Карпатські математичні
публікації. Т.3, №2

Carpathian Mathematical
Publications. V.3, №2

УДК 512.536.7

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ON THE CLOSURE OF THE EXTENDED BICYCLIC SEMIGROUP

Fihel I.R., Gutik O.V. *On the closure of the extended bicyclic semigroup*, Carpathian Mathematical Publications, **3**, 2 (2011), 131–157.

In the paper we study the semigroup $\mathscr{C}_{\mathbb{Z}}$ which is a generalization of the bicyclic semigroup. We describe main algebraic properties of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and prove that every non-trivial congruence $\mathfrak{C}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a group congruence, and moreover the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is isomorphic to a cyclic group. Also we show that the semigroup $\mathscr{C}_{\mathbb{Z}}$ as a Hausdorff semitopological semigroup admits only the discrete topology. Next we study the closure $\operatorname{cl}_T(\mathscr{C}_{\mathbb{Z}})$ of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup T. We show that the non-empty remainder of $\mathscr{C}_{\mathbb{Z}}$ in a topological inverse semigroup T consists of a group of units $H(1_T)$ of T and a two-sided ideal I of T in the case when $H(1_T)\neq\varnothing$ and $I\neq\varnothing$. In the case when T is a locally compact topological inverse semigroup and $I\neq\varnothing$ we prove that an ideal I is topologically isomorphic to the discrete additive group of integers and describe the topology on the subsemigroup $\mathscr{C}_{\mathbb{Z}}\cup I$. Also we show that if the group of units $H(1_T)$ of the semigroup T is non-empty, then $H(1_T)$ is either singleton or $H(1_T)$ is topologically isomorphic to the discrete additive group of integers.

1 Introduction and preliminaries

In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [6, 7, 9, 10]. If Y is a subspace of a topological space X and $A\subseteq Y$, then by $\operatorname{cl}_Y(A)$ we shall denote the topological closure of A in Y. We denote by $\mathbb{N}$ the set of positive integers.

An algebraic semigroup S is called *inverse* if for any element $x\in S$ there exists the unique $x^{-1}\in S$ such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. The element x^{-1} is called the *inverse of* $x\in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S\to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from universal and identity congruence on S, and *group* if the quotient semigroup $S/\mathfrak{C}$ is a group.

If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S, then the

2000 *Mathematics Subject Classification:* 22A15, 20M18, 20M20, 54H15.
Key words and phrases: topological semigroup, semitopological semigroup, topological inverse semigroup, bicyclic semigroup, closure, locally compact space, ideal, group of units.

semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order.

Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [7]):

$$\begin{aligned}
&a\mathscr{R}b \text{ if and only if } aS^1 = bS^1;\\
&a\mathscr{L}b \text{ if and only if } S^1a = S^1b;\\
&a\mathscr{J}b \text{ if and only if } S^1aS^1 = S^1bS^1;\\
&\mathscr{D} = \mathscr{L}\circ\mathscr{R} = \mathscr{R}\circ\mathscr{L};\\
&\mathscr{H} = \mathscr{L}\cap\mathscr{R}.
\end{aligned}$$

A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

A *semitopological* (resp. *topological*) *semigroup* is a Hausdorff topological space together with a separately (resp. jointly) continuous semigroup operation [6, 18]. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup*. A topology τ on a (inverse) semigroup S which turns S to be a topological (inverse) semigroup is called a (*inverse*) *semigroup topology* on S.

An element s of a topological semigroup S is called *topologically periodic* if for every open neighbourhood $U(s)$ of s in S there exists a positive integer $n \geqslant 2$ such that $s^n \in U(s)$. Obviously, if there exists a subgroup $H(e)$ with a neutral element e in S, then $s \in H(e)$ is topologically periodic if and only if for every open neighbourhood $U(e)$ of e in S there exists a positive integer n such that $s^n \in U(e)$.

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by elements p and q subject only to the condition $pq = 1$. The distinct elements of $\mathscr{C}(p,q)$ are exhibited in the following useful array:

$$\begin{array}{ccccc}
1 & p & p^2 & p^3 & \cdots\\
q & qp & qp^2 & qp^3 & \cdots\\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots\\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p,q)$ under h is a cyclic group (see [7, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known Andersen's result [1] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete semigroup topology and a topological semigroup S can contain the bicyclic semigroup $\mathscr{C}(p,q)$ as a dense subsemigroup only as an open subset [8]. Also Bertman and West in [5] proved that the bicyclic semigroup as a Hausdorff semitopological semigroup admits only the discrete topology. The problem of an embedding of the bicycle semigroup into compact-like topological semigroups solved in the papers [2, 3, 4, 11, 13] and the closure of the bicycle semigroup in topological semigroups studied in [8].

Let $\mathbb{Z}$ be the additive group of integers. On the Cartesian product $\mathscr{C}_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}$ we define the semigroup operation as follows:

$$(a,b) \cdot (c,d) = \begin{cases} (a-b+c,d), & \text{if } b < c; \\ (a,d), & \text{if } b = c; \\ (a,d+b-c), & \text{if } b > c, \end{cases} \tag{1}$$

for $a,b,c,d \in \mathbb{Z}$. The set $\mathscr{C}_{\mathbb{Z}}$ with such defined operation is called the *extended bicycle semigroup* [19].

In this paper we study the semigroup $\mathscr{C}_{\mathbb{Z}}$. We describe main algebraic properties of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and prove that every non-trivial congruence $\mathfrak{C}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a group congruence, and moreover the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is isomorphic to a cyclic group. Also we show that the semigroup $\mathscr{C}_{\mathbb{Z}}$ as a Hausdorff semitopological semigroup admits only the discrete topology. Next we study the closure $\operatorname{cl}_T(\mathscr{C}_{\mathbb{Z}})$ of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup T. We show that the non-empty remainder of $\mathscr{C}_{\mathbb{Z}}$ in a topological inverse semigroup T consists of a group of units $H(1_T)$ of T and a two-sided ideal I of T in the case when $H(1_T) \neq \varnothing$ and $I \neq \varnothing$. In the case when T is a locally compact topological inverse semigroup and $I \neq \varnothing$ we prove that an ideal I is topologically isomorphic to the discrete additive group of integers and describe the topology on the subsemigroup $\mathscr{C}_{\mathbb{Z}} \cup I$. Also we show that if the group of units $H(1_T)$ of the semigroup T is non-empty, then $H(1_T)$ is either singleton or $H(1_T)$ is topologically isomorphic to the discrete additive group of integers.

2 Algebraic properties of the semigroup $\mathscr{C}_{\mathbb{Z}}$

Proposition 2.1. *The following statements hold:*

(i) $E(\mathscr{C}_{\mathbb{Z}}) = \{(a,a) \mid a \in \mathbb{Z}\}$, *and* $(a,a) \leqslant (b,b)$ *in* $E(\mathscr{C}_{\mathbb{Z}})$ *if and only if* $a \geqslant b$ *in* $\mathbb{Z}$, *and hence* $E(\mathscr{C}_{\mathbb{Z}})$ *is isomorphic to the linearly ordered semilattice* $(\mathbb{Z}, \max)$*;*

(ii) $\mathscr{C}_{\mathbb{Z}}$ *is an inverse semigroup, and the elements* (a,b) *and* (b,a) *are inverse in* $\mathscr{C}_{\mathbb{Z}}$*;*

(iii) for any idempotents $e,f \in \mathscr{C}_{\mathbb{Z}}$ *there exists* $x \in \mathscr{C}_{\mathbb{Z}}$ *such that* $x \cdot x^{-1} = e$ *and* $x^{-1} \cdot x = f$*;*

(iv) elements (a,b) *and* (c,d) *of the semigroup* $\mathscr{C}_{\mathbb{Z}}$ *are:*

(a) $\mathscr{R}$*-equivalent if and only if* $a = c$*;*

(b) $\mathscr{L}$*-equivalent if and only if* $b = d$*;*

(c) $\mathscr{H}$-equivalent *if and only if* $a = c$ *and* $b = d$;

(d) $\mathscr{D}$-equivalent *for all* $a, b, c, d \in \mathbb{Z}$;

(e) $\mathscr{J}$-equivalent *for all* $a, b, c, d \in \mathbb{Z}$;

(v) $\mathscr{C}_{\mathbb{Z}}$ *is a bisimple semigroup and hence it is simple;*

(vi) *if* $(a, b) \cdot (c, d) = (x, y)$ *in* $\mathscr{C}_{\mathbb{Z}}$ *then* $x - y = a - b + c - d$.

(vii) *every maximal subgroup of* $\mathscr{C}_{\mathbb{Z}}$ *is trivial.*

(viii) *for every integer* n *the subsemigroup* $\mathscr{C}_{\mathbb{Z}}[n] = \{(a, b) \mid a \geqslant n \ \& \ b \geqslant n\}$ *of* $\mathscr{C}_{\mathbb{Z}}$ *is isomorphic to the bicyclic semigroup* $\mathscr{C}(p, q)$, *and moreover an isomorphism* $h\colon \mathscr{C}_{\mathbb{Z}}[n] \to \mathscr{C}(p, q)$ *is defined by the formula* $((a, b))\, h = q^{a-n}p^{b-n}$;

(ix) $\mathscr{L}\mathscr{I}_{\mathscr{C}_{\mathbb{Z}}} = \{\mathscr{L}^a \mid a \in \mathbb{Z}\}$, *where* $\mathscr{L}^a = \{(x, y) \in \mathscr{C}_{\mathbb{Z}} \mid y \geqslant a\}$, *is the family of all left ideals of the semigroup* $\mathscr{C}_{\mathbb{Z}}$;

(x) $\mathscr{R}\mathscr{I}_{\mathscr{C}_{\mathbb{Z}}} = \{\mathscr{R}^a \mid a \in \mathbb{Z}\}$, *where* $\mathscr{R}^a = \{(x, y) \in \mathscr{C}_{\mathbb{Z}} \mid x \geqslant a\}$, *is the family of all right ideals of the semigroup* $\mathscr{C}_{\mathbb{Z}}$.

