ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2021, **13** (1), 127–133
doi:10.15330/cmp.13.1.127-133

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2021, Т.13, №1, С.127–133

On hereditary irreducibility of some monomial matrices over local rings

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We consider monomial matrices over a commutative local principal ideal ring R of type $M(t,k,n) = \Phi \begin{pmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{pmatrix}$, $0 < k < n$, where t is a generating element of Jacobson radical $J(R)$ of R, Φ is the companion matrix to $\lambda^n - 1$ and I_k is the identity $k \times k$ matrix. In this paper, we indicate a criterion of the hereditary irreducibility of $M(t,k,n)$ in the case $t^{\left[\frac{k\cdot(n-k)}{n}\right]+1} \neq 0$.

Key words and phrases: local ring, Jacobson radical, irreducible matrix, monomial matrix, hereditary irreducible matrix.

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Introduction

We say that $n \times n$ matrices A and B over a commutative ring R with identity are similar over R if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. It is well known [10, p. 238] that two square matrices over a field are similar if and only if their canonical rational forms are equal. The problem of classifying, up to similarity, all matrices over a commutative ring (which is not a field) is usually very difficult; in most cases it is "unsolvable" (wild), as in the case of rings of residue classes [3]. It has been solved only for square matrices of small degree over some principle ideal rings (for example, see [2,11,12]). The ring of rational integers is one of the most important cases. Let $\mathbb{Z}$, $\mathbb{Z}_m$ and I_p be the ring of integers, ring of residues modulo $m \geq 2$ and the ring of p-adic integers, respectively. Let A, B be $n \times n$ matrices over the ring $\mathbb{Z}$. Denote their images under the reduction homomorphism modulo m by A_m and B_m, respectively. It is well known (see [13]) that the similarity (over $\mathbb{Z}_m$) of A_m and B_m for all $m \geq 2$ not implies the similarity of A and B over $\mathbb{Z}$. However, H. Applegate and H. Onishi [1] proved that $n \times n$ matrices A and B over I_p are similar over I_p if and only if A_{p^r}, B_{p^r} are similar over $\mathbb{Z}_{p^r}$ for all $r \geq 1$. In such situation, an important place is occupied by matrices over commutative local principle ideals rings (like $\mathbb{Z}_{p^r}$).

The knowledge of all, up to similarity, irreducible matrices of any degree over a commutative ring with identity is also still far from complete. It is well known that if characteristic polynomial of a square matrix over a commutative ring with identity is irreducible, then the

УДК 512.643.8
2020 *Mathematics Subject Classification:* 15B33, 15A30.

The paper was written during the research stay of the first author at the University of Presov under the National Scholarship Programme of the Slovak Republic. The second author acknowledges the support of the Slovak VEGA Grant № 1/0097/18

matrix is irreducible. Converse is true for fields [10, p. 243, Ex. 20] but, in general, it is not true for commutative rings. This paper is devoted to one class of square monomial matrices of any size over commutative rings, which first arose in studying indecomposable representations of finite p-groups over commutative local rings [9].They were studied more extensively (and more generally) in [4–6].

Let R be a commutative ring with Jacobson radical $J(R) \neq 0$ and t be a non-zero element from $J(R)$. Consider an $n \times n$ matrix over R of the following form

$$M(t,k,n) := \Phi_n \begin{pmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & t \\ 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \end{pmatrix},$$

where $0 < k < n$, Φ_n is the companion matrix to the polynomial $x^n - 1$ and I_s is the identity $s \times s$ matrix. Let (n,k) denotes the greatest common divisor of n and k. In [7] it was shown that if $(n,k) > 1$, then for any positive divisor $d > 1$ of the number (n,k) the matrix $M(t,k,n)$ is similar over R to a matrix of the following form $\left(\begin{smallmatrix} M(t,k',n') & B \\ 0 & A \end{smallmatrix}\right)$, where $k' = \frac{k}{d}$ and $n' = \frac{n}{d}$.

The matrix $M(t,k,n)$ is said to be *hereditary reducible* if it is similar to a matrix

$$\begin{pmatrix} M(t,k',n') & B \\ 0 & A \end{pmatrix}, \quad 0 \le k' \le n',\ 0 < n' < n,$$

and *hereditary irreducible* if otherwise.

