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A POISSON TYPE FORMULA FOR HARDY CLASSES ON HEISENBERG'S GROUP

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The Hardy type class of complex functions with infinite many variables defined on the Schrödinger irreducible unitary orbit of reduced Heisenberg group, generated by the Gauss density, is investigated. A Poisson integral type formula for their analytic extensions on an open ball is established. Taylor coefficients for analytic extensions are described by the associated symmetric Fock space.

Introduction

Hardy type spaces $\mathcal{H}^2$ for irreducible representations of locally compact groups were introduced in [4]. In this work we concentrate on an important partial case of such spaces, defined by the Schrödinger irreducible unitary representation of reduced Heisenberg group.

The Hardy type space $\mathcal{H}^2$ on the reduced Heisenberg group $\mathbb{H}$, which acts irreducibly and unitarily over the complex Hilbert space $L^2(\mathbb{R})$ with the help of Schrödinger's representation, is associated, in according to its definition, with a Gauss type density $\hbar$ on $\mathbb{R}$ and the Haar measure on $\mathbb{H}$. The Schrödinger representation of $\mathbb{H}$ contains the complex cyclic subgroup $\mathbb{T} = \{\tau = e^{\mathrm{i}\vartheta} \colon \vartheta \in [0, 2\pi)\}$, which means that the essential assumption of the work [4] is satisfied. We consider the Poisson type integral representation of analytic functions, belonging to $\mathcal{H}^2$, on the open ball

$$\Omega_{L^2(\mathbb{R})} = \Big\{\xi \in L^2(\mathbb{R}) \colon \|\xi\|_{L^2(\mathbb{R})} < 2\sqrt{\pi}\Big\}.$$

The Hilbert space of Taylor coefficients for the space $\mathcal{H}^2$ is unitary equivalent to the Hermitian dual $\Gamma^*(\mathbb{R})$ of the symmetric Fock space $\Gamma(\mathbb{R})$, generated by the complex Hilbert space $L^2(\mathbb{R})$. The corresponding isometry $\Gamma^*(\mathbb{R}) \simeq \mathcal{H}^2$ is described in Theorem 2.

We establish in Theorem 3 the Poissson type integral formula

$$\mathfrak{P}[f](\xi) = \int_{\mathbb{H}} \mathfrak{P}\left[\xi, (x, y, \tau)\right] f(x, y, \tau) \; dx\, dy\, d\tau, \qquad \xi \in \Omega_{L^2(\mathbb{R})},$$

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which for corresponding functions $f \in L^2(\mathbb{H})$, defined on $\mathbb{H}$, produces their unique analytic extensions $\mathfrak{P}[f]$ on $\Omega_{L^2(\mathbb{R})}$.

A theory of Hardy spaces $\mathcal{H}^p$ with $p \geq 1$ of infinitely many variables was advanced in [2]. Many Hardy type spaces on infinite-dimensional Banach domains important in applications have been studied in [6]. Hilbertian Hardy type classes $\mathcal{H}^2$, being reproducing kernel spaces, have been investigated in [5].

We refer for infinite dimensional holomorphy to [3] and for Heisenberg groups to [7].

1 Preliminaries and denotations

Consider the complex space $L^2(\mathbb{R})$ of all quadratically integrable functions on $\mathbb{R}$ with the scalar product $\langle \xi \mid \zeta \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \xi(t)\bar{\zeta}(t)\,dt$ and the norm $\|\xi\|_{L^2(\mathbb{R})} = \langle \xi \mid \xi \rangle^{1/2}_{L^2(\mathbb{R})}$, where $\xi, \zeta \in L^2(\mathbb{R})$. In $L^2(\mathbb{R})$ we consider the orthonormal basis

$$\varphi(t) = \frac{e^{-t^2/2}}{\sqrt[4]{\pi}} \frac{\varepsilon_j(t)}{\sqrt{2^j j!}}, \qquad \varepsilon_j(t) = (-1)^j e^{t^2} \frac{d^j}{dt^j} e^{-t^2}, \qquad t \in \mathbb{R}, \quad j \in \mathbb{Z}_+,$$

where ε_j denotes the Hermite polynomial of degree j.