Proof. The proofs of statements (i), (ii), (iii), (iv), (vi), (vii) and $(viii)$ are trivial. Statement (v) follows from statement (iii) and Lemma 1.1 of [16].

Simple verifications (see: formula (1)) show that

$$(a, b)\mathscr{C}_{\mathbb{Z}} = \{(x, y) \in \mathscr{C}_{\mathbb{Z}} \mid x \geqslant a\} \qquad \text{and} \qquad \mathscr{C}_{\mathbb{Z}}(a, b) = \{(x, y) \in \mathscr{C}_{\mathbb{Z}} \mid y \geqslant b\}$$

for every $(a, b) \in \mathscr{C}_{\mathbb{Z}}$. This completes the proof of statements (ix) and (x). $\square$

Proposition 2.2. *Every non-trivial congruence* $\mathfrak{C}$ *on the semigroup* $\mathscr{C}_{\mathbb{Z}}$ *is a group congruence, and moreover the quotient semigroup* $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ *is isomorphic to a cyclic group.*

Proof. First we shall show that if two distinct idempotents (a, a) and (b, b) of $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent then the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is a group. Without loss of generality we can assume that $(a, a) \leqslant (b, b)$, i.e., $a \geqslant b$ in $\mathbb{Z}$. Then we have that

$$\begin{aligned}
(a, b) \cdot (b, b) \cdot (b, a) &= (a, a); \\
(a, b) \cdot (a, a) \cdot (b, a) &= (a + (a - b), a + (a - b)); \\
(a, b) \cdot (a + (a - b), a + (a - b)) \cdot (b, a) &= (a + 2(a - b), a + 2(a - b)); \\
\cdots \quad \cdots \quad \cdots & \qquad \cdots \quad \cdots \\
(a, b) \cdot (a + j(a - b), a + j(a - b)) \cdot (b, a) &= (a + (j + 1)(a - b), a + (j + 1)(a - b)); \\
\cdots \quad \cdots \quad \cdots & \qquad \cdots \quad \cdots
\end{aligned}$$

This implies that for every non-negative integers i and j we have that

$$(a + i(a - b), a + i(a - b))\, \mathfrak{C}\, (a + j(a - b), a + j(a - b))\,.$$

If $b \geqslant k$ in $\mathbb{Z}$ for some integer k, then by Proposition 2.1$(viii)$ we get that any two distinct idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}[k]$ of $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent and hence Proposition 2.1$(viii)$ and Corollary 1.32 from [7] imply that for every integer n all idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}[n]$ are $\mathfrak{C}$-equivalent. This implies that all idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}[n]$ are $\mathfrak{C}$-equivalent. Since the semigroup $\mathscr{C}_{\mathbb{Z}}$ is inverse we conclude that the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ contains only one idempotent and hence by Lemma II.1.10 from [17] the semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is a group.

Suppose that two distinct elements (a,b) and (c,d) of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent. Since $\mathscr{C}_{\mathbb{Z}}$ is an inverse semigroup, Lemma III.1.1 from [17] implies that $(a,a)\mathfrak{C}(c,c)$ and $(b,b)\mathfrak{C}(d,d)$. Since $(a,b) \neq (c,d)$ we have that either $(a,a) \neq (c,c)$ or $(b,b) \neq (d,d)$, and hence by the first part of the proof we get that all idempotents of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent.

Next we shall show that if $\mathfrak{C}_{mg}$ be a least group congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$, then the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}_{mg}$ is isomorphic to the additive group of integers $\mathbb{Z}$.

By Proposition 2.1(i) and Lemma III.5.2 from [17] we have that elements (a,b) and (c,d) are $\mathfrak{C}_{mg}$-equivalent in $\mathscr{C}_{\mathbb{Z}}$ if and only if there exists an integer n such that $(a,b)\cdot(n,n) = (c,d)\cdot(n,n)$. Then Proposition 2.1$(i)$ implies that $(a,b)\cdot(g,g) = (c,d)\cdot(g,g)$ for any integer g such that $g \geqslant n$ in $\mathbb{Z}$. If $g \geqslant b$ and $g \geqslant d$ in $\mathbb{Z}$, then the semigroup operation in $\mathscr{C}_{\mathbb{Z}}$ implies that $(a,b)\cdot(g,g) = (g-b+a,g)$ and $(c,d)\cdot(g,g) = (g-d+c,g)$, and since $\mathbb{Z}$ is the additive group of integers we get that $a-b = c-d$. Converse, suppose that (a,b) and (c,d) are elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ such that $a-b = c-d$. Then for any element $g \in \mathbb{Z}$ such that $g \geqslant b$ and $g \geqslant d$ in $\mathbb{Z}$ we have that $(a,b)\cdot(g,g) = (g-b+a,g)$ and $(c,d)\cdot(g,g) = (g-d+c,g)$, and since $a-b = c-d$ we get that $(a,b)\mathfrak{C}_{mg}(c,d)$. Therefore, $(a,b)\mathfrak{C}_{mg}(c,d)$ in $\mathscr{C}_{\mathbb{Z}}$ if and only if $a-b = c-d$.

We determine a map $\mathfrak{f}\colon \mathscr{C}_{\mathbb{Z}} \to \mathbb{Z}$ by the formula $((a,b))\mathfrak{f} = a-b$, for $a,b \in \mathbb{Z}$. Proposition 2.1(vi) implies that such defined map $\mathfrak{f}\colon \mathscr{C}_{\mathbb{Z}} \to \mathbb{Z}$ is a homomorphism. Then we have that $(a,b)\mathfrak{C}_{mg}(c,d)$ if and only if $((a,b))\mathfrak{f} = ((c,d))\mathfrak{f}$, for $(a,b),(c,d) \in \mathscr{C}_{\mathbb{Z}}$, and hence the homomorphism $\mathfrak{f}$ generates the least group congruence $\mathfrak{C}_{mg}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$.

If $\mathfrak{c}$ is any congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$ then the mapping $\mathfrak{c} \mapsto \mathfrak{c} \vee \mathfrak{C}_{mg}$ maps the congruence $\mathfrak{c}$ onto a group congruence $\mathfrak{c}\vee\mathfrak{C}_{mg}$, where $\mathfrak{C}_{mg}$ is the least group congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$ (cf. [17, Section III]). Therefore every homomorphic image of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a homomorphic image of the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$, i.e., it is a homomorphic image of the additive group of integers $\mathbb{Z}$. This completes the proof of the theorem. □

3 The semigroup $\mathscr{C}_{\mathbb{Z}}$: topologizations and closures of $\mathscr{C}_{\mathbb{Z}}$ in topological semigroups

Theorem 1. *Every Hausdorff topology τ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ such that $(\mathscr{C}_{\mathbb{Z}},\tau)$ is a semitopological semigroup is discrete, and hence $\mathscr{C}_{\mathbb{Z}}$ is a discrete subspace of any semitopological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a subsemigroup.*

Proof. We fix an arbitrary idempotent (a,a) of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and suppose that (a,a) is a non-isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Since the maps $\lambda_{(a,a)}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$ and

$\rho_{(a,a)}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$ defined by the formulae $((x,y))\,\lambda_{(a,a)} = (a,a)\cdot(x,y)$ and $((x,y))\,\rho_{(a,a)} = (x,y)\cdot(a,a)$ are continuous retractions we conclude that $(a,a)\mathscr{C}_{\mathbb{Z}}$ and $\mathscr{C}_{\mathbb{Z}}(a,a)$ are closed subsets in the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. We put

$$\mathsf{DL}_{(a,a)}\,[(a,a)] = \{(x,y)\in\mathscr{C}_{\mathbb{Z}} \mid (x,y)\cdot(a,a) = (a,a)\}\,.$$

Simple verifications show that

$$\mathsf{DL}_{(a,a)}\,[(a,a)] = \{(x,x)\in\mathscr{C}_{\mathbb{Z}} \mid x \leqslant a \ \text{ in } \ \mathbb{Z}\}\,,$$

and since right translations are continuous maps in $(\mathscr{C}_{\mathbb{Z}},\tau)$ we get that $\mathsf{DL}_{(a,a)}\,[(a,a)]$ is a closed subset of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Then there exists an open neighbourhood $W_{(a,a)}$ of the point (a,a) in the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$ such that

$$W_{(a,a)} \subseteq \mathscr{C}_{\mathbb{Z}} \setminus \big((a+1,a+1)\mathscr{C}_{\mathbb{Z}} \cup \mathscr{C}_{\mathbb{Z}}(a+1,a+1) \cup \mathsf{DL}_{(a-1,a-1)}(a-1,a-1)\big).$$

Since $(\mathscr{C}_{\mathbb{Z}},\tau)$ is a semitopological semigroup we conclude that there exists an open neighbourhood $V_{(a,a)}$ of the idempotent (a,a) in the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$ such that the following conditions hold:

$$V_{(a,a)} \subseteq W_{(a,a)}, \qquad (a,a)\cdot V_{(a,a)} \subseteq W_{(a,a)} \qquad \text{and} \qquad V_{(a,a)}\cdot(a,a) \subseteq W_{(a,a)}.$$

Hence at least one of the following conditions holds:

(a) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x,y)\in\mathscr{C}_{\mathbb{Z}}$ such that $x< y \leqslant a$; or

(b) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x,y)\in\mathscr{C}_{\mathbb{Z}}$ such that $y< x \leqslant a$.

In case (a) we have that

$$(a,a)\cdot(x,y) = \big(a, a+(y-x)\big) \notin W_{(a,a)},$$

because $y-x \geqslant 1$, and in case (b) we have that

$$(x,y)\cdot(a,a) = \big(a+(x-y), a\big) \notin W_{(a,a)},$$

because $x-y \geqslant 1$, a contradiction. The obtained contradiction implies that the set $V_{(a,a)}$ is singleton, and hence the idempotent (a,a) is an isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$.