1 On irreduciblity of $M(t,k,n)$ over discrete valuation domain

Let R be a discrete valuation domain. This mean that R is a local principal ideal domain, which are not a field. A nonconstant polynomial $f(x)$ over R is said to be reducible over R if it can be written as a product of two nonconstant polynomials over R, otherwise $f(x)$ is called irreducible over R. One of the oldest sufficient condition of irreducibility for polynomials with coefficients in a discrete valuation domain was given by G. Dumas [8].

Theorem 1. *Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ be a polynomial over a discrete valuation domain R with valued field $(F;v)$. If the following conditions are fulfilled*

1) $v(a_0) = 0$,

2) $\frac{v(a_n)}{n} < \frac{v(a_i)}{i}$, $i = 1, \dots, n-1$,

3) $(v(a_n);n) = 1$,

then the polynomial $f(x)$ is irreducible over F (and also over R).

In particular, if t is a generator element of $J(R)$ and k is a positive integer relatively prime to n, then $f(x) = x^n - t^k$ is irreducible over R. Obviously, $(-1)^n f(x)$ is the characteristic polynomial $|M(t,k,n) - xI_n|$ of the matrix $M(t,k,n)$.

Theorem 2. *Let n and k be positive integers, $k < n$. Let R be a discrete valuation domain, t be a generator element of $J(R)$. The matrix $M(t,k,n)$ is reducible (over R) if and only if $(n,k) > 1$.*

Proof. Sufficiency follows from [7, p. 2, Thm. 1]. Assume now that $(n,k) = 1$ and the matrix $M(t,k,n)$ is reducible. Then $M(t,k,n)$ is similar to a matrix $\left(\begin{smallmatrix} C & B \\ 0 & A \end{smallmatrix}\right)$ for some $n' \times n'$ matrix C, $0 < n' < n$. Then the characteristic polynomial $(-1)^n(x^n - t^k)$ of the matrix $M(t,k,n)$ is reducible, which is impossible. □

2 On hereditary irreduciblity of $M(t,k,n)$ over commutative local principal ideal rings

Now we will assume that R is a commutative local principal ideal ring (not necessary domain), which is not a field.

Theorem 3. *Let n and k be positive integers, $k < n$. Let R be a commutative local principal ideal ring, t be a generator element of $J(R)$, $t^{\left[\frac{k\cdot(n-k)}{n}\right]+1} \neq 0$. The matrix $M(t,k,n)$ is similar (over R) to a matrix of the form*

$$N = \begin{pmatrix} M(t,k',n') & B \\ 0 & A \end{pmatrix}$$

for some integers k' and n', $0 \leq k' \leq n'$, $0 < n' < n$, if and only if $(n,k) > 1$, $k' = \frac{k}{d}$, $n' = \frac{n}{d}$ for some common divisor $d > 1$ of integers k, n.

Proof. Sufficiency follows from [7, p. 2, Thm. 1]. Assume now that there exists an invertible $n \times n$ matrix $C = (c_{ij})_{i,j=1}^{n}$ over R such that $C^{-1}M(t,k,n)C = N$, or equivalently, $M(t,k,n)C = CN$, i.e.

$$\begin{pmatrix} \overbrace{0 & \dots & 0}^{k} & 0 & \dots & 0 & t \\ 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \end{pmatrix} C = C \begin{pmatrix} M(t,k',n') & B \\ 0 & A \end{pmatrix}. \tag{1}$$

For $i,j \in \{1,\dots,n\}$, the scalar equality $(M(t,k,n)C)_{ij} = (CN)_{ij}$ is denoted by $(1,ij)$.

Put $c_i = (c_{i1},\dots,c_{in'})$. We write the equalities $(1,1j)$, $(1,2j),\dots,(1,nj)$, where, in all cases, j runs from 1 to n', respectively in the form

$$tc_n = c_1M(t,k',n'),\ c_1 = c_2M(t,k',n'),\dots,\ c_k = c_{k+1}M(t,k',n'),$$

$$tc_{k+1} = c_{k+2}M(t,k',n'),\dots,\ tc_{n-1} = c_nM(t,k',n').$$

If $k' = 0$, then $M(t,k',n') = tD$ for some $n' \times n'$ matrix D over R and

$$tc_n = tc_1D,\ c_1 = tc_2D,\dots,\ c_k = tc_{k+1}D, tc_{k+1} = tc_{k+2}D,\dots,\ tc_{n-1} = tc_nD.$$

Since $t \neq 0$, we have

$$c_n \equiv c_1D \pmod{J(R)},\ c_1 \in J(R),\ \dots,\ c_k \in J(R),$$

$$c_{k+1} \equiv c_{k+2}D \pmod{J(R)},\dots,\ c_{n-1} \equiv c_nD \pmod{J(R)}.$$

Then $c_n \equiv c_1D \equiv 0 \pmod{J(R)}$, $c_{n-1} \equiv c_1D^2 \equiv 0 \pmod{J(R)}$, $\dots$, $c_{k+1} \equiv c_1D^{n-k} \equiv 0 \pmod{J(R)}$. This implies that $\det C \in J(R)$, which is impossible.