Let $L^2(\mathbb{R}^n) = \bigotimes_h^n L^2(\mathbb{R}) = \overline{\mathrm{span}}_{\mathbb{C}}\big\{\xi_1 \otimes \ldots \otimes \xi_n \colon \xi_1, \ldots, \xi_n \in L^2(\mathbb{R})\big\}$ be the n-folds Hilbert tensor product of $L^2(\mathbb{R})$ endowed with the scalar product

$$\big\langle \xi_1 \otimes \ldots \otimes \xi_n \,\big|\, \zeta_1 \otimes \ldots \otimes \zeta_n \big\rangle_{\bigotimes_h^n L^2(\mathbb{R})} = \langle \xi_1 \mid \zeta_1 \rangle_{L^2(\mathbb{R})} \ldots \langle \xi_n \mid \zeta_n \rangle_{L^2(\mathbb{R})}$$

and the norm $\|\omega\|_{\bigotimes_h^n L^2(\mathbb{R})} = \langle \omega \mid \omega \rangle^{1/2}_{\bigotimes_h^n L^2(\mathbb{R})}$ with $\omega \in \bigotimes_h^n L^2(\mathbb{R})$. Then the set of elements $\big\{\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_n} \colon (i_1, \ldots, i_n) \in \mathbb{Z}_+^n\big\}$ forms the Hermite orthonormal basis in $\bigotimes_h^n L^2(\mathbb{R})$. If $\mathfrak{s} \colon \{1, \ldots, n\} \longmapsto \{\mathfrak{s}(1), \ldots, \mathfrak{s}(n)\}$ runs through all n-elements permutations, then the codomain of the orthogonal projector

$$\mathfrak{s}_n \colon \bigotimes_h^n L^2(\mathbb{R}) \ni \xi_1 \otimes \ldots \otimes \xi_n \longmapsto \xi_1 \odot \ldots \odot \xi_n := \frac{1}{n!} \sum\nolimits_{\mathfrak{s}} \xi_{\mathfrak{s}(1)} \otimes \ldots \otimes \xi_{\mathfrak{s}(n)},$$

called a symmetric Hilbertian tensor power of $L^2(\mathbb{R})$, we denote by $\bigodot_h^n L^2(\mathbb{R})$. A symmetric Fock space is defined as the orthogonal sum

$$\Gamma := \bigoplus_{n \in \mathbb{Z}_+} \big[\bigodot_h^n L^2(\mathbb{R})\big], \qquad \bigodot_h^0 L^2(\mathbb{R}) = \mathbb{C},$$

with the scalar product and norm, respectively

$$\langle \psi \mid \omega \rangle_\Gamma = \sum_{n \in \mathbb{Z}_+} \langle \psi_n \mid \omega_n \rangle_{\bigotimes_h^n L^2(\mathbb{R})}, \qquad \|\psi\|_\Gamma = \langle \psi \mid \psi \rangle_\Gamma^{1/2},$$

where $\psi = \sum_n \psi_n$, $\omega = \sum_n \omega_n \in \Gamma$ and $\psi_n, \omega_n \in \bigodot_h^n L^2(\mathbb{R})$. We will use the following short denotations

$$\xi^{\otimes n} := \xi \otimes \ldots \otimes \xi \in \bigodot_h^n L^2(\mathbb{R}) \quad \text{with} \quad \xi \in L^2(\mathbb{R}),$$

$$(k) := (k_1, \ldots, k_n) \in \mathbb{Z}_+^n, \quad |(k)| := k_1 + \cdots + k_n, \quad (k)! := k_1! \ldots k_n!.$$

Consider the systems of elements

$$\Phi_n = \left\{ \varphi_{(j)}^{\otimes(k)} \in \bigodot_h^n L^2(\mathbb{R})\colon (j),(k) \in \mathbb{Z}_+^n,\ |(k)| = n \right\}, \qquad \Phi = \left\{ \Phi_n\colon\ n \in \mathbb{Z}_+ \right\},$$

where we denote $\varphi_{(j)}^{\otimes(k)} := \varphi_{j_1}^{\otimes k_1} \odot \ldots \odot \varphi_{j_n}^{\otimes k_n}$ and $\varphi_{(j)}^{\otimes(k)} = 1$ for all (j) if $n = |(k)| = 0$. As is known (see e.g. [1, 2.2.2]), the system Φ forms an orthogonal basis in the symmetric Fock space Γ such that $\|\varphi_{(j)}^{\otimes(k)}\|_{\Gamma(\mathbb{R})} = \sqrt{(k)!/|(k)|!}$ (see e.g. [1, 2.2.2]).

In what follows $\mathbb{H} = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$ will stands for a reduced Heisenberg group with the multiplication

$$(x, y, e^{\mathrm{i}\vartheta}) \cdot (u, v, e^{\mathrm{i}\eta}) = \left(x + u, y + v, e^{\mathrm{i}(\vartheta+\eta)} e^{\mathrm{i}(yu - xv)/2}\right)$$

endowed with the Haar measure $dx\,dy\,d\tau$, where $2\pi d\tau = d\vartheta$.

Let $L^2(\mathbb{H})$ be the space of all quadratically Haar integrable complex functions f on $\mathbb{H}$ with the norm $\|f\|_{L^2(\mathbb{H})} = \left(\int_{\mathbb{H}} |f(x,y,\tau)|^2\, dx\,dy\,d\tau\right)^{1/2}$ and let $L^\infty(\mathbb{H})$ be the space of all essentially bounded complex functions on $\mathbb{H}$.