Let (a,b) be an arbitrary element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and suppose that (a,b) is a non-isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Since all right translations are continuous maps in $(\mathscr{C}_{\mathbb{Z}},\tau)$ and every idempotent (a,a) of $\mathscr{C}_{\mathbb{Z}}$ is an isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$ we conclude that

$$\mathsf{DL}_{(b,a)}\,[(a,a)] = \big\{(x,y)\in\mathscr{C}_{\mathbb{Z}} \mid (x,y)\cdot(b,a) = (a,a)\big\}$$

is a closed-and-open subset of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Simple verifications show that

$$\mathsf{DL}_{(b,a)}\,[(a,a)] = \big\{(x,y)\in\mathscr{C}_{\mathbb{Z}} \mid x-y = a-b \ \text{ and } \ x \leqslant a\big\}.$$

Then we have that

$$\{(a,b)\} = \mathsf{DL}_{(b,a)}\left[(a,a)\right] \setminus \mathsf{DL}_{(b-1,a-1)}\left[(a-1,a-1)\right],$$

and hence (a,b) is an isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. This completes the proof of the theorem. □

Theorem 1 implies the following:

Corollary 3.1. *Every Hausdorff semigroup topology τ on $\mathscr{C}_{\mathbb{Z}}$ is discrete, and hence $\mathscr{C}_{\mathbb{Z}}$ is a discrete subspace of any topological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a subsemigroup.*

Since every discrete topological space is locally compact, Theorem 1 and Theorem 3.3.9 from [9] imply the following:

Corollary 3.2. *Let T be a semitopological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a subsemigroup. Then $\mathscr{C}_{\mathbb{Z}}$ is an open subsemigroup of T.*

Lemma 3.1. *Let T be a Hausdorff semitopological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup. Let $f \in T\setminus\mathscr{C}_{\mathbb{Z}}$ be an idempotent of the semigroup T which satisfies the property: there exists an idempotent $(n,n)\in\mathscr{C}_{\mathbb{Z}}$, $n\in\mathbb{Z}$, such that $(n,n)\leqslant f$. Then the following statements hold:*

(*i*) *there exists an open neighbourhood $U(f)$ of f in T such that $U(f)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$;*

(*ii*) *f is the unit of T.*

Proof. (*i*) Let $W(f)$ be an arbitrary open neighbourhood of the idempotent f in T. We fix an arbitrary element $(n,n)\in\mathscr{C}_{\mathbb{Z}}$, $n\in\mathbb{Z}$. By Corollary 3.2 the element (n,n) is an isolated point in T, and since T is a semitopological semigroup we have that there exists an open neighbourhood $U(f)$ of f in T such that

$$U(f)\subseteq W(f), \qquad U(f)\cdot\{(n,n)\}=\{(n,n)\} \qquad \text{and} \qquad \{(n,n)\}\cdot U(f)=\{(n,n)\}.$$

If the set $U(f)$ contains a non-idempotent element $(x,y)\in\mathscr{C}_{\mathbb{Z}}$, then Proposition 2.1($vi$) implies that $(x,y)\cdot(n,n),(n,n)\cdot(x,y)\notin E(\mathscr{C}_{\mathbb{Z}})$, a contradiction. The obtained contradiction implies the statement of the assertion.

(*ii*) First we show that $f\cdot(k,l)=(k,l)\cdot f=(k,l)$ for every $(k,l)\in\mathscr{C}_{\mathbb{Z}}$.

Suppose the contrary: there exists an element $(k,l)\in\mathscr{C}_{\mathbb{Z}}$ such that $x=f\cdot(k,l)\neq(k,l)$ for some $x\in T$. Let $U(x)$ be an open neighbourhood of x in T such that $(k,l)\notin U(x)$. Since T is a semitopological semigroup we get that there exists an open neighbourhood $V(f)$ of f in T such that $V(f)\cdot\{(k,l)\}\subseteq U(x)$. Again, since for an arbitrary integer a the maps $\lambda_{(a,a)}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ and $\rho_{(a,a)}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ defined by the formulae $((x,y))\,\lambda_{(a,a)}=(a,a)\cdot(x,y)$ and $((x,y))\,\rho_{(a,a)}=(x,y)\cdot(a,a)$ are continuous retractions we conclude that statement (i) implies that there exists an open neighbourhood $W(f)$ of f in T such that $W(f)\subseteq V(f)$, $W(f)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$ and the following condition holds:

$$(p,p)\in W(f)\cap\mathscr{C}_{\mathbb{Z}} \qquad \text{if and only if} \qquad p\geqslant k.$$

Then $(p,p)\cdot(k,l)=(k,l)\notin U(x)$ for every $(p,p)\in W(f)\cap\mathscr{C}_{\mathbb{Z}}$, a contradiction. The obtained contradiction implies that $f\cdot(k,l)=(k,l)$ for every $(k,l)\in\mathscr{C}_{\mathbb{Z}}$. Similar arguments show that $(k,l)\cdot f=(k,l)$ for every $(k,l)\in\mathscr{C}_{\mathbb{Z}}$.

Next we show that $f\cdot x=x\cdot f=x$ for every $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$. Suppose the contrary: there exists an element $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$ such that $y=f\cdot x\neq x$ for some $y\in T$. Let $U(x)$ and $U(y)$ be open neighbourhoods of x and y in T, respectively, such that $U(x)\cap U(y)=\varnothing$. Since T is a semitopological semigroup we get that there exists an open neighbourhood $V(x)$ of x in T such that $V(x)\subseteq U(x)$ and $f\cdot V(x)\subseteq U(y)$. Again, since $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$ we have that the set $V(x)\cap\mathscr{C}_{\mathbb{Z}}$ is infinite, and the previous part of the proof of the statement implies that $f\cdot(V(x)\cap\mathscr{C}_{\mathbb{Z}})\subseteq(V(x)\cap\mathscr{C}_{\mathbb{Z}})$. But we have that $V(x)\cap U(y)=\varnothing$, a contradiction. The obtained contradiction implies the equality $f\cdot x=x$. Similar arguments show that $x\cdot f=x$ for every $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$. □

Remark 3.1. *We observe that the assertion (i) of Lemma 3.1 holds for right-topological and left-topological monoids.*

Lemma 3.2. *Let T be a Hausdorff topological monoid with the unit 1_T which contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup. Then the following assertions hold:*

- (i) *there exists an open neighbourhood $U(1_T)$ of the unit 1_T in T such that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$;*

and if the group of units $H(1_T)$ of T is non-singleton, then:

- (ii) *for every $x\in H(1_T)$ there exists an open neighbourhood $U(x)$ in T such that $a-b=c-d$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$;*
- (iii) *for distinct $x,y\in H(1_T)$ there exist open neighbourhoods $U(x)$ and $U(y)$ of x and y in T, respectively, such that $a-b\neq c-d$ for every $(a,b)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$ and for every $(c,d)\in U(y)\cap\mathscr{C}_{\mathbb{Z}}$;*
- (iv) *the group $H(1_T)$ is torsion free;*
- (v) *the group of units $H(1_T)$ of T is a discrete subgroup in T;*
- (vi) *the group of units $H(1_T)$ of T is isomorphic to the infinite cyclic group;*
- (vii) *every non-identity element of the group of units $H(1_T)$ in the semigroup T is not topologically periodic.*

Proof. Statement (i) follows from Lemma 3.1(i).

(ii) In the case $H(1_T)=\{1_T\}$ statement (i) implies our assertion. Hence we suppose that $H(1_T)\neq\{1_T\}$ and let $x\in H(1_T)\setminus\{1_T\}$. By statement (i) there exists an open neighbourhood $U(1_T)$ of the unit 1_T in T such that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$. Then the continuity of the semigroup operation in T implies that there exist open neighbourhoods $U(x)$ and $U(x^{-1})$ in the topological space T of x and the inverse element x^{-1} of x in $H(1_T)$, respectively, such that

$$U(x)\cdot U(x^{-1})\subseteq U(1_T)\qquad\text{and}\qquad U(x^{-1})\cdot U(x)\subseteq U(1_T).$$

Since $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$ we have that Proposition 2.1(vi) implies that $a-b+u-v=c-d+u-v$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$ and some $(u,v)\in U(x^{-1})\cap\mathscr{C}_{\mathbb{Z}}$, and hence $a-b=c-d$.

(iii) Suppose the contrary: there exist distinct $x,y\in H(1_T)$ and for all open neighbourhoods $U(x)$ and $U(y)$ of x and y in T, respectively, there are $(a,b)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$ and $(c,d)\in U(y)\cap\mathscr{C}_{\mathbb{Z}}$ such that $a-b=c-d$. The Hausdorffness of T implies that without loss of generality we can assume that $U(x)\cap U(y)=\varnothing$. Then statement ($i$) and the continuity of the semigroup operation in T imply that there exist open neighbourhoods $V(1_T)$, $V(x)$ and $V(y)$ of 1_T, x and y in T, respectively, such that

$$V(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}}),\, V(x)\subseteq U(x),\; V(y)\subseteq U(y),\; V(1_T)\cdot V(x)\subseteq U(x)$$
$$\text{and}\;\; V(1_T)\cdot V(y)\subseteq U(y).$$

Since by Theorem 1.7 from [6, Vol. 1] the sets $(a,a)T$ and $T(a,a)$ are closed in T for every idempotent $(a,a)\in\mathscr{C}_{\mathbb{Z}}$ and both neighbourhoods $V(x)$ and $V(y)$ contain infinitely many elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ we conclude that for every $(p,p)\in V(1_T)\cap\mathscr{C}_{\mathbb{Z}}$ there exist $(k,l)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$ and $(m,n)\in V(y)\cap\mathscr{C}_{\mathbb{Z}}$ such that

$$p>k>m, \qquad p>l>n \qquad \text{and} \qquad k-l=m-n.$$

Then we get that

$$(p,p)\cdot(k,l)=(p,p+(l-k)) \qquad \text{and} \qquad (p,p)\cdot(m,n)=(p,p+(n-m)),$$

a contradiction. The obtained contradiction implies our assertion.