If $k' = n'$, then the $n' \times n'$ matrix $M(t,k',n')$ is invertible over R. But $M(t,k,n)^k$ is an $n \times n$ matrix over tR. This implies that $M(t,k',n')^k$ is an $n' \times n'$ matrix over $tR = J(R)$, which is also impossible.

Finally, assume that $0 < k' < n'$. Let $\phi(i,x) \equiv i \pmod{x}$ and $\phi(i,x) \in \{1,\dots,x\}$, where i and x are integers, $x > 1$. Put

$$\alpha_i = \begin{cases} 0, & \text{if } \phi(i,n) \leqslant k, \\ 1, & \text{if } \phi(i,n) > k \end{cases} \quad \text{and} \quad \beta_j = \begin{cases} 0, & \text{if } \phi(j,n') \leqslant k', \\ 1, & \text{if } \phi(j,n') > k' \end{cases}$$

for all integers i, j. Let us rewrite (1) in the form

$$\operatorname{diag}\left[t^{\alpha_0}, t^{\alpha_1}, \ldots, t^{\alpha_{n-1}}\right] \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix} C = C \left(\begin{array}{c|c} \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix} \operatorname{diag}[t^{\beta_1}, \ldots, t^{\beta_{n'}}] & B \\ \hline 0 & A \end{array} \right). \tag{2}$$

It follows from (2) that for integers $i = 0, \ldots, n-1$, $j = 1, \ldots, n'$, we have

$$t^{\alpha_i} c_{\phi(i,n)\,\phi(j,n')} = t^{\beta_j} c_{\phi(i+1,n)\,\phi(j+1,n')}. \tag{3}$$

Obviously, last equation holds for any integers i, j.

Since $k' < n'$, we deduce that $\phi(k'+1, n') = k'+1$, $\beta_{k'+1} = 1$. Using (3), for $j = k'+1$ we obtain

$$t^{\alpha_i} c_{\phi(i,n) k'+1} = t^{\beta_{k'+1}} c_{\phi(i+1,n)\,\phi(k'+2,n')} = t c_{\phi(i+1,n)\,\phi(k'+2,n')}.$$

Thus, $c_{i\,k'+1} = t c_{i+1\,\phi(k'+2,n')}$, $i = 1, \ldots, k$, and $c_{i\,k'+1} \in J(R)$, $i = 1, \ldots, k$. Since $0 < k' < n'$, we deduce that $\phi(k', n') = k'$, $\beta_{k'} = 0$. Using (3), for $j = k'$ we obtain

$$t^{\alpha_i} c_{\phi(i,n) k'} = t^{\beta_{k'}} c_{\phi(i+1,n)\,k'+1} = c_{\phi(i+1,n)\,k'+1}.$$

Thus, $t c_{i\,k'} = c_{i+1\,k'+1}$, $i = k+1, \ldots, n-1$, and $c_{i+1\,k'+1} \in J(R)$, $i = k+1, \ldots, n-1$, or $c_{i\,k'+1} \in J(R)$, $i = k+2, \ldots, n$. If $i \neq k+1$, then $c_{i\,k'+1} \in J(R)$, $i = 1, \ldots, n$. A matrix $C = (c_{ij})$ is invertible, therefore $\delta_1 = c_{k+1\,k'+1} \in R^*$.