The Schrödinger representation U from $\mathbb{H}$ into the C^*-algebra $\mathscr{L}\left(L^2(\mathbb{R})\right)$ of bounded linear operators on $L^2(\mathbb{R})$ has the form

$$U_{x,y,\tau}\xi(t) = \tau e^{\mathrm{i}xy/2} e^{\mathrm{i}yt} \xi(t + x), \qquad x, y, t \in \mathbb{R}, \quad \tau \in \mathbb{T}, \quad \xi \in L^2(\mathbb{R}).$$

It is unitary and irreducible. Easy to see that its codomain $U(\mathbb{H}) = \left\{U_{x,t,\tau}\colon (x,t,\tau) \in \mathbb{H}\right\}$ contains the cyclic group $\mathbb{T}$. Moreover, due to the Stone-von Neumann Theorem every irreducible unitary representation $\tilde{U}$ of $\mathbb{H}$ on a Hilbert space H, such that $\tilde{U}(0,0,\tau)\xi = \tau\xi$ for all $\tau \in \mathbb{T}$ and $\xi \in H$, is unitarily equivalent to the Schrödinger representation U.

Lemma 1.1. *The Gauss density function*

$$\hbar(t) := \pi^{-1/4} e^{-t^2/2}, \qquad t \in \mathbb{R}$$

has the property $\hbar \in L^2(\mathbb{R}) \bigcap L^\infty(\mathbb{H})$ *and each basis element* $\varphi_{(j)}^{\otimes(k)} \in \Phi$ *generates the continuous function*

$$\mathbb{H} \ni (x,y,\tau) \longmapsto \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_{\Gamma} = \langle U_{x,y,\tau}\hbar \mid \varphi_{j_1}\rangle_{L^2(\mathbb{R})}^{k_1} \ldots \langle U_{x,y,\tau}\hbar \mid \varphi_{j_n}\rangle_{L^2(\mathbb{R})}^{k_n} \qquad (1)$$

which belongs to $L^2(\mathbb{H})$ *and for all* $(j) \in \mathbb{Z}_+^n$ *and* $(k) \in \mathbb{Z}_+^n$ *with* $|(k)| = n$

$$\left(\int_{\mathbb{H}} \left| \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_{\Gamma} \right|^2 dx\,dy\,d\tau \right)^{1/2} \le \sqrt{\frac{2^{1+n}\pi^{1+n/2}}{n}} \qquad (2)$$

Proof. We have $\hbar \in L^2(\mathbb{R}) \bigcap L^\infty(\mathbb{H})$, since

$$\|\hbar\|_{L^\infty(\mathbb{H})} = \|\hbar\|_{L^\infty(\mathbb{R})} = \sup_{t\in\mathbb{R}} |\hbar(t)| = \frac{1}{\sqrt[4]{\pi}}, \qquad \|\hbar\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} \left| \frac{e^{-t^2/2}}{\sqrt[4]{\pi}} \right|^2 dt \right)^{1/2} = 1.$$

Applying the Fourier transform by the variable $t \in \mathbb{R}$, we can define a linear mapping $\hbar*\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{H})$ in the following form

$$\begin{aligned} (\hbar * \varphi_j)(x,y,\tau) &:= \langle U_{x,y,\tau}\hbar \mid \varphi_j \rangle_{L^2(\mathbb{R})} = \frac{\tau e^{\mathrm{i}xy/2}}{\sqrt[4]{\pi}\sqrt{2^j j!}} \int_{\mathbb{R}} e^{\mathrm{i}yt} \left[e^{-(x+t)^2/2} e^{-t^2/2} \varepsilon_j(t) \right] dt \\ &= \sqrt{2\pi} \frac{\tau e^{\mathrm{i}xy/2}}{\sqrt[4]{\pi}\sqrt{2^j j!}} (-1)^j (x - \mathrm{i}y)^j e^{-x^2/2 + (x-\mathrm{i}y)^2/4} \end{aligned} \qquad (3)$$

for any $\varphi_j \in \Phi_1$ and $j \in \mathbb{Z}_+$. For all (k) such that $|(k)| = n$ it follows

$$\left|\left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_\Gamma\right|^2 \leq \left(2\sqrt{\pi}\right)^n e^{-n(x^2+y^2)/2} \prod_{l=1}^n \left[\frac{(x^2+y^2)^{j_l}}{2^{j_l}(j_l)!}\right]^{k_l}.$$