(iv) Suppose the contrary: there exist $x\in H(1_T)\setminus\{1_T\}$ and a positive integer n such that $x^n=1_T$. Then by statement (i) there exists an open neighbourhood $U(1_T)$ of the unit 1_T in T such that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$. The continuity of the semigroup operation in T and statement (ii) imply that there exists an open neighbourhood $V(x)$ of x in T such that $a-b=c-d$ for all $(a,b),(c,d)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$ and $\underbrace{V(x)\cdot\ldots\cdot V(x)}_{n\text{-times}}\subseteq U(1_T)$. We fix an arbitrary element $(a,b)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$. If $(a,b)^n=(x,y)$, then Proposition 2.1(vi) implies that $x-y=n\cdot(a-b)$ and since $x\neq 1_T$ we get that $(x,y)\notin U(1_T)$, a contradiction. The obtained contradiction implies statement (iv).

(v) Statement (iv) implies that the group of units $H(1_T)$ is infinite.

We fix an arbitrary $x\in H(1_T)$ and suppose that x is not an isolated point of $H(1_T)$. Then by statement (ii) there exists an open neighbourhood $U(x)$ in T such that $a-b=c-d$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Since the point x is not isolated in $H(1_T)$ we conclude that there exists $y\in H(1_T)$ such that $y\in U(x)$. Hence the set $U(x)$ is an open neighbourhood of y in T. Statement (iii) implies that there exist open neighbourhoods $W(x)\subseteq U(x)$ and $W(y)\subseteq U(x)$ of x and y in T, respectively, such that $a-b\neq c-d$ for every $(a,b)\in W(x)\cap\mathscr{C}_{\mathbb{Z}}$ and for every $(c,d)\in W(y)\cap\mathscr{C}_{\mathbb{Z}}$. This contradicts the choice of the neighbourhood $U(x)$. The obtained contradiction implies that every $x\in H(1_T)$ is an isolated point of $H(1_T)$.

(vi) Since the group of units $H(1_T)$ is not trivial, i.e., the group $H(1_T)$ is non-singleton, we fix an arbitrary $x\in H(1_T)\setminus\{1_T\}$. Then by statement ($iv$) we have that $x^n\neq 1_T$ for any positive integer n. Statement (ii) implies that there exists an open neighbourhood $U(x)$ in T such that $a-b=c-d$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. We define the map $\varphi\colon H(1_T)\to\mathbb{Z}$ by the following way: $(x)\varphi=k$ if and only if $a-b=k$ for every $(a,b)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Then statement (iv) and Proposition 2.1(vi) imply that the map $\varphi\colon H(1_T)\to\mathbb{Z}$ is an injective homomorphism. Obviously that $(H(1_T))\,\varphi$ is a subgroup in the additive group of integers. We fix the least positive integer $p\in(H(1_T))\,\varphi$. Then the element p generates the subgroup $(H(1_T))\,\varphi$ in the additive group of integers $\mathbb{Z}$, and hence the group $(H(1_T))\,\varphi$ is cyclic.

(vii) We fix an arbitrary element $x\in H(1_T)\setminus\{1_T\}$. Suppose the contrary: x is a topologically periodic element of S. Then there exist open neighbourhoods $U(1_T)$ and $U(x)$ of 1_T and x in T, respectively, such that $U(1_T)\cap U(x)=\varnothing$. Statements (i) and (iii) imply that without loss of generality we can assume that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$, and $a-b=c-d\neq 0$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Then the topologically periodicity of x implies that there exists a positive integer n such that $x^n\in U(1_T)$. Since the semigroup operation in T is continuous we conclude that there exists an open neighbourhood $V(x)$ of x in T such that $\underbrace{V(x)\cdot\ldots\cdot V(x)}_{n\text{-times}}\subseteq U(1_T)$. We fix an arbitrary element $(a,b)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$. Then we have that $(a,b)^n\in U(1_T)\cap\mathscr{C}_{\mathbb{Z}}$ and hence $n(a-b)=0$, a contradiction. The obtained contradiction implies assertion (vii). □

Proposition 3.1. *Let G be non-trivial subgroup of the additive group of integers $\mathbb{Z}$ and $n\in\mathbb{Z}$. Then the subsemigroup H which is generated by the set $\{n\}\cup G$ is a cyclic subgroup of $\mathbb{Z}$.*

Proof. Without loss of generality we can assume that $n\in\mathbb{Z}\setminus G$ and $n>0$.

Since every subgroup of a cyclic group is cyclic (see [14, P. 47]), we have that G is a cyclic subgroup in $\mathbb{Z}$. We fix a generating element k of G such that $k>0$. Then we have that

$$(\underbrace{n+\cdots+n}_{(k-1)\text{-times}})-(\underbrace{k+\cdots+k}_{n\text{-times}})+n=0,$$

and hence we have that $-n\in H$. Since $\mathbb{Z}$ is a commutative group we conclude that H is a subgroup in $\mathbb{Z}$, which is generated by elements n and k, and hence H is a cyclic subgroup in $\mathbb{Z}$. □

Proposition 3.2. *Let T be a Hausdorff topological monoid with the unit 1_T which contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup. Then the following assertions hold:*

(i) *if the set $L_{\mathscr{C}_{\mathbb{Z}}}=\{x\in T\setminus\mathscr{C}_{\mathbb{Z}}\mid$ there exists $y\in\mathscr{C}_{\mathbb{Z}}$ such that $x\cdot y\in\mathscr{C}_{\mathbb{Z}}\}$ is non-empty, then $L_{\mathscr{C}_{\mathbb{Z}}}$ is a subsemigroup of T, and moreover if $a\in L_{\mathscr{C}_{\mathbb{Z}}}$, then there exists an open neighbourhood $U(a)$ of a in T such that $n_1-m_1=n_2-m_2$ for all $(n_1,m_1),(n_2,m_2)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$;*

(ii) *if the set $R_{\mathscr{C}_{\mathbb{Z}}}=\{x\in T\setminus\mathscr{C}_{\mathbb{Z}}\mid$ there exists $y\in\mathscr{C}_{\mathbb{Z}}$ such that $y\cdot x\in\mathscr{C}_{\mathbb{Z}}\}$ is non-empty, then $R_{\mathscr{C}_{\mathbb{Z}}}$ is a subsemigroup of T, and moreover if $a\in R_{\mathscr{C}_{\mathbb{Z}}}$, then there exists an open neighbourhood $U(a)$ of a in T such that $n_1-m_1=n_2-m_2$ for all $(n_1,m_1),(n_2,m_2)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$;*

(*iii*) *if the set* $L_{\mathscr{C}_{\mathbb{Z}}}$ *(resp.,* $R_{\mathscr{C}_{\mathbb{Z}}}$*) is non-empty, then for every* $a \in L_{\mathscr{C}_{\mathbb{Z}}}$ *(resp.,* $a \in R_{\mathscr{C}_{\mathbb{Z}}}$*) there exist an open neighbourhood* $U(a)$ *of* a *in* T *and an integer* n_a *such that* $p \leqslant n_a$ *and* $q \leqslant n_a$ *for all* $(p,q) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$*;*

(*iv*) $L_{\mathscr{C}_{\mathbb{Z}}} = R_{\mathscr{C}_{\mathbb{Z}}}$*;*

(*v*) ${\uparrow}\mathscr{C}_{\mathbb{Z}} = \mathscr{C}_{\mathbb{Z}} \cup L_{\mathscr{C}_{\mathbb{Z}}}$ *is a subsemigroup of* T *and* $\mathscr{C}_{\mathbb{Z}}$ *is a minimal ideal in* ${\uparrow}\mathscr{C}_{\mathbb{Z}}$*;*

(*vi*) *if for an element* $a \in T \setminus \mathscr{C}_{\mathbb{Z}}$ *there is an open neighbourhood* $U(a)$ *of* a *in* T *and the following conditions hold:*

  (*a*) $m_1 - m_2 = n_1 - n_2$ *for all* $(m_1,n_1),(m_2,n_2) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$*; and*

  (*b*) *there exists an integer* n_a *such that* $n \leqslant n_a$ *and* $m \leqslant n_a$ *for every* $(m,n) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$*,*

  then $a \in L_{\mathscr{C}_{\mathbb{Z}}}$*;*

(*vii*) *if* $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}} \neq \varnothing$*, then* I *is an ideal of* T*;*

(*viii*) *the set*

$$\begin{aligned}{\uparrow}(a,b) &= \{x \in T \mid x \cdot (b,b) = (a,b)\}\\ &= \{x \in T \mid (a,a) \cdot x = (a,b)\}\\ &= \{x \in T \mid (a,a) \cdot x \cdot (b,b) = (a,b)\}\end{aligned}$$

is closed-and-open in T *for every* $(a,b) \in \mathscr{C}_{\mathbb{Z}}$*;*

(*ix*) *the set* ${\uparrow}(a,b) \cap L_{\mathscr{C}_{\mathbb{Z}}}$ *is either singleton or empty;*

(*x*) $L_{\mathscr{C}_{\mathbb{Z}}}$ *is isomorphic to a submonoid of the additive group of integers* $\mathbb{Z}$*, and moreover if a maximal subgroup of* $L_{\mathscr{C}_{\mathbb{Z}}}$ *is non-singleton, then* $L_{\mathscr{C}_{\mathbb{Z}}}$ *is isomorphic to the additive group of integers* $\mathbb{Z}$*;*

(*xi*) ${\uparrow}\mathscr{C}_{\mathbb{Z}}$ *is an open subset in* T*, and hence if* $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}} \neq \varnothing$*, then the ideal* I *is a closed subset in* T*;*

(*xii*) *if the semigroup* T *contains a non-singleton group of units* $H(1_T)$*, then* $H(1_T) = T \setminus (\mathscr{C}_{\mathbb{Z}} \cup I)$*.*

Proof. (*i*) We observe that since $\mathscr{C}_{\mathbb{Z}}$ is an inverse semigroup we conclude that $x \in L_{\mathscr{C}_{\mathbb{Z}}}$ if and only if there exists an idempotent $e \in \mathscr{C}_{\mathbb{Z}}$ such that $x \cdot e \in \mathscr{C}_{\mathbb{Z}}$, for $x \in T$.