Let $\delta_i = c_{\phi(k+i,n)\,\phi(k'+i,n')}$ for any integer i. It follows from (3) that

$$t^{\alpha_{k+i}} \delta_i = t^{\beta_{k'+i}} \delta_{i+1}. \tag{4}$$

Any element $\delta \in R$ can be written in the form $\delta = t^d \theta$, where d is a nonnegative integer, $\theta \in R^*$ and if $t^d \theta = t^{d'} \theta' \neq 0$ for a nonnegative integer d' and $\theta' \in R^*$, then $d = d'$ and $\theta \equiv \theta'$ $(\mathrm{mod}\ J(R))$ (see [14, p. 245]). For any integer i, let $\delta_i = t^{d_i} \theta_i$, where d_i is a nonnegative integer and $\theta_i \in R^*$. Since δ_1 is invertible in R, we must have $d_1 = 0$. It follows from (4) that

$$t^{\alpha_{k+i} + d_i} \theta_i = t^{\beta_{k'+i} + d_{i+1}} \theta_{i+1}. \tag{5}$$

Let $d'_1 = d_1 = 0$ and $d'_{i+1} = \sum_{j=1}^{i} \alpha_{k+j} - \sum_{j=1}^{i} \beta_{k'+j}$ for any positive integer i.

If $t^{\alpha_{k+j} + d_j} \neq 0$, $j = 1, \ldots, i$, from (5) we conclude that $\alpha_{k+j} + d_j = \beta_{k'+j} + d_{j+1}$, $j = 1, \ldots, i$, and $d'_{i+1} = \sum_{j=1}^{i} \alpha_{k+j} - \sum_{j=1}^{i} \beta_{k'+j} = \sum_{j=1}^{i} d_{j+1} - \sum_{j=1}^{i} d_j = d_{i+1} - d_1 = d_{i+1}$. This also means that $d'_j = d_j$, $j = 1, \ldots, i+1$. So, if $t^{\alpha_{k+j} + d_j} \neq 0$, $j = 1, \ldots, i$, then $d'_j = d_j$, $j = 1, \ldots, i+1$. But $\alpha_{k+j} = 1$, $\alpha_{k+j} - \beta_{k'+j} \in \{0, 1\}$, $j = 1, \ldots, n-k$, and $\alpha_{k+j} = 0$, $\alpha_{k+j} - \beta_{k'+j} \in \{-1, 0\}$, $j = n-k+1, \ldots, n$. So, for any integer $0 < i \leq n$ we have

$$\begin{aligned} d'_{i+1} = \sum_{j=1}^{i} \alpha_{k+j} - \sum_{j=1}^{i} \beta_{k'+j} = \sum_{j=1}^{i} (\alpha_{k+j} - \beta_{k'+j}) &\leq \sum_{j=1}^{n-k} (\alpha_{k+j} - \beta_{k'+j}) \\ &= \sum_{j=1}^{n-k} \alpha_{k+j} - \sum_{j=1}^{n-k} \beta_{k'+j} = n - k - \sum_{j=1}^{n-k} \beta_{k'+j}. \end{aligned}$$

Let $n - k = n'q + r$ for some $q \geq 0$, $0 \leq r < n'$. Then

$$\begin{aligned} d'_{i+1} \leq n - k - \sum_{j=1}^{n'q} \beta_{k'+j} - \sum_{j=1}^{r} \beta_{k'+j} &= n - k - q(n' - k') + \overbrace{\underbrace{1 + \cdots + 1}_{n'-k'} + 0 + \cdots + 0}^{r} \\ &= n - k - q(n' - k') - \min(r, n' - k'). \end{aligned}$$

Since $n'-k'>0$, we conclude that if $r>0$, then

$$\min(r,n'-k') = r\times\min\left(1,\frac{n'-k'}{r}\right) > r\min\left(1,\frac{n'-k'}{n'}\right) = r\frac{n'-k'}{n'}.$$

If $r=0$, then $\min(r,n'-k') = 0 = 0\times\frac{n'-k'}{n'} = r\frac{n'-k'}{n'}$. So,

$$\begin{aligned} d'_{i+1} \le n-k-q(n'-k')-r\frac{n'-k'}{n'} &= n-k-qn'\frac{n'-k'}{n'}-r\frac{n'-k'}{n'} \\ &= n-k-(qn'+r)\frac{n'-k'}{n'} = n-k-(n-k)\frac{n'-k'}{n'}. \end{aligned}$$

Suppose $\frac{n'-k'}{n'} \ge \frac{n-k}{n}$. Then

$$d'_{i+1} \le n-k-(n-k)\frac{n-k}{n} = (n-k)\frac{n-(n-k)}{n} = \frac{k(n-k)}{n}.$$