Since,

$$\int_0^\infty e^{-nu} \prod_{l=1}^n \left(\frac{u^{j_l}}{j_l!}\right)^{k_l} du = \frac{m!}{(j_1!)^{k_1}...(j_n!)^{k_n}} \int_0^\infty e^{-nu} \frac{u^m}{m!}\, du$$
$$= \frac{m!}{(j_1!)^{k_1}...(j_n!)^{k_n}} \frac{1}{n^m} \int_0^\infty e^{-nu} \frac{(un)^m}{m!}\, du \leq \frac{1}{n}$$

with $m = \sum\limits_{l=1}^n j_l k_l$ and

$$\int_{-\infty}^\infty \int_{-\infty}^\infty f\left(\frac{x^2+y^2}{2}\right) dx\, dy = 4 \int_0^\infty \int_0^{\pi/2} f(u)\, du\, d\vartheta = 2\pi \int_0^\infty f(u)\, du,$$

where $x^2 = 2u\cos^2\vartheta$ and $y^2 = 2u\sin^2\vartheta$, we obtain that the functions (1) belong to $L^2(\mathbb{H})$ and the estimation (2) holds.

□

2 The main unitary isometry

Since the Schrödinger unitary representation of $\mathbb{H}$ over $L^2(\mathbb{R})$ contains the complex cyclic subgroup $\mathbb{T}$, we can apply the general result of [4] to the case of reduced Heisenberg group.

Definition 2.1. *The closure in $L^2(\mathbb{H})$ of the complex linear span of all functions* (1)*, generated by the Fock symmetric basis Φ (respectively by Φ_n with $n \in \mathbb{N}$), we will denote by $\mathcal{H}^2$ (respectively by $\mathcal{H}^2_n$).*

Now we consider the unitary representation of the diagonal form

$$U^{\otimes n}\colon \mathbb{H} \ni (x,y,\tau) \longmapsto U^{\otimes n}_{x,y,\tau} \in \mathscr{L}\left(\textstyle\bigodot_h^n L^2(\mathbb{R})\right) \quad \text{with} \quad U^{\otimes n}_{x,y,\tau} := \textstyle\bigotimes^n U_{x,y,\tau}, \quad n \in \mathbb{Z}_+,$$

where $U^{\otimes 0}$ is the unit in $\mathbb{C}$ and $\mathscr{L}\left(\bigodot_h^n L^2(\mathbb{R})\right)$ denotes the C^*-algebra of all bounded linear operators on $\bigodot_h^n L^2(\mathbb{R})$.

The next axillary statements immediately follow from [4] and the previous Lemma 1.1.

Lemma 2.1. (i) *The unitary representation $U^{\otimes n}$ is irreducible for any $n \in \mathbb{N}$.*

(ii) *For any $\varphi_{(j)}^{\otimes(k)} \in \Phi$ with $|(k)| = n$ the constants*

$$\aleph_n := \sqrt{\frac{(k)!}{n!}} \left(\int_{\mathbb{H}} \left|\left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_\Gamma\right|^2 dx\, dy\, d\tau\right)^{-1/2} \tag{4}$$

are dependent on an index $n \in \mathbb{N}$ but independent of indexes $(j),(k) \in \mathbb{Z}_+^n$.

(iii) *For any element* $\psi_n \in \bigodot_h^n L^2(\mathbb{R})$ *uniquely corresponds the function*

$$\widehat{\psi}_n \colon \mathbb{H} \ni (x,y,\tau) \longmapsto \widehat{\psi}_n(x,y,\tau) := \aleph_n \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \psi_n \right\rangle_\Gamma,$$

which belongs to $\mathcal{H}^2$, *and the equality*

$$\int_{\mathbb{H}} \widehat{\psi}_n(x,y,\tau)\overline{\widehat{\omega}}_n(x,y,\tau)\,dx\,dy\,d\tau = \left\langle \omega_n \mid \psi_n \right\rangle_\Gamma, \qquad \psi_n,\omega_n \in \bigodot_h^n L^2(\mathbb{R}), \tag{5}$$

holds. The mapping of the form

$$\mathfrak{h} \colon \bigodot_h^n L^2(\mathbb{R}) \ni \psi_n \longmapsto \widehat{\psi}_n \in \mathcal{H}_n^2 \tag{6}$$

is an antilinear unitary equivalence between $\bigodot_h^n L^2(\mathbb{R})$ *and* $\mathcal{H}_n^2$. *The following inequality holds for all* $\psi_n,\omega_n \in \bigodot_h^n L^2(\mathbb{R})$,

$$\begin{aligned}\left|\int_{\mathbb{H}} \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \psi_n \right\rangle_\Gamma \overline{\left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \omega_n \right\rangle_\Gamma} dx\,dy\,d\tau\right| \\ \le n!(n-1)!\,2^n\pi^{1+n/2}\|\psi_n\|_\Gamma\|\omega_n\|_\Gamma.\end{aligned} \tag{7}$$