We fix an arbitrary $x \in L_{\mathscr{C}_{\mathbb{Z}}}$. Let (n,n) be an idempotent in $\mathscr{C}_{\mathbb{Z}}$ such that $(a,b) = x \cdot (n,n) \in \mathscr{C}_{\mathbb{Z}}$. Then by Corollary 3.1 we have that (n,n) and (a,b) are isolated points in T, and the continuity of the semigroup operation in T implies that there exists an open neighbourhood $U(x)$ of x in T such that

$$U(x) \cdot \{(n,n)\} = \{(a,b)\} \in \mathscr{C}_{\mathbb{Z}}.$$

Then Proposition 2.1(vi) implies that $p-q=a-b$ for all $(p,q)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Also, since

$$(p,q)(n,n)=\begin{cases}(p-q+n,n), & \text{if } q\leqslant n;\\ (p,q), & \text{if } q\geqslant n\end{cases} \tag{2}$$

we have that $q\leqslant n=b$.

Suppose that $x,y\in L_{\mathscr{C}_{\mathbb{Z}}}$, and (i,i) and (j,j) are idempotents in $\mathscr{C}_{\mathbb{Z}}$ such that $x\cdot(i,i)=(k,l)\in\mathscr{C}_{\mathbb{Z}}$ and $y\cdot(j,j)\in\mathscr{C}_{\mathbb{Z}}$, $i,j,k,l\in\mathbb{Z}$. We fix an arbitrary integer d such that $d\geqslant\max\{k,j\}$. Then we have that

$$\begin{aligned}(y\cdot x)\cdot((i,i)\cdot(l,k)\cdot(d,d)) &= y\cdot(x\cdot(i,i)\cdot(l,k)\cdot(d,d))\\ &= y\cdot((k,l)\cdot(l,k)\cdot(d,d))\\ &= y\cdot((k,k)\cdot(d,d))\\ &= y\cdot(d,d)\\ &= y\cdot((j,j)\cdot(d,d))\\ &= (y\cdot(j,j))\cdot(d,d)\in\mathscr{C}_{\mathbb{Z}}.\end{aligned}$$

This implies that $L_{\mathscr{C}_{\mathbb{Z}}}$ is a subsemigroup of T and completes the proof of our assertion.

The proof of assertion (ii) is similar to (i).

Statement (i) and formula (2) imply assertion (iii). In the case $a\in R_{\mathscr{C}_{\mathbb{Z}}}$ the proof is similar.

(iv) Let be $L_{\mathscr{C}_{\mathbb{Z}}}\neq\varnothing$. We fix an arbitrary element $a\in L_{\mathscr{C}_{\mathbb{Z}}}$. Then there exists an idempotent $(i_a,i_a)\in\mathscr{C}_{\mathbb{Z}}$ such that $a\cdot(i_a,i_a)=(i,j)\in\mathscr{C}_{\mathbb{Z}}$. Assertion (iii) implies that there exist an open neighbourhood $U(a)$ of a in T and an integer n_a such that $n-m=i-j$, $n\leqslant n_a$ and $m\leqslant n_a$ for all $(n,m)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$. Without loss of generality we can assume that $i_a\geqslant n_a$.

We shall show that $(i_a,i_a)\cdot a\in\mathscr{C}_{\mathbb{Z}}$. Suppose the contrary: $(i_a,i_a)\cdot a=b\in T\setminus\mathscr{C}_{\mathbb{Z}}$. Assertion (iii) implies that there exist integers

$$n_0(a)=\max\{n\mid(n,m)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}\}\qquad\text{and}\qquad m_0(a)=\max\{m\mid(n,m)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}\}.$$

Since $i_a\geqslant n_a$ we have that

$$(i_a,i_a)\cdot(n_0(a),m_0(a))=(i_a,i_a-n_0(a)+m_0(a)).$$

Let $W(b)$ be an open neighbourhood of b in T such that $(i_a,i_a-n_0(a)+m_0(a))\notin W(b)$. Then the continuity of the semigroup operation in T implies that there exists an open neighbourhood $V(a)$ of a in T such that

$$V(a)\subseteq U(a)\qquad\text{and}\qquad\{(i_a,i_a)\}\cdot V(a)\subseteq W(b).$$

We fix an arbitrary element $(n,m)\in V(a)\cap\mathscr{C}_{\mathbb{Z}}$. Then we have that

$$(i_a,i_a)\cdot(n,m)=(i_a,i_a-n+m)=(i_a,i_a-n_0(a)+m_0(a)),$$

a contradiction. The obtained contradiction implies that $a\in R_{\mathscr{C}_{\mathbb{Z}}}$, and hence we have that $L_{\mathscr{C}_{\mathbb{Z}}}\subseteq R_{\mathscr{C}_{\mathbb{Z}}}$.

The proof of the inclusion $R_{\mathscr{C}_{\mathbb{Z}}} \subseteq L_{\mathscr{C}_{\mathbb{Z}}}$ is similar.

Statement (v) follows from statements $(i) - (iv)$ and Proposition 2.1(v).

(vi) Let $U(a)$ be an open neighbourhood of a in T such that conditions (a) and (b) hold, and let n_a be such integer as in condition (b). Then for all $(m_1, n_1), (m_2, n_2) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$ we have that

$$(m_1, n_1) \cdot (n_a, n_a) = (m_1 - n_1 + n_a, n_a) = (m_2 - n_2 + n_a, n_a) = (m_2, n_2) \cdot (n_a, n_a),$$

and hence the continuity of the semigroup operation in T implies that $a \in L_{\mathscr{C}_{\mathbb{Z}}}$.

(vii) Statements (i) and (iii) imply that $a \cdot (m, n) \in I$ and $(m, n) \cdot a \in I$ for all $a \in I$ and $(m, n) \in \mathscr{C}_{\mathbb{Z}}$.

Fix arbitrary elements $a, b \in I$. We consider the following two cases:

$$1) \;\; a \cdot b \in \mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad 2) \;\; a \cdot b \in L_{\mathscr{C}_{\mathbb{Z}}}.$$

In case 1) we put $a \cdot b = (m, n) \in \mathscr{C}_{\mathbb{Z}}$. Then the continuity of the semigroup operation in T implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of a and b in T, respectively, such that

$$U(a) \cdot U(b) = \{(m, n)\}.$$

Since a and b are accumulation points of $\mathscr{C}_{\mathbb{Z}}$ in T, we conclude that there exist $(m_a, n_a) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$ and $(m_b, n_b) \in U(b) \cap \mathscr{C}_{\mathbb{Z}}$. Hence we have that

$$(m_a, n_a) \cdot b \in \{(m_a, n_a)\} \cdot U(b) \subseteq U(a) \cdot U(b) = \{(m, n)\}$$

and

$$a \cdot (m_b, n_b) \in U(a) \cdot \{(m_b, n_b)\} \subseteq U(a) \cdot U(b) = \{(m, n)\}$$

This implies that $a, b \in L_{\mathscr{C}_{\mathbb{Z}}}$, a contradiction.

Suppose case 2) holds and $a \cdot b = x \in L_{\mathscr{C}_{\mathbb{Z}}}$. Then by statements (i) and (iii) we have that there exist an open neighbourhood $U(x)$ of x in T and an integer n_x such that $m_1 - n_1 = m_2 - n_2$, $m_1 \leqslant n_x$ and $n_1 \leqslant n_x$ for all $(m_1, n_1), (m_2, n_2) \in U(x) \cap \mathscr{C}_{\mathbb{Z}}$. Also, the continuity of the semigroup operation in T implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of a and b in T, respectively, such that

$$U(a) \cdot U(b) \subseteq U(x).$$

Since $U(a) \cap \mathscr{C}_{\mathbb{Z}} \neq \varnothing$ and $U(b) \cap \mathscr{C}_{\mathbb{Z}} \neq \varnothing$, we can find arbitrary elements $(m_a, n_a) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$ and $(m_b, n_b) \in U(b) \cap \mathscr{C}_{\mathbb{Z}}$. Then by Proposition 2.1(vi) we have that

$$x_a - y_a + m_b - n_b = m_1 - n_1 \qquad \text{and} \qquad m_a - n_a + x_b - y_b = m_1 - n_1$$

for all $(x_a, y_a) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$ and $(x_b, y_b) \in U(b) \cap \mathscr{C}_{\mathbb{Z}}$. This implies that there exist integers k_a and k_b such that

$$x_a - y_a = k_a \qquad \text{and} \qquad x_b - y_b = k_b$$

for all $(x_a, y_a) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$ and $(x_b, y_b) \in U(b) \cap \mathscr{C}_{\mathbb{Z}}$. Then by statement (vi) we have that $a, b \in L_{\mathscr{C}_{\mathbb{Z}}}$, a contradiction.

The obtained contradictions imply that $a \cdot b \in I$, and hence we get that the set I is an ideal of T.

$(viii)$ Proposition 2.1(vi) and assertion (vi) imply the following equalities:

$$\{x \in T \mid x \cdot (b,b) = (a,b)\} = \{x \in T \mid (a,a) \cdot x = (a,b)\} = \{x \in T \mid (a,a) \cdot x \cdot (b,b) = (a,b)\}.$$

Since by Corollary 3.1 every element (a,b) of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is an isolated point in T, the continuity of the semigroup operation in T implies that $\uparrow(a,b)$ is a closed-and-open subset in T.

(ix) Suppose that the set $\uparrow(a,b) \cap L_{\mathscr{C}_{\mathbb{Z}}}$ is non-empty. Assuming that the set $\uparrow(a,b) \cap L_{\mathscr{C}_{\mathbb{Z}}}$ is non-singleton implies that there exist distinct $x, y \in \uparrow(a,b) \cap L_{\mathscr{C}_{\mathbb{Z}}}$. Then the Hausdorffness of T implies that there exist disjoint open neighbourhoods $U(x)$ and $U(y)$ of x and y in T, respectively. By the continuity of the semigroup operation in T we can find open neighbourhoods $V(1_T)$, $V(x)$ and $V(y)$ of 1_T, x and y in T, respectively, such that the following conditions hold:

$$V(x) \subseteq U(x), \quad V(y) \subseteq U(y), \quad V(1_T) \cdot V(x) \subseteq U(x) \quad \text{and} \quad V(1_T) \cdot V(y) \subseteq U(y).$$

By assertions $(i)-(iii)$ we can find the integers n, n_1, n_2, m_1 and m_2 such that

$$(n,n) \in V(1_T), \quad (n_1,n_2) \in V(x), \quad (m_1,m_2) \in V(y), \quad n_1 - n_2 = m_1 - m_2,$$
$$n \geqslant n_1 \quad \text{and} \quad n \geqslant m_1.$$

Then we have that

$$(n,n) \cdot (n_1,n_2) = (n, n - n_1 + n_2) = (n, n - m_1 + m_2) = (n,n) \cdot (m_1,m_2),$$

and hence $(V(1_T) \cdot V(x)) \cdot (V(1_T) \cdot V(y)) \neq \varnothing$, a contradiction. The obtained contradiction implies that $x = y$.

