathbf{a})(\top^m(P,Q,\mathbf{a})f)(\mathbf{x})$ for $m \in \mathbb{Z}_+$. Now we have the following result.

Theorem 6. *Let Φ be an arbitrary Young function, P and Q be polynomials and $\mathbf{a} \in \mathbb{R}^n$, $f \in L^\Phi(\mathbb{R}^n)$ (or $\mathbf{a} \in \mathbb{C}^n$, $f \in L^\Phi(\mathbb{R}^n) \cap \mathcal{F}(\mathcal{E}'(\mathbb{R}^n))$). Then $\mathrm{sp}(\top^m(P,Q,\mathbf{a})f) \subset \mathrm{sp}(f)$ $\forall m \in \mathbb{N}$ and*

$$\liminf_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \ge \sup\{|P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}}| : \mathbf{x} \in \mathrm{sp}(f)\}. \tag{18}$$

Proof. It is easy to see $\mathrm{sp}(\top^m(P,Q,\mathbf{a})f) \subset \mathrm{sp}(f)$ $\forall m \in \mathbb{Z}_+$ from the fact that $\mathrm{sp}(D^\alpha f) \subset \mathrm{sp}(f)$ and $\mathrm{sp}(D^\alpha f(.+\mathbf{a})) \subset \mathrm{sp}(f)$ for all $\alpha \in \mathbb{Z}_+^n$. Now we prove (18). By the definition of $\top^m(P,Q,\mathbf{a})f$, one has $\mathcal{F}(\top^m(P,Q,\mathbf{a})f) = (P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})\mathcal{F}(\top^{m-1}(P,Q,\mathbf{a})f)$ and then

$$\mathcal{F}(\top^m(P,Q,\mathbf{a})f) = (P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m \widehat{f} \quad \forall m \in \mathbb{N}. \tag{19}$$

We consider $\varrho \in \text{sp}(f)$ satisfying $|P(\varrho) + Q(\varrho)e^{i\mathbf{a}\varrho}| > 0$. Then, for sufficiently small $\varepsilon > 0$, we obtain $\inf\{|P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}}| : \mathbf{x} \in B(\varrho, \varepsilon)\} > 0$ and there is a function $\psi \in C^\infty(\mathbb{R}^n)$, $\text{supp}\psi \subset B(\varrho, \varepsilon)$ such that $\langle \widehat{f}, \psi \rangle \neq 0$. We define the function $\mathcal{G}_m$ as follows $\mathcal{G}_m = \mathcal{F}(\psi(\mathbf{x})/(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m)$. Then $\mathcal{G}_m$ is well defined and by (19), we get

$$\begin{aligned}\langle \top^m(P,Q,\mathbf{a})f, \mathcal{G}_m\rangle &= \langle \widehat{\top^m(P,Q,\mathbf{a})f}, \mathcal{F}^{-1}(\mathcal{G}_m)\rangle = \langle (P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m \widehat{f}, \mathcal{F}^{-1}(\mathcal{G}_m)\rangle \\ &= \langle \widehat{f}, (P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m \mathcal{F}^{-1}(\mathcal{G}_m)\rangle = \langle \widehat{f}, \psi\rangle\end{aligned}$$

and apply Lemma 1 to conclude that

$$|\langle \widehat{f}, \psi\rangle| = |\langle \top^m(P,Q,\mathbf{a})f, \mathcal{G}_m\rangle| \leq \|\top^m(P,Q,\mathbf{a})f\|_\Phi \|\mathcal{G}_m\|_{(\overline{\Phi})}.$$

Therefore, it follows from $\langle \widehat{f}, \psi\rangle \neq 0$ that

$$\liminf_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \geq 1/\limsup_{m\to\infty} \|\mathcal{G}_m\|_{(\overline{\Phi})}^{1/m}. \tag{20}$$

By the same argument as in the proof of Theorem 2 we get

$$\limsup_{m\to\infty} \|\mathcal{G}_m\|_{(\overline{\Phi})}^{1/m} \leq \sup_{\mathbf{x}\in B(\varrho,\varepsilon)} |(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^{-1}|$$

and then it follows from (20) that

$$\liminf_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \geq 1/\sup_{\mathbf{x}\in B(\varrho,\varepsilon)} |(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^{-1}|.$$

Letting $\varepsilon \to 0$, we obtain $\liminf_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \geq |P(\varrho) + Q(\varrho)e^{i\mathbf{a}\varrho}|$ and then

$$\liminf_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \geq \sup\{|P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}}| : \mathbf{x} \in \text{sp}(f)\}.$$

□

We have the following generalization of (2).