Proof. Elements $\psi_n,\omega \in \bigodot_h^n L^2(\mathbb{R})$ present in the form of their Fourier decompositions on the orthogonal basis Φ_n,

$$\psi_n = \sum_{(k),(j)\in\mathbb{Z}_+^n} \alpha_{(j)}^{(k)} \varphi_{(j)}^{\otimes(k)} \frac{n!}{(k)!}, \qquad \omega_n = \sum_{(m),(i)\in\mathbb{Z}_+^n} \beta_{(i)}^{(m)} \varphi_{(i)}^{\otimes(m)} \frac{n!}{(m)!}$$

with the Fourier coefficients $\alpha_{(j)}^{(k)}, \beta_{(i)}^{(m)} \in \mathbb{C}$, where $|(k)| = |(m)| = n$. Hence,

$$\begin{aligned}&\left|\int_{\mathbb{H}} \left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \psi_n \right\rangle_\Gamma \overline{\left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \omega_n \right\rangle_\Gamma} dx\,dy\,d\tau\right| \\ &\le \sum |\alpha_{(j)}^{(k)}\beta_{(i)}^{(m)}| \frac{n!^2}{(k)!(m)!} \left|\int_{\mathbb{H}} \left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_\Gamma \overline{\left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \varphi_{(i)}^{\otimes(m)} \right\rangle_\Gamma} dx\,dy\,d\tau\right|.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\sum_{\substack{(k),(j)\\(m),(i)}} |\alpha_{(j)}^{(k)}\beta_{(i)}^{(m)}| \frac{n!^2}{(k)!(m)!} \left|\int_{\mathbb{H}} \left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_\Gamma \overline{\left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \varphi_{(i)}^{\otimes(m)} \right\rangle_\Gamma} dx\,dy\,d\tau\right| \\ &\qquad \le \sum_{(k),(j),(m),(i)} |\alpha_{(j)}^{(k)}\beta_{(i)}^{(m)}| \frac{n!^2}{(k)!(m)!} \\ &\times \left(\int_{\mathbb{H}} \left| \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_\Gamma \right|^2 dx\,dy\,d\tau\right)^{1/2} \left(\int_{\mathbb{H}} \left| \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_\Gamma \right|^2 dx\,dy\,d\tau\right)^{1/2}.\end{aligned}$$

Applying the inequality (2) and the Cauchy-Schwarz inequality one more, we have

$$\begin{aligned}&\left|\int_{\mathbb{H}} \left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \psi_n \right\rangle_\Gamma \overline{\left\langle (U_{x,y,\tau}\hbar)^{\otimes|(k)|} \mid \omega_n \right\rangle_\Gamma} dx\,dy\,d\tau\right| \le \frac{\pi\,(2\sqrt{\pi})^n}{n} \sum |\alpha_{(j)}^{(k)}\beta_{(i)}^{(m)}| \frac{n!^2}{(k)!(m)!} \\ &\qquad \le \frac{\pi\,(2\sqrt{\pi})^n}{n} \left(\sum |\alpha_{(j)}^{(k)}|^2 \frac{n!^2}{(k)!^2}\right)^{1/2} \left(\sum |\beta_{(i)}^{(m)}|^2 \frac{n!^2}{(m)!^2}\right)^{1/2}.\end{aligned}$$

It follows (7), as $\max_{(k)\in\mathbb{Z}_+^n}\dfrac{n!}{(k)!}=\max_{(m)\in\mathbb{Z}_+^n}\dfrac{n!}{(m)!}=n!$ and $\|\psi_n\|_\Gamma^2=\sum_{(k),(j)}\bigl|\alpha_{(j)}^{(k)}\bigr|^2\dfrac{n!}{(k)!}$, $\|\omega_n\|_\Gamma^2=\sum_{(m),(i)}\bigl|\beta_{(i)}^{(m)}\bigr|^2\dfrac{n!}{(m)!}$. So, the integral $\int_{\mathbb{H}}\bigl\langle(U_{x,y,\tau}\hbar)^{\otimes|(k)|}\mid\psi_n\bigr\rangle_\Gamma\overline{\bigl\langle(U_{x,y,\tau}\hbar)^{\otimes|(k)|}\mid\omega_n\bigr\rangle_\Gamma}dx\,dy\,d\tau$ is a Hermitian bounded form on $\bigodot_h^n L^2(\mathbb{R})$, which is antilinear by $\psi_n\in\bigodot_h^n L^2(\mathbb{R})$ and linear by $\omega_n\in\bigodot_h^n L^2(\mathbb{R})$. Therefore, the statements (i) and (iii) directly follow from [4]. □