(x) Statement (vii) implies that $T \setminus (I \cup \mathscr{C}_{\mathbb{Z}}) = L_{\mathscr{C}_{\mathbb{Z}}}$. Let $\mathbb{Z}$ be the additive group of integers. We define a map $\mathfrak{h}\colon L_{\mathscr{C}_{\mathbb{Z}}} \to \mathbb{Z}$ as follows:

$$(x)\mathfrak{h} = n \qquad \begin{array}{l}\text{if and only if there exists a neighbourhood } U(x) \text{ of } x \text{ in } T \text{ such that}\\ a - b = n, \text{ for all } (a,b) \in U(x) \cap \mathscr{C}_{\mathbb{Z}},\end{array}$$

where $x \in L_{\mathscr{C}_{\mathbb{Z}}}$. We observe that assertions $(i)-(v)$ imply that the map $\mathfrak{h}$ is well defined. Also, Proposition 2.1 implies that $\mathfrak{h}\colon L_{\mathscr{C}_{\mathbb{Z}}} \to \mathbb{Z}$ is a monomorphism, and hence $L_{\mathscr{C}_{\mathbb{Z}}}$ is a submonoid of $\mathbb{Z}$. In the case when a maximal subgroup of $L_{\mathscr{C}_{\mathbb{Z}}}$ is non-singleton Proposition 3.1 implies that $(L_{\mathscr{C}_{\mathbb{Z}}})\,\mathfrak{h}$ is a cyclic subgroup of $\mathbb{Z}$. This completes the proof of our assertion.

(xi) Assertion (v) implies that

$$\uparrow\mathscr{C}_{\mathbb{Z}} = \{x \in T \mid \text{there exists } y \in \mathscr{C}_{\mathbb{Z}} \text{ such that } x \cdot y \in \mathscr{C}_{\mathbb{Z}}\} = \bigcup_{(a,b) \in \mathscr{C}_{\mathbb{Z}}} \uparrow(a,b).$$

Then assertion $(viii)$ implies that $\uparrow\mathscr{C}_{\mathbb{Z}}$ is an open subset in T and hence by assertion (vii) we get that the ideal I is a closed subset of T.

Assertion (xii) follows from (x). $\square$

4 On a closure of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a locally compact topological inverse semigroup

For every non-negative integer k by $k\mathbb{Z}$ we denote a subgroup of the additive group of integers $\mathbb{Z}$ which is generated by an element $k \in \mathbb{Z}$. We observe if $k = 0$ then the group $k\mathbb{Z}$ is trivial. Also, we denote $G_0 = \mathbb{Z}$ and $G_1(k) = k\mathbb{Z}$ for a positive integer k.

The following five examples illustrate distinct structures of a closure of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a locally compact topological inverse semigroup.

Example 1. *Let be $S_1 = G_1(0) \sqcup \mathscr{C}_{\mathbb{Z}}$. Then $G_1(0)$ is a trivial group and we put $\{e_1\} = G_1(0)$. We extend the semigroup operation from $\mathscr{C}_{\mathbb{Z}}$ onto S_1 as follows:*

$$e_1 \cdot (a,b) = (a,b) \cdot e_1 = (a,b) \in \mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad e_1 \cdot e_1 = e_1,$$

i.e., S_1 is the semigroup $\mathscr{C}_{\mathbb{Z}}$ with the adjoined unit e_1. We fix an arbitrary decreasing sequence $\{m_i\}_{i\in\mathbb{N}}$ of negative integers and for every positive integer n we put

$$U_n(e_1) = \{e_1\} \cup \{(m_i, m_i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant n\}.$$

Then we determine a topology τ_1 on S_1 as follows:

1) *all elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are isolated points in (S_1, τ_1); and*
2) *the family $\mathscr{B}_1(e_1) = \{U_n(e_1) \mid n \in \mathbb{N}\}$ is a base of the topology τ_1 at the point $e_1 \in G_1(0) \subseteq S_1$.*

Then for every positive integer n we have that

$$U_n(e_1) \cdot U_n(e_1) = U_n(e_1) \qquad \text{and} \qquad (U_n(e_1))^{-1} = U_n(e_1).$$

Let (m,n) be an arbitrary element of the semigroup $\mathscr{C}_{\mathbb{Z}}$. We fix a positive integer $i_{(m,n)}$ such that $m_{i_{(m,n)}} \leqslant m$ and $m_{i_{(m,n)}} \leqslant n$. Then we have that

$$U_{i_{(m,n)}}(e_1) \cdot \{(m,n)\} = \{(m,n)\} \qquad \text{and} \qquad \{(m,n)\} \cdot U_{i_{(m,n)}}(e_1) = \{(m,n)\}.$$

Hence we get that (S_1, τ_1) is a topological inverse semigroup. Obviously, (S_1, τ_1) is a Hausdorff locally compact space.

Example 2. *Let k and n be any positive integers such that $n \in \{1, \ldots, k\}$ is a divisor of k and we put $k = n \cdot s$, where s is some positive integer. We put $S_2 = G_1(k) \sqcup \mathscr{C}_{\mathbb{Z}}$. Later an element of the group $G_1(k) = k\mathbb{Z}$ will be denote by ki, where $i \in \mathbb{Z}$. We extend the semigroup operation from $\mathscr{C}_{\mathbb{Z}}$ onto S_2 by the following way:*

$$ki \cdot (a,b) = (-ki + a, b) \in \mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad (a,b) \cdot ki = (a, b + ki) \in \mathscr{C}_{\mathbb{Z}},$$

for arbitrary $(a,b) \in \mathscr{C}_{\mathbb{Z}}$ and $ki \in G_1(k)$. To see that the extended binary operation is associative we need only check six possibilities, the other being evident.

Then for arbitrary $ki_1, ki_2 \in G_1(k)$ and $(a,b), (c,d) \in \mathscr{C}_{\mathbb{Z}}$ we have that:

1) $(ki_1 \cdot ki_2) \cdot (a,b) = (ki_1 + ki_2)(a,b) = (-ki_1 - ki_2 + a, b) = ki_1 \cdot (-ki_2 + a, b) = ki_1 \cdot (ki_2 \cdot (a,b))$;

2) $(a,b) \cdot (ki_1 \cdot ki_2) = (a,b) \cdot (ki_1 + ki_2) = (a, b + ki_1 + ki_2) = (a, b + ki_1) \cdot ki_2 = ((a,b) \cdot ki_1) \cdot ki_2$;

3) $(ki_1 \cdot (a,b)) \cdot ki_2 = (-ki_1 + a, b) \cdot ki_2 = (-ki_1 + a, b + ki_2) = ki_1 \cdot (a, b + ki_2) = ki_1 \cdot ((a,b) \cdot ki_2)$;

4) $(ki_1 \cdot (a,b)) \cdot (c,d) = (-ki_1 + a, b) \cdot (c,d) = \begin{cases} (-ki_1 + a - b + c, d), & \textit{if } b \leqslant c; \\ (-ki_1 + a, b - c + d), & \textit{if } b \geqslant c \end{cases}$
$= \begin{cases} ki_1 \cdot (a - b + c, d), & \textit{if } b \leqslant c; \\ ki_1 \cdot (a, b - c + d), & \textit{if } b \geqslant c \end{cases} = ki_1 \cdot ((a,b) \cdot (c,d))$;

5) $((a,b) \cdot (c,d)) \cdot ki_1 = \begin{cases} (a - b + c, d) \cdot ki_1, & \textit{if } b \leqslant c; \\ (a, b - c + d) \cdot ki_1, & \textit{if } b \geqslant c \end{cases}$
$= \begin{cases} (a - b + c, d + ki_1), & \textit{if } b \leqslant c; \\ (a, b - c + d + ki_1), & \textit{if } b \geqslant c \end{cases} = (a,b) \cdot (c, d + ki_1) = (a,b) \cdot ((c,d) \cdot ki_1)$;

6) $((a,b) \cdot ki_1) \cdot (c,d) = (a, b + ki_1) \cdot (c,d) = \begin{cases} (a - b - ki_1 + c, d), & \textit{if } b + ki_1 \leqslant c; \\ (a, b + ik_1 - c + d), & \textit{if } b + ki_1 \geqslant c \end{cases}$
$= \begin{cases} (a - b - ki_1 + c, d), & \textit{if } b \leqslant -ki_1 + c; \\ (a, b + ki_1 - c + d), & \textit{if } b \geqslant -ki_1 + c \end{cases} = (a,b) \cdot (-ki_1 + c, d) = (a,b) \cdot (ki_1 \cdot (c,d))$.

Also simple verifications show that S_2 is an inverse semigroup.

Let ki be an arbitrary element of the group $G_1(k)$. For every positive integer j we denote

$$U_j^n(ki) = \{ki\} \cup \{(-nq, -nq + ki) \mid q \geqslant j, q \in \mathbb{N}\}.$$

We determine a topology τ_2 on S_2 as follows:

1) *all elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are isolated points in (S_2, τ_2);* and

2) *the family $\mathscr{B}_2(ki) = \left\{U_j^n(ki) \mid j \in \mathbb{N}\right\}$ is a base of the topology τ_2 at the point $ki \in G_1(k) \subseteq S_2$.*

Then for every positive integer j we have that

$$U_j^n(ki_1) \cdot U_{j-i_1 s}^n(ki_2) \subseteq U_j^n(ki_1 + ki_2) \qquad \text{and} \qquad \left(U_j^n(ki_1)\right)^{-1} = U_j^n(-ki_1),$$

for $ki_1, ki_2 \in G_1(k)$.