Theorem 7. *Let Φ be an arbitrary Young function, $\mathbf{a} \in \mathbb{C}^n$, P, Q be polynomials and $f \in L^\Phi(\mathbb{R}^n)$. Assume that* $\text{sp}(f)$ *is compact. Then $\top^m(P,Q,\mathbf{a})f \in L^\Phi(\mathbb{R}^n)$ for all m, and*

$$\lim_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} = \sup\{|P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}}| : \mathbf{x} \in \text{sp}(f)\}.$$

Proof. Put $K = \text{sp}(f)$. For any $\varepsilon > 0$ we choose a function $\mathcal{J} \in C^\infty(\mathbb{R}^n)$ satisfying $\mathcal{J}(\mathbf{x}) = 0$ $\forall\, \mathbf{x} \notin K_\varepsilon$, and $\mathcal{J}(\mathbf{x}) = 1\ \forall\, \mathbf{x} \in K_{\varepsilon/2}$. Then it follows from (19) that $\widehat{\top^m(P,Q,\mathbf{a})f} = \mathcal{J}(\mathbf{x})(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m \widehat{f}$ and then

$$\top^m(P,Q,\mathbf{a})f = (2\pi)^{-n/2} f * \mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m).$$

Hence, using Lemma 2, we deduce that $\top^m(P,Q,\mathbf{a})f \in L^\Phi(\mathbb{R}^n)$ and

$$\|\top^m(P,Q,\mathbf{a})f\|_\Phi \leq (2\pi)^{-n/2} \|f\|_\Phi \, ||\mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m)||_1.$$

Therefore,

$$\limsup_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \leq \limsup_{m\to\infty} \|\mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{a}\mathbf{x}})^m)\|_1^{1/m}. \tag{21}$$

Similarly as in the proof of Theorem 3, we get

$$\limsup_{m\to\infty} \|\mathcal{F}^{-1}(\mathcal{J}(\mathbf{x})(P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}})^m)\|_1^{1/m} \le \sup\{|P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| : \mathbf{x}\in K_\varepsilon\}.$$

Then it follows from (21) that $\limsup_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \le \sup\{|P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| : \mathbf{x}\in K_\varepsilon\}$. Letting $\varepsilon\to 0$ with the note that K is compact, we obtain

$$\limsup_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \le \sup\{|P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| : \mathbf{x}\in \mathrm{sp}(f)\}.$$

From this and (18), we arrive

$$\lim_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} = \sup\{|P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| : \mathbf{x}\in \mathrm{sp}(f)\}.$$

□

Theorem 8. *Let Φ be an arbitrary Young function, $\mathbf{a}\in\mathbb{C}^n$, P,Q be polynomials, $f\in L^\Phi(\mathbb{R}^n)$ and $\Omega := \{\mathbf{x}\in\mathbb{R}^n : |P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| \le 1\}$. Assume that Ω is the compact set. Then $\mathrm{sp}(f)\subset\Omega$ if and only if $\top^m(P,Q,\mathbf{a})f\in L^\Phi(\mathbb{R}^n)$ for all $m\in\mathbb{Z}_+$ and*

$$\liminf_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \le 1. \tag{22}$$

Proof. Necessary. Assume that $\mathrm{sp}(f)\subset\Omega$. Hence, $\mathrm{sp}(f)$ is also compact. Then it follows from Theorem 7 that

$$\lim_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} = \sup\{|P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| : \mathbf{x}\in \mathrm{sp}(f)\}.$$

This implies, by $\mathrm{sp}(f)\subset\Omega$, that $\lim_{m\to\infty} \|\top^m(P,Q,\mathbf{a})f\|_\Phi^{1/m} \le 1$.

Sufficiency. Assume that (22) holds. Then it follows from Theorem 7 that $\sup\{|P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| : \mathbf{x}\in\mathrm{sp}(f)\}\le 1$ and then $\mathrm{sp}(f)\subset\Omega$. □

Theorem 9. *Let Φ be an arbitrary Young function and K be an arbitrary compact set in $\mathbb{R}^n$. Then for any $\tau>0$ there exists a constant $C_{\tau,K}<\infty$ independent of Φ such that*

$$\|\top(P,Q,\mathbf{a})f\|_\Phi \le C_{\tau,K}\|f\|_\Phi \sup_{\mathbf{x}\in K_{(\tau)}} |P(\mathbf{x})+Q(\mathbf{x})e^{i\mathbf{ax}}| \tag{23}$$

for all $\mathbf{a}\in\mathbb{C}^n$, $f\in\mathcal{E}_\Phi(K)$ and for all polynomials $P(\mathbf{x}),Q(\mathbf{x})$, where $\mathcal{E}_\Phi(K)=\{f\in L^\Phi(\mathbb{R}^n) : \mathrm{sp}(f)\subset K\}$.