Let us consider properties of the systems

$$\widehat{\Phi}:=\Bigl\{\widehat{\Phi}_n\colon n\in\mathbb{Z}_+\Bigr\},\qquad \widehat{\Phi}_n:=\Bigl\{\widehat{\varphi}_{(j)}^{(k)}\colon\ (j),(k)\in\mathbb{Z}_+^n,\ |(k)|=n\Bigr\},$$

which is generated by the orthogonal basis Φ of the Fock space Γ, and where is denoted

$$\widehat{\varphi}_{(j)}^{(k)}:=\widehat{\varphi_{(j)}^{\otimes(k)}}\quad\text{with}\quad \widehat{\varphi_{(j)}^{\otimes(k)}}(x,y,\tau)=\aleph_{|(k)|}\bigl\langle U_{x,y,\tau}\hbar\mid\varphi_{j_1}\bigr\rangle_{L^2(\mathbb{R})}^{k_1}\ldots\bigl\langle U_{x,y,\tau}\hbar\mid\varphi_{j_n}\bigr\rangle_{L^2(\mathbb{R})}^{k_n}$$

with $\aleph_0=1$. Using that the Schrödinger representation U of $\mathbb{H}$ contains the cyclic subgroup $\mathbb{T}$, we similarly to [4] obtain the following.

Theorem 1. *The system $\widehat{\Phi}$ forms an orthogonal basis in $\mathcal{H}^2$ and the subsystem $\widehat{\Phi}_n$ does so in $\mathcal{H}_n^2$. If $m\neq n$ then $\mathcal{H}_m^2$ is orthogonal to $\mathcal{H}_n^2$ in $L^2(\mathbb{H})$ and the orthogonal Hilbertian decomposition*

$$\mathcal{H}^2=\bigoplus_{n\in\mathbb{Z}_+}\mathcal{H}_n^2,\qquad \mathcal{H}_0^2=\mathbb{C},$$

holds. The surjective mapping (which is a linear extension of (6))

$$\mathfrak{h}\colon\Gamma\ni\psi=\bigoplus_{n\in\mathbb{Z}_+}\psi_n\longmapsto\widehat{\psi}=\sum_{n\in\mathbb{Z}_+}\widehat{\psi}_n\in\mathcal{H}^2,\tag{8}$$

where $\widehat{\psi}_0=\psi_0\in\mathbb{C}$, realizes an antilinear unitary equivalence between the Fock space Γ and the space $\mathcal{H}^2$. Moreover, the following equality holds

$$\int_{\mathbb{H}}\widehat{\psi}(x,y,\tau)\overline{\widehat{\omega}}(x,y,\tau)\,dx\,dy\,d\tau=\bigl\langle\omega\mid\psi\bigr\rangle_\Gamma,\qquad\psi,\omega\in\Gamma.\tag{9}$$

3 Integral formulas for analytic extensions

Lemma 3.1. *For any function $f=\sum_{n\in\mathbb{Z}_+}f_n\in\mathcal{H}^2$ with $f_n\in\mathcal{H}_n^2$ its integral transform*

$$\mathfrak{C}[f](\xi)=\int_{\mathbb{H}}\mathfrak{C}\left[\xi,(x,y,\tau)\right]f(x,y,\tau)\ dx\,dy\,d\tau,\qquad\xi\in\Omega_{L^2(\mathbb{R})},\tag{10}$$

with the Cauchy type reproducing kernel of the form

$$\mathfrak{C}\left[\xi,(x,y,\tau)\right]=1+\sum_{n\in\mathbb{N}}\frac{n}{2^{1+n}\pi^{1+3n/4}}\left[\tau e^{\mathrm{i}xy}\int_{\mathbb{R}}\xi(t)e^{\mathrm{i}yt-(t+x)^2/2}\,dt\right]^n,\qquad(x,y,\tau)\in\mathbb{H},\tag{11}$$

is a unique analytic extension of f on $\Omega_{L^2(\mathbb{R})}$ with the Taylor coefficients at the origin

$$\frac{d_0^n\mathfrak{C}[f](\xi)}{n!}=\frac{n}{2^{1+n}\pi^{1+3n/4}}\int_{\mathbb{H}}\left[\tau e^{\mathrm{i}xy}\int_{\mathbb{R}}\xi(t)e^{\mathrm{i}yt-(t+x)^2/2}\,dt\right]^n f_n(x,y,\tau)\ dx\,dy\,d\tau,\tag{12}$$

where $\xi\in L^2(\mathbb{R})$ and $n\in\mathbb{N}$.