Let (a,b) be an arbitrary element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and $ki \in G_1(k)$. Then we have that

$$U_j^n(ki) \cdot \{(a,b)\} = \{(a - ki, b)\} \qquad \text{and} \qquad \{(a,b)\} \cdot U_j^n(ki) = \{(a, b + ki)\},$$

for every positive integer j such that $nj \geqslant \max\{-b; ki - a\}$.

Therefore (S_2, τ_2) is a topological inverse semigroup, and moreover the topological space (S_2, τ_2) is Hausdorff and locally compact.

Example 3. *We put $S_3 = \mathscr{C}_{\mathbb{Z}} \sqcup G_0$ and extend the semigroup operation from the semigroup $\mathscr{C}_{\mathbb{Z}}$ onto S_3 by the following way:*

$$(a,b)\cdot n = n\cdot(a,b) = n+b-a \in G_0,$$

for all $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ and $n\in G_0$. To see that the extended binary operation is associative we need only check two possibilities, the other being evident.

Then for arbitrary $m,n\in G_0$ and $(a,b),(c,d)\in\mathscr{C}_{\mathbb{Z}}$ we have that:

1) $(n\cdot(a,b))\cdot(c,d) = (n+b-a)\cdot(c,d) = n+b-a+d-c = \begin{cases} n\cdot(a-b+c,d), & \text{if } b\leqslant c;\\ n\cdot(a,b-c+d), & \text{if } b\geqslant c\end{cases}$ $= n\cdot((a,b)\cdot(c,d))$;

2) $(m\cdot n)\cdot(a,b) = m+n+b-a = m\cdot(n+b-a) = m\cdot(n\cdot(a,b))$.

This completes the proof of the associativity of such defined binary operation on S_3. Also, we observe that S_3 with such defined semigroup operation is an inverse semigroup.

For every positive integer n and every element $k\in G_0$ we put:

$$U_n(k) = \begin{cases} \{k\}\cup\{(a,a+k)\mid a=n,n+1,n+2,\ldots\}, & \text{if } k\geqslant 0;\\ \{k\}\cup\{(a-k,a)\mid a=n,n+1,n+2,\ldots\}, & \text{if } k\leqslant 0.\end{cases}$$

We determine a topology τ_3 on S_3 as follows:

1) *all elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are isolated points in (S_3,τ_3); and*

2) *the family $\mathscr{B}_3(k)=\{U_n(k)\mid n\in\mathbb{N}\}$ is a base of the topology τ_3 at the point $k\in G_0\subseteq S_3$.*

Then for all $k_1,k_2\in G_0$ we have that

$$U_{2n}(k_1)\cdot U_{2n}(k_2)\subseteq U_n(k_1+k_2),$$

for every positive integer $n\geqslant\max\{|k_1|,|k_2|\}$, and

$$(U_i(k_1))^{-1} = U_i(-k_1),$$

for every positive integer i. Also, for arbitrary $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ and $k\in G_0$ we have that

$$(a,b)\cdot U_{2n}(k)\subseteq U_n(k+b-a) \qquad \text{and} \qquad U_{2n}(k)\cdot(a,b)\subseteq U_n(k+b-a),$$

for every positive integer $n\geqslant\max\{|a|,|b|,|k|\}$.

This completes the proof that (S_3,τ_3) is a topological inverse semigroup. Obviously, (S_3,τ_3) is a Hausdorff locally compact space.

Example 4. *Let be $S_4 = G_1(0)\sqcup S_3$, where the group $G_1(0)$ and the semigroup S_3 are defined in Example 1 and Example 3, respectively. We extend the semigroup operation from S_3 onto S_4 as follows:*

$$e_1\cdot x = x\cdot e_1 = x\in\mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad e_1\cdot e_1 = e_1,$$

i.e., S_4 is the semigroup S_3 with the adjoined unit e_1.

Let τ_4 be a topology on S_4 which is generated by the family $\tau_1 \cup \tau_3$ (see Examples 1 and 3). Then for every element $k_0 \in G_0$ and every positive integers n_1 and n_0 we have that the following inclusions hold:

$$U_{n_1}(e_1) \cdot U_{n_0}(k_0) \subseteq U_{n_0}(k_0) \qquad \text{and} \qquad U_{n_0}(k_0) \cdot U_{n_1}(e_1) \subseteq U_{n_0}(k_0),$$

where $U_{n_1}(e_1) \in \mathscr{B}_1(e_1)$ and $U_{n_0}(k_0) \in \mathscr{B}_3(k_0)$ (see Examples 1 and 3). These inclusions and Examples 1 and 3 imply that (S_4, τ_4) is a Hausdorff topological inverse semigroup. Obviously, (S_4, τ_4) is a locally compact space.

Example 5. *Let k and n be such positive integers as in Example 2. We put $S_5 = G_1(k) \sqcup \mathscr{C}_{\mathbb{Z}} \sqcup G_0$ and extend semigroup operation from S_2 and S_3 onto S_5 as follows. Later we denote elements of groups $G_1(K)$ and G_0 by $(ki)^1$ and $(n)^0$, respectively. We put*

$$(ki)^1 \cdot (n)^0 = (n)^0 \cdot (ki)^1 = (ki + n)^0 \in G_0,$$

for all $(ki)^1 \in G_1(k)$ and $(n)^0 \in G_0$. To see that the extended binary operation is associative we need only check twelve possibilities, the other either are evident or are proved in Examples 2 and 3.

Then for arbitrary $(ki_1)^1, (ki_2)^1 \in G_1(k)$, $(n_1)^0, (n_2)^0 \in G_0$ and $(a,b) \in \mathscr{C}_{\mathbb{Z}}$ we have that:

1) $((n_1)^0 \cdot (n_2)^0) \cdot (ki_1)^1 = (n_1+n_2)^0 \cdot (ki_1)^1 = (n_1+n_2+ki_1)^0 = (n_1)^0 \cdot (n_2+ki_1)^0$ $= (n_1)^0 \cdot ((n_2)^0 \cdot (ki_1)^1)$;

2) $((n_1)^0 \cdot (ki_1)^1) \cdot (n_2)^0 = (n_1+ki_1)^0 \cdot (n_2)^0 = (n_1+ki_1+n_2)^0 = (n_1)^0 \cdot (ki_1+n_2)^0$ $= (n_1)^0 \cdot ((ki_1)^1 \cdot (n_2)^0)$;

3) $((n_1)^0 \cdot (ki_1)^1) \cdot (ki_2)^1 = (n_1+ki_1)^0 \cdot (ki_2)^1 = (n_1+ki_1+ki_2)^0 = (n_1)^0 \cdot (ki_1+ki_2)^1$ $= (n_1)^0 \cdot ((ki_1)^1 \cdot (ki_2)^1)$;

4) $((n_1)^0 \cdot (ki_1)^1) \cdot (a,b) = (n_1+ki_1)^0 \cdot (a,b) = (n_1+ki_1+b-a)^0 = (n_1)^0 \cdot (-ki_1+a,b)$ $= (n_1)^0 \cdot ((ki_1)^1 \cdot (a,b))$;

5) $((n_1)^0 \cdot (a,b)) \cdot (ki_1)^1 = (n_1+b-a)^0 \cdot (ki_1)^1 = (n_1+b-a+ki_1)^0 = (n_1)^0 \cdot (a, b+ki_1)$ $= (n_1)^0 \cdot ((a,b) \cdot (ki_1)^1)$;

6) $((ki_1)^1 \cdot (n_1)^0) \cdot (n_2)^0 = (ki_1+n_1)^0 \cdot (n_2)^0 = (ki_1+n_1+n_2)^0 = (ki_1)^1 \cdot (n_1+n_2)^0$ $= (ki_1)^1 \cdot ((n_1)^0 \cdot (n_2)^0)$;

7) $((ki_1)^1 \cdot (n_1)^0) \cdot (ki_2)^1 = (ki_1+n_1)^0 \cdot (ki_2)^1 = (ki_1+n_1+ki_2)^0 = (ki_1)^1 \cdot (n_1+ki_2)^0$ $= (ki_1)^1 \cdot ((n_1)^0 \cdot (ki_2)^1)$;

8) $((ki_1)^1 \cdot (n_1)^0) \cdot (a,b) = (ki_1+n_1)^0 \cdot (a,b) = (ki_1+n_1+b-a)^0 = (ki_1)^1 \cdot (n_1+b-a)^0$ $= (ki_1)^1 \cdot ((n_1)^0 \cdot (a,b))$;

9) $((ki_1)^1 \cdot (ki_2)^1) \cdot (n_1)^0 = (ki_1+ki_2)^1 \cdot (n_1)^0 = (ki_1+ki_2+n_1)^0 = (ki_1)^1 \cdot (ki_2+n_1)^0$ $= (ki_1)^1 \cdot ((ki_2)^1 \cdot (n_1)^0)$;

10) $((ki_1)^1 \cdot (a,b)) \cdot (n_1)^0 = (-ki_1 + a, b) \cdot (n_1)^0 = (ki_1 + b - a + n_1)^0$
$= (ki_1)^1 \cdot (b - a + n_1)^0 = (ki_1)^1 \cdot ((a,b) \cdot (n_1)^0)$;

11) $((a,b) \cdot (n_1)^0) \cdot (ki_1)^1 = (b - a + n_1)^0 \cdot (ki_1)^1 = (b - a + n_1 + ki_1)^0 = (a,b) \cdot (n_1 + ki_1)^0$
$= (a,b) \cdot ((n_1)^0 \cdot (ki_1)^1)$;

12) $((a,b) \cdot (ki_1)^1) \cdot (n_1)^0 = (a, b + ki_1)^0 \cdot (n_1)^0 = (b + ki_1 - a + n_1)^0 = (a,b) \cdot (ki_1 + n_1)^0$
$= (a,b) \cdot ((ki_1)^1 \cdot (n_1)^0)$.

This completes the proof of the associativity of such defined binary operation on S_5. Also, we observe that S_5 with such defined semigroup operation is an inverse semigroup.