So, $\alpha_{k+j}+d'_j \le 1+\frac{k(n-k)}{n}$, $\alpha_{k+j}+d'_j \le \left[\frac{k(n-k)}{n}\right]+1$ and $t^{\alpha_{k+j}+d'_j} \ne 0$, $j=2,\dots,i+1$. But $d'_1 = d_1$, $t^{\alpha_{k+1}+d_1} = t^{1+0} \ne 0$, $d'_2 = d_2$, $t^{\alpha_{k+2}+d_2} \ne 0$, $d'_3 = d_3$, and so on. Thus, $d'_j = d_j$, $j=1,\dots,i+2$. Since $0<i\le n$, we conclude that

$$d_{n+1} = d'_{n+1} = \sum_{j=1}^{n}\alpha_{k+j} - \sum_{j=1}^{n}\beta_{k'+j} = n-k-\sum_{j=1}^{n}\beta_{k'+j}.$$

Let $n = n'q+r$ for some $q\ge 0$, $0\le r<n'$. Then

$$d_{n+1} = n-k-\sum_{j=1}^{n'q}\beta_{k'+j} - \sum_{j=1}^{r}\beta_{k'+j} = n-k-q(n'-k')-\min(r,n'-k').$$

If $r>0$ or $\frac{n'-k'}{n'} > \frac{n-k}{n}$, then at least one of the following two inequalities is strong

$$\begin{aligned} n-k-q(n'-k')-\min(r,n'-k') &\le n-k-qn'\frac{n'-k}{n'}-r\frac{n'-k'}{n'} \\ &= n-k-(qn'+r)\frac{n'-k'}{n'} = n-k-n\frac{n'-k'}{n'} \\ &\le n-k-n\frac{n-k}{n} = 0. \end{aligned}$$

Thus, $d_{n+1}<0$, which is impossible. So, n' divides n and $\frac{n'-k'}{n'} = \frac{n-k}{n}$.

Thus, $\frac{n}{n'} = \frac{n-k}{n'-k'}$ and in the case $\frac{n'-k'}{n'} \ge \frac{n-k}{n}$ the theorem holds.

Now assume $\frac{n'-k'}{n'} < \frac{n-k}{n}$. Let $d'_{-i} = \sum_{j=0}^{i}\beta_{k'-j} - \sum_{j=0}^{i}\alpha_{k-j}$ for any nonnegative integer i. If $t^{\beta_{k'-j}+d_{-j+1}} \ne 0$, $j=0,\dots,i$, from (5) we conclude that $\alpha_{k-j}+d_{-j} = \beta_{k'-j}+d_{-j+1}$, $j=0,\dots,i$, and $d'_{-i} = \sum_{j=0}^{i}\beta_{k'-j} - \sum_{j=0}^{i}\alpha_{k-j} = \sum_{j=0}^{i}d_{-j} - \sum_{j=0}^{i}d_{-j+1} = d_{-i}-d_1 = d_{-i}$. This also means that $d'_{-j} = d_{-j}$, $j=-1,0,\dots,i$. So, if $t^{\beta_{k'-j}+d_{-j+1}} \ne 0$, $j=0,\dots,i$, then $d'_{-j} = d_{-j}$, $j=-1,0,\dots,i$. But $\alpha_{k-j}=0$, $\beta_{k'-j}-\alpha_{k-j} \in \{0,1\}$, $j=0,\dots,k-1$, and $\alpha_{k-j}=1$, $\beta_{k'-j}-\alpha_{k-j} \in \{-1,0\}$, $j=k,\dots,n-1$. So, for any integer $0\le i<n$ we have

$$\begin{aligned} d'_{-i} = \sum_{j=0}^{i}\beta_{k'-j} - \sum_{j=0}^{i}\alpha_{k-j} &= \sum_{j=0}^{i}(\beta_{k'-j}-\alpha_{k-j}) \\ &\le \sum_{j=0}^{k-1}(\beta_{k'-j}-\alpha_{k-j}) = \sum_{j=0}^{k-1}\beta_{k'-j} - \sum_{j=0}^{k-1}\alpha_{k-j} = \sum_{j=0}^{k-1}\beta_{k'-j}. \end{aligned}$$

Let $k = n'q + r$ for some $q \geq 0$, $0 \leq r < n'$. Then

$$d'_{-i} \leq \sum_{j=0}^{n'q-1} \beta_{k'-j} + \sum_{j=0}^{r-1} \beta_{k'-j} = q(n'-k') + \overbrace{\underbrace{0+\cdots+0}_{k'}+1+\cdots+1}^{r}$$
$$= q(n'-k') + \max(0, r-k').$$