Proof. Applying the property that $\aleph_n$ is independent on indexes $(j),(k)\in\mathbb{Z}_+^n$ and after the formula (3) at $j=0$, we conclude that these constants can be calculated by the formulas

$$\aleph_n^{-2}=\frac{n!}{n!}\int_{\mathbb{H}}\left|\left\langle(U_{x,y,\tau}\hbar)^{\otimes n}\mid\varphi_0^{\otimes n}\right\rangle_\Gamma\right|^2 dx\,dy\,d\tau=\int_{\mathbb{R}^2}\left|\frac{\sqrt{2\pi}}{\sqrt[4]{\pi}}e^{-x^2/2+(x-\mathrm{i}y)^2/4}\right|^{2n}dx\,dy$$
$$=\left(\frac{2\pi}{\sqrt{\pi}}\right)^n\int_{\mathbb{R}}\frac{e^{-ny^2/2}\sqrt{2\pi}}{\sqrt{n}}\,dy=\left(\frac{2\pi}{\sqrt{\pi}}\right)^n\frac{2\pi}{n}=\frac{2^{1+n}\pi^{1+n/2}}{n}.$$

It follows that the radius of convergence of the power series (12) is the inverse of

$$\lim_{n\to\infty}\sqrt[n]{\aleph_n^2}=\lim_{n\to\infty}\sqrt[n]{\frac{n}{2^{1+n}\pi^{1+n/2}}}=\frac{1}{2\sqrt{\pi}}.$$

Therefore the power series of the form

$$\mathfrak{C}\left[\cdot,(x,y,\tau)\right]:\xi\longmapsto 1+\sum_{n\in\mathbb{N}}\frac{n}{2^{1+n}\pi^{1+n/2}}\left\langle\xi\mid U_{x,y,\tau}\hbar\right\rangle_{L^2(\mathbb{R})}^n$$
$$=1+\sum_{n\in\mathbb{N}}\frac{n}{2^{1+n}\pi^{1+n/2}}\left[\frac{\tau e^{\mathrm{i}xy}}{\sqrt[4]{\pi}}\int_{\mathbb{R}}\xi(t)e^{\mathrm{i}yt-(t+x)^2/2}\,dt\right]^n$$

is an analytic $L^\infty(\mathbb{H})$-valued function by the variable $\xi\in\Omega_{L^2(\mathbb{R})}$. For any function $f\in L^2(\mathbb{H})$ the linear functional $F\colon L^\infty(\mathbb{H})\ni g\longmapsto\int_{\mathbb{H}}fg\,dx\,dy\,d\tau$ is continuous. Since $\mathfrak{C}[f](\xi)=F\circ\mathfrak{C}(\xi,\cdot)$, the function $\mathfrak{C}[f]$, determined by the formula (10), is analytic by $\xi\in\Omega_{L^2(\mathbb{R})}$ in view of [3, 3.1.2]. Therefore, differentiating $\mathfrak{C}[f]$ in the formula (10) at the origin, we obtain

$$\frac{d_0^n\mathfrak{C}[f](\xi)}{n!}=\frac{n}{2^{1+n}\pi^{1+n/2}}\int_{\mathbb{H}}\left\langle\xi^{\otimes n}\mid(U_{x,y,\tau}\hbar)^{\otimes n}\right\rangle_\Gamma f_n(x,y,\tau)\;dx\,dy\,d\tau=\mathfrak{C}[f_n](\xi)$$

for all $\xi\in\Omega_{L^2(\mathbb{R})}$. By the Cauchy-Schwarz inequality

$$\left|\mathfrak{C}[f_n](\xi)\right|\le\frac{n}{2^{1+n}\pi^{1+n/2}}\int_{\mathbb{H}}\left|\left\langle\xi^{\otimes n}\mid(U_{x,y,\tau}\hbar)^{\otimes n}\right\rangle_\Gamma f_n(x,y,\tau)\right|\,dx\,dy\,d\tau$$
$$\le\frac{n}{2^{1+n}\pi^{1+n/2}}\|\xi\|_{L^2(\mathbb{R})}^n\|f_n\|_{L^2_\chi(\mathbb{H})}\qquad\text{for all}\qquad\xi\in L^2(\mathbb{R}).$$

Hence, every element $f_n\in\mathcal{H}_n^2$ has a unique continuous extension to a n-homogenous polynomial $\mathfrak{C}[f_n]$ defined on $L^2(\mathbb{R})$, which takes the form (12). As is known [3, 2.4.2], continuous Taylor coefficients uniquely define the analytic function $\mathfrak{C}[f]$ on $\Omega_{L^2(\mathbb{R})}$. So, uniqueness of the analytic extension $\mathfrak{C}[f]$ is proved. □

Definition 3.1. *Following to [4] we mean the space of analytic functions*

$$\mathcal{H}^2\left(\Omega_{L^2(\mathbb{R})}\right):=\left\{\mathfrak{C}[f]\colon f\in\mathcal{H}^2\right\},$$

defined by the formula (10) *with the finite norm*

$$\|\mathfrak{C}[f]\|_{\mathcal{H}^2}:=\sup_{r\in[0,1)}\left(\int_{\mathbb{H}}\left|\mathfrak{C}[f](rU_{x,y,\tau}\hbar)\right|^2\,dx\,dy\,d\tau\right)^{1/2},$$

the Hardy type space associated with the Heisenberg group $\mathbb{H}$ *and the Gaussian density* $\hbar$.