Proof. Necessity. We choose a function $\mathcal{A}\in C_0^\infty(\mathbb{R}^n)$ such that $\mathcal{A}(\mathbf{z})=1$ if $\mathbf{z}\in K_{\tau/4}$ and $\mathcal{A}(\mathbf{z})=0$ if $\mathbf{z}\notin K_{\tau/2}$. It follows from $\mathcal{F}(\top(P,Q,\mathbf{a})f)=(P(\mathbf{z})+Q(\mathbf{z})e^{i\mathbf{az}})\widehat{f}$ and $\mathrm{sp}(f)\subset K$ that $\mathcal{F}(\top(P,Q,\mathbf{a})f)=\mathcal{A}(\mathbf{z})(P(\mathbf{z})+Q(\mathbf{z})e^{i\mathbf{az}})\widehat{f}$, and then

$$\top(P,Q,\mathbf{a})f=(2\pi)^{-n/2}f*\mathcal{F}^{-1}(\mathcal{A}(\mathbf{z})(P(\mathbf{z})+Q(\mathbf{z})e^{i\mathbf{az}})).$$

Therefore, by Lemma 2 we have

$$\|\top(P,Q,\mathbf{a})f\|_\Phi \le (2\pi)^{-n/2}\|f\|_\Phi\|\mathcal{F}^{-1}(\mathcal{A}(\mathbf{z})(P(\mathbf{z})+Q(\mathbf{z})e^{i\mathbf{az}}))\|_1=(2\pi)^{-n/2}\|f\|_\Phi\|\mathcal{J}\|_1, \tag{24}$$

where $\mathcal{J}(\mathbf{x}) := (\mathcal{F}^{-1}(\mathcal{A}(\mathbf{z})((P(\mathbf{z}) + Q(\mathbf{z})e^{i\mathbf{az}})))(\mathbf{x})$. Hence, for $\sigma \in \mathbb{Z}_+^n$, $\sigma \le (2, 2, \dots, 2)$ we get the following estimate

$$\begin{aligned}\sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\sigma \mathcal{J}(\mathbf{x})| &= (2\pi)^{-n/2} \sup_{\mathbf{x}\in\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} e^{ix\mathbf{z}} D^\sigma(\mathcal{A}(\mathbf{z})(P(\mathbf{z}) + Q(\mathbf{z})e^{i\mathbf{az}}))\, d\mathbf{z} \Big| \\ &= (2\pi)^{-n/2} \sup_{\mathbf{x}\in\mathbb{R}^n} \Big| \int_{K_{\tau/2}} e^{ix\mathbf{z}} D^\sigma(\mathcal{A}(\mathbf{z})(P(\mathbf{z}) + Q(\mathbf{z})e^{i\mathbf{az}}))\, d\mathbf{z} \Big| \\ &\le (2\pi)^{-n/2} \int_{K_{\tau/2}} |D^\sigma(\mathcal{A}(\mathbf{z})(P(\mathbf{z}) + Q(\mathbf{z})e^{i\mathbf{az}}))|\, d\mathbf{z}.\end{aligned}$$

Then it follows from Leibniz's rule that

$$\begin{aligned}\sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\sigma \mathcal{J}(\mathbf{x})| &\le (2\pi)^{-n/2} \int_{K_{\tau/2}} \Big| \sum_{\gamma\le\sigma} \frac{\sigma!}{\gamma!(\sigma-\gamma)!} D^\gamma \mathcal{A}(\mathbf{z}) D^{\sigma-\gamma}(P(\mathbf{z}) + Q(\mathbf{z})e^{i\mathbf{az}}) \Big|\, d\mathbf{z} \\ &\le (2\pi)^{-n/2} \sum_{\gamma\le\sigma} \left(\frac{\sigma!}{\gamma!(\sigma-\gamma)!} \sup_{\mathbf{x}\in K_{\tau/2}} |D^{\sigma-\gamma}(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}})| \int_{K_{\tau/2}} |D^\gamma \mathcal{A}(\mathbf{z})|\, d\mathbf{z} \right) \\ &\le (2\pi)^{-n/2} \max_{\nu\le(2,2,\dots,2)} \sup_{\mathbf{x}\in K_{\tau/2}} |D^\nu(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}})| \\ &\quad \times \sum_{\gamma\le\sigma} \left(\frac{\sigma!}{\gamma!(\sigma-\gamma)!} \int_{K_{\tau/2}} |D^\gamma \mathcal{A}(\mathbf{z})|\, d\mathbf{z} \right). \end{aligned} \tag{25}$$