Since $k' > 0$, we conclude that if $r > 0$, then

$$\max(0, r-k') = r\max\left(0, \frac{r-k'}{r}\right) = r\max\left(0, 1-\frac{k'}{r}\right)$$
$$< r\max\left(0, 1-\frac{k'}{n'}\right) = r\max\left(0, \frac{n'-k'}{n'}\right) = r\frac{n'-k'}{n'}.$$

If $r = 0$, then $\max(0, r-k') = \max(0, -k') = 0 = 0\frac{n'-k'}{n'} = r\frac{n'-k'}{n'}$. So,

$$d'_{-i} \leq qn'\frac{n'-k'}{n'} + r\frac{n'-k'}{n'} = (n'q+r)\frac{n'-k'}{n'} = k\frac{n'-k'}{n'} \leq \frac{k(n-k)}{n}.$$

Thus, $\beta_{k'-j-1} + d'_{-j} < 1 + \frac{k(n-k)}{n}$, $\beta_{k'-j-1} + d'_{-j} \leq \left[\frac{k(n-k)}{n}\right] + 1$ and $t^{\beta_{k'-j-1}+d'_{-j}} \neq 0$, $j = 0, \ldots, i$. But $d'_1 = d_1 = 0$, $t^{\beta_{k'}+d_1} = t^{0+0} \neq 0$, $d'_0 = d_0$, $t^{\beta_{k'-1}+d_0} \neq 0$, $d'_{-1} = d_{-1}$, and so on. So, $d'_{-j} = d_{-j}$, $j = 0, \ldots, i+1$. Since $0 \leq i < n$, we conclude that

$$d_{-n+1} = d'_{-n+1} = \sum_{j=0}^{n-1} \beta_{k'-j} - \sum_{j=0}^{n-1} \alpha_{k-j} = \sum_{j=0}^{n-1} \beta_{k'-j} - (n-k).$$

Let $n = n'q + r$ for some $q \geq 0$, $0 \leq r < n'$. Then

$$d_{-n+1} = \sum_{j=0}^{n'q-1} \beta_{k'-j} + \sum_{j=0}^{r-1} \beta_{k'-j} - (n-k) = q(n'-k') + \max(0, r-k') - (n-k)$$
$$\leq qn'\frac{n'-k'}{n'} + r\frac{n'-k'}{n'} - (n-k) = (qn'+r)\frac{n'-k'}{n'} - (n-k)$$
$$= n\frac{n'-k'}{n'} - (n-k) < n\frac{n-k}{n} - (n-k) = 0.$$

Thus, $d_{-n+1} < 0$, which is impossible. □

In [5, p. 186], it was shown that the matrix $M(t, 4, 7)$ over a commutative local principal ideal ring R, where t is a generating element of $J(R)$, is hereditary reducible if $t^2 = 0$. It follows from Theorem 3 that $M(t, 4, 7)$ over the ring R is hereditary irreducible if $t^{\left[\frac{4\cdot 3}{7}\right]+1} = t^2 \neq 0$. Moreover, if $t^3 = 0$, then the characteristic polynomial $-x^7$ of the matrix $M(t, 4, 7)$ is reducible.

Acknowledgment

The authors are grateful to V. Bondarenko for collaboration, which supplied the key ideas for our main result.

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Received 22.07.2020

Тилищак О.А., Демко М. *Про спадкову незвідність деяких мономіальних матриць над локальними кільцями* // Карпатські матем. публ. — 2021. — Т.13, №1. — С. 127–133.

Розглядаються мономіальні матриці над локальним кільцем R головних ідеалів вигляду $M(t,k,n) = \Phi \begin{pmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{pmatrix}$, $0 < k < n$, де t — твірний елемент радикалу Джекобсона $J(R)$ кільця R, Φ — супровідна матриця многочлена $\lambda^n - 1$ і I_k — одинична $k \times k$ матриця. В роботі встановлено критерій спадкової незвідності $M(t,k,n)$ у випадку, коли $t^{\left[\frac{k\cdot(n-k)}{n}\right]+1} \neq 0$.

Ключові слова і фрази: локальне кільце, радикал Джекобсона, незвідна матриця, мономіальна матриця, спадково незвідна матриця.