Applying Lemma 3.1 similarly to [4] we can prove the following statement.

Theorem 2. *The following is an antilinear surjective isometry*

$$(\mathfrak{C}\circ\mathfrak{h})\colon \Gamma\ni\psi\longmapsto\mathfrak{C}[\widehat{\psi}]\in\mathcal{H}^2\left(\Omega_{L^2(\mathbb{R})}\right).$$

Now consider an analogue of analytic extension in a Poisson form. For this purpose we need the positive function

$$\mathfrak{C}(\xi,\xi)=1+\sum_{n\in\mathbb{N}}\frac{n}{2^{1+n}\pi^{1+n/2}}\|\xi\|^{2n}_{L^2(\mathbb{R})},\qquad \xi\in\Omega_{L^2(\mathbb{R})},$$

where the series is convergent by Lemma 3.1.

Theorem 3. *The integral transform*

$$\mathfrak{P}[f](\xi):=\int_{\mathbb{H}}\mathfrak{P}\left[\xi,(x,y,\tau)\right]f(x,y,\tau)\;dx\,dy\,d\tau,\qquad f\in\mathcal{H}^2$$

with the Poisson type reproducing kernel

$$\mathfrak{P}\left[\xi,(x,y,\tau)\right]:=\frac{\left|\mathfrak{C}\left[\xi,(x,y,\tau)\right]\right|^2}{\mathfrak{C}(\xi,\xi)}>0,\qquad \xi\in\Omega_{L^2(\mathbb{R})},\quad (x,y,\tau)\in\mathbb{H}$$

satisfies the equalities

$$\mathfrak{P}[f]=\mathfrak{C}[f],\qquad \mathfrak{P}\left[\operatorname{Re}f\right]=\operatorname{Re}\mathfrak{C}[f].$$

Proof. If we put $g(x,y,\tau):=\dfrac{\overline{\mathfrak{C}}\left[\xi,(x,y,\tau)\right]}{\mathfrak{C}(\xi,\xi)}f(x,y,\tau)$ with $\xi\in\Omega_{L^2(\mathbb{R})}$, then $g\in\mathcal{H}^2$, since the function $\dfrac{\mathfrak{C}\left[\xi,(x,y,\tau)\right]}{\mathfrak{C}(\xi,\xi)}$ is uniformly bounded by the variables $(x,y,\tau)\in\mathbb{H}$ for any fixed element $\xi\in\Omega_{L^2(\mathbb{R})}$ via Lemma 3.1. For instance, we obtain $\mathfrak{C}[f](\xi)=\mathfrak{C}[g](\xi)$. Therefore,

$$\begin{aligned}\mathfrak{C}[f](\xi)&=\int_{\mathbb{H}}\mathfrak{C}\left[\xi,(x,y,\tau)\right]g(x,y,\tau)\;d\chi(x,y,\tau)\\&=\int_{\mathbb{H}}\mathfrak{P}\left[\xi,(x,y,\tau)\right]f(x,y,\tau)\;d\chi(x,y,\tau)=\mathfrak{P}[f](\xi)\end{aligned}$$

via Theorem 3.1, that it was necessary to prove. □

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Лопушанський О.В., Олексієнко М.В. *Формула типу Пуассона для класів Харді на групах Хейзенберґа* // Карпатські математичні публікації. — 2010. — Т.2, №1. — С. 87–95.

Досліджується клас типу Харді комплексних функцій нескінченої кількості змінних, визначених на унітарній орбіті породженій ґауссівською функцією густини при незвідному представленні Шредінґера редукованої групи Хейзенберґа. Встановлено інтегральну формулу типу Пуассона для їх аналітичних розширень у відкриту кулю. Коефіцієнти Тейлора аналітичних розширень описано за допомогою симетричних просторів Фока.

Лопушанский О.В., Олексиенко М.В. *Формула типа Пуассона для классов Харди на группах Хейзенберга* // Карпатские математические публикации. — 2010. — Т.2, №1. — С. 87–95.

Исследуется класс типа Харди комплексных функций бесконечного числа переменных, определённых на унитарной орбите породждённой гауссовской функцией плотности при неприводимом представлении Шрёдингера редуцированной группы Хейзенберга. Установлено интегральную формулу типа Пуассона для их аналитических расширений в открытую кулю. Коэффициєнты Тейлора аналитичесних расширений описаны при помощи ассоциированых симметрических пространств Фока.