Because the derivatives of the analytic function $(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}})$ can be estimated in $K_{\tau/2}$ by the maximum of the modulus in $K_{(\tau)}$, there exists a constant $A_\tau < \infty$ independent of f, $P(\mathbf{x})$, $Q(\mathbf{x})$, $\mathbf{a}$, and Φ such that

$$\sup_{\mathbf{x}\in K_{\tau/2}} |D^\nu(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}})| \le A_\tau \sup_{\mathbf{x}\in K_{(\tau)}} |(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}})| \tag{26}$$

for all $\nu \in \mathbb{Z}_+^n$, $\nu \le (2, 2, \dots, 2)$. From (25) and (26), we have

$$\begin{aligned}\sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\sigma \mathcal{J}(\mathbf{x})| &\le (2\pi)^{-n/2} \sum_{\gamma\le\sigma} \left(\frac{\sigma!}{\gamma!(\sigma-\gamma)!} A_\tau \sup_{\mathbf{x}\in K_{(\tau)}} |(P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}})| \int_{K_{\tau/2}} |D^\gamma \mathcal{A}(\mathbf{z})| d\mathbf{z} \right) \\ &\le (2\pi)^{-n/2} 2^{2n} A_\tau C \sup_{\mathbf{x}\in K_{(\tau)}} |P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}}|, \end{aligned} \tag{27}$$

where

$$C := \max_{\gamma\le(2,2,\dots,2)} \int_{K_{\tau/2}} |D^\gamma \mathcal{A}(\mathbf{z})|\, d\mathbf{z}.$$

Then it follows from (27) and $\|\mathcal{J}\|_1 \le \pi^n \sup_{\mathbf{x}\in\mathbb{R}^n} |(1 + x_1^2)(1 + x_2^2) \dots (1 + x_n^2)\mathcal{J}(\mathbf{x})|$ that

$$\|\mathcal{J}\|_1 \le C_{\tau,K} \sup_{\mathbf{x}\in K_{(\tau)}} |P(\mathbf{x}) + Q(\mathbf{x})e^{i\mathbf{ax}}|, \tag{28}$$

where $C_{\tau,K}$ is independent of $f, P(\mathbf{x}), Q(\mathbf{x}), \mathbf{a}, \Phi$. Combining (24) and (28), we obtain (23). □

Acknowledgments. The authors would like to thank the referee very much for the valuable remarks. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2018.300.

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Received 27.04.2020
Revised 20.02.2021

Банґ Г.Г., Гуй В.Н. *Деякі спектральні формули для функцій, породжених диференціальними та інтегральними операторами в просторах Орліча* // Карпатські матем. публ. — 2021. — Т.13, №2. — С. 326–339.

У цій статті ми досліджуємо поведінку послідовності L^Φ-норм функцій, які породжені диференціальними та інтегральними операторами за допомогою їхнього спектра (носій перетворення Фур'є функції f називають її спектром і позначають $\mathrm{sp}(f)$). Для деякого полінома Q ми вводимо поняття Q-примітивів, яке стає поняттям примітивів, якщо $Q(x) = x$, і вивчаємо поведінку послідовності норм Q-примітивів функцій у просторі Орліча $L^\Phi(\mathbb{R}^n)$. Ми отримали наступний головний результат: *нехай Φ − довільна функція Юнга, $Q(\mathbf{x})$ − поліном та $(\mathcal{Q}^m f)_{m=0}^\infty \subset L^\Phi(\mathbb{R}^n)$ задовольняє $\mathcal{Q}^0 f = f, Q(D)\mathcal{Q}^{m+1} f = \mathcal{Q}^m f$ для $m \in \mathbb{Z}_+$. Припустимо, що $\mathrm{sp}(f)$ є компактом і $\mathrm{sp}(\mathcal{Q}^m f) = \mathrm{sp}(f)$ для всіх $m \in \mathbb{Z}_+$. Тоді*

$$\lim_{m\to\infty} \|\mathcal{Q}^m f\|_\Phi^{1/m} = \sup_{\mathbf{x}\in \mathrm{sp}(f)} \big|1/Q(\mathbf{x})\big|.$$

Подано також відповідні результати для функцій, що породжені диференціальними та інтегральними операторами.

Ключові слова і фрази: простір Орліча, нерівність в апроксимації, перетворення Фур'є, узагальнена функція.