Let τ_5 be a topology on S_5 which is generated by the family $\tau_2 \cup \tau_3$ (see Examples 2 and 3). Also Examples 2 and 3 imply that it is sufficient to show that the semigroup operation in S_5 is continuous in cases $(ki)^1 \cdot (n)^0$ and $(n)^0 \cdot (ki)^1$, where $(n)^0 \in G_0$ and $(ki)^1 \in G_1(k)$. Then for every positive integer $p \geqslant \max\{|ki|, |n|\}$ we have that

$$U_{2p}\big((ki)^1\big) \cdot U_{2p}\big((n)^0\big) \subseteq U_p\big((ki+n)^0\big) \qquad \text{and} \qquad U_{2p}\big((n)^0\big) \cdot U_{2p}\big((ki)^1\big) \subseteq U_p\big((ki+n)^0\big).$$

This completes the proof that (S_5, τ_5) is a topological inverse semigroup. Obviously, (S_5, τ_5) is a locally compact space.

Theorem 2. *Let T be a Hausdorff topological inverse semigroup. If T contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup and $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}} \neq \varnothing$, then the following assertions hold:*

(i) $E(T)$ is a countable linearly ordered semilattice;

(ii) $E(T) \cap (T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}})$ is a singleton set;

(iii) $T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ is a subgroup in T.

Proof. (i) By Proposition II.3 from [8] we have that $\operatorname{cl}_T(E(\mathscr{C}_{\mathbb{Z}})) = E(T)$ and since the closure of a linearly ordered subsemilattice in a topological semilattice is a linearly ordered subsemilattice too (see [12, Lemma 1]) we get that $E(T)$ is a linearly ordered semilattice. Then the semilattice operation in $E(T)$ implies that the sets $E(T) \setminus \bigcup\limits_{e \in E(\mathscr{C}_{\mathbb{Z}})} {\downarrow}e$ and $E(T) \setminus \bigcup\limits_{e \in E(\mathscr{C}_{\mathbb{Z}})} {\uparrow}e$ are either singleton or empty. This completes the proof of our assertion.

Assertion (ii) follows from assertion (i).

(iii) Since T is an inverse semigroup and $\overline{e}$ is a minimal idempotent in $E(T)$ we conclude that the $\mathscr{H}$-class $H_{\overline{e}}$ which contains $\overline{e}$ coincides with the ideal $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$. Indeed, if there exist $x \in I$ and an $\mathscr{H}$-class $H_x \subseteq I$ in T such that $x \in H_x \neq H_{\overline{e}}$, then since T is an inverse semigroup we have that there exists an idempotent $e \in T$ such that either $xx^{-1} = e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$ or $x^{-1}x = e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$. If $xx^{-1} = e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$, then we have that $x = xx^{-1}x = ex \in eT$, and since T is an inverse semigroup Theorem 1.17 from [7] implies $e \in xT$, a contradiction. Similar arguments show that $x^{-1}x \neq e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$. Hence assertion (ii) implies that $xx^{-1} = x^{-1}x = \overline{e}$ and hence $x \in H_x = H_{\overline{e}}$. □

The following theorem describes the structure of a closure of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a locally compact topological inverse semigroup T, i.e., it gives the description of the non-empty ideal $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ in the remainder of $\mathscr{C}_{\mathbb{Z}}$ in T.

Theorem 3. *Let T be a Hausdorff locally compact topological inverse semigroup. If T contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup and $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}} \neq \varnothing$, then the following assertions hold:*

(i) *${\downarrow}e_n$ is a compact subsemilattice in $E(T)$ for every idempotent $e_n = (n,n) \in \mathscr{C}_{\mathbb{Z}}$, $n \in \mathbb{Z}$;*

(ii) *$T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ is isomorphic to the discrete additive group of integers;*

(iii) *if $\overline{e}$ is a unit of $T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$, then the map $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}} \to T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ which is defined by the formula $((a,b))\mathfrak{h} = (a,b) \cdot \overline{e}$ is the natural homomorphism generated by the minimal group congruence $\mathfrak{C}_{mg}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$;*

(iv) *the subsemigroup $S = \mathscr{C}_{\mathbb{Z}} \cup I$ is topologically isomorphic to the topological inverse semigroup (S_3, τ_3) from Example 3.*

Proof. (i) We show that ${\downarrow}e_0$ is a compact subset in $E(T)$ for $e_0 = (0,0)$. By assertion (ii) of Theorem 2 we get that the set $E(T) \cap (T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}})$ is singleton and we put $\{\overline{e}\} = E(T) \cap (T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}})$. Then $\overline{e}$ is a smallest idempotent in $E(T)$. By Theorem 1.5 from [6, Vol. 1] we have that $E(T)$ is a closed subset in T, and hence by Theorem 3.3.9 from [9] we get that $E(T)$ is a locally compact space. Suppose the contrary: ${\downarrow}e_0$ is not a compact subset in $E(T)$. Since Corollary 3.1 implies that every element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is an isolated point in T and hence so it is in $E(T)$, we get that there exists an open neighbourhood $U(\overline{e})$ of $\overline{e}$ in $E(T)$ such that the set ${\downarrow}e_0 \setminus U(\overline{e})$ is an infinite discrete subspace of $E(T)$, $U(\overline{e}) \subseteq E(T) \setminus {\uparrow}e_0$ and $\operatorname{cl}_{E(T)}(U(\overline{e})) = U(\overline{e})$ is a compact subset of $E(T)$. Then for every positive integer i there exists an integer $j \geqslant i$ such that $(j,j) \notin U(\overline{e})$ and $(j+1,j+1) \in U(\overline{e})$. Then the semigroup operation in $\mathscr{C}_{\mathbb{Z}}$ implies that by induction we can construct an infinite subset $M \subseteq {\downarrow}e_0 \setminus \{\overline{e}\}$ of $E(T)$ such that $M \subseteq U(\overline{e}) \setminus \{\overline{e}\}$ and $\{(0,1)\} \cdot M \cdot \{(1,0)\} \subseteq {\downarrow}e_0 \setminus U(\overline{e})$. Since the set $U(\overline{e})$ is compact and the set $M \subseteq U(\overline{e}) \setminus \{\overline{e}\}$ contains only isolated points from $E(\mathscr{C}_{\mathbb{Z}})$, we conclude that $\overline{e} \in \operatorname{cl}_T(M)$. Since ${\downarrow}e_0 \setminus U(\overline{e})$ is a closed subset of $E(T)$ we have that the continuity of the semigroup operation in T and Proposition 1.4.1 from [9] imply that

$$\overline{e} \in \{(0,1)\} \cdot \operatorname{cl}_T(M) \cdot \{(1,0)\} \subseteq \operatorname{cl}_T\left(\{(0,1)\} \cdot M \cdot \{(1,0)\}\right) \subseteq {\downarrow}e_0 \setminus U(\overline{e}),$$

which contradicts $\overline{e} \in U(\overline{e})$. The obtained contradiction implies that the set ${\downarrow}e_0 \setminus U(\overline{e})$ is finite, and hence the set ${\downarrow}e_0$ is compact. Since for every integer n the set ${\downarrow}e_n \setminus {\downarrow}e_0$ is either finite or empty and e_n is an isolated point in $E(T)$ we conclude that ${\downarrow}e_n$ is a compact subsemilattice of $E(T)$.

(ii) By assertion (i) we have that $\overline{e}$ is an accumulation point of the subsemigroup $\mathscr{C}_{\mathbb{N}}[0]$ in T. Since by Theorem 3.3.9 from [9] a closed subset of a locally compact space is a locally compact subspace too, and by Proposition 2.1$(viii)$ the semigroup $\mathscr{C}_{\mathbb{N}}[0]$ is isomorphic to the bicyclic semigroup, Proposition V.3 from [8] implies that the subset $\operatorname{cl}_T(\mathscr{C}_{\mathbb{N}}[0]) \setminus \mathscr{C}_{\mathbb{N}}[0]$

is a non-singleton subgroup of T. By Corollary 3.1 we get that $\mathscr{C}_{\mathbb{Z}}$ is an open discrete subsemigroup of T and hence we get that $\operatorname{cl}_T(\mathscr{C}_{\mathbb{N}}[0]) \setminus \mathscr{C}_{\mathbb{N}}[0] \subseteq \operatorname{cl}_T(\mathscr{C}_{\mathbb{Z}}) \setminus \mathscr{C}_{\mathbb{Z}}$.

By assertion (iii) of Theorem 2 we have that $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ is a non-singleton subgroup in T. Since T is a topological inverse semigroup we get that I is a topological group. Then by Proposition 3.2(xi) we have that I is a closed subset of T and hence by Theorem 3.3.9 from [9] we get that I is a locally compact topological group.

Later we show that $(a,b)\cdot\overline{e} = \overline{e}\cdot(a,b)$ for every $(a,b)\in\mathscr{C}_{\mathbb{Z}}$. Suppose the contrary: there exists $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ such that $(a,b)\cdot\overline{e} \neq \overline{e}\cdot(a,b)$. Without loss of generality we can assume that $a \leqslant b$ in $\mathbb{Z}$. Then the Hausdorffness of the space T implies that there exist open neighbourhoods $U((a,b)\cdot\overline{e})$ and $U(\overline{e}\cdot(a,b))$ of the points $(a,b)\cdot\overline{e}$ and $\overline{e}\cdot(a,b)$ in T such that $U((a,b)\cdot\overline{e})\cap U(\overline{e}\cdot(a,b)) = \varnothing$. Then the continuity of the semigroup operation of T implies that there exists an open neighbourhood $V(\overline{e})$ of $\overline{e}$ in T such that the following conditions hold:

$$\{(a,b)\}\cdot V(\overline{e}) \subseteq U((a,b)\cdot\overline{e}) \qquad \text{and} \qquad V(\overline{e})\cdot\{(a,b)\} \subseteq U(\overline{e}\cdot(a,b)).$$

By assertion (i) we get that without loss of generality we can assume that $V(\overline{e})\cap E(T)$ is a compact subset in T and there exists a positive integer $n_0 \geqslant \max\{a,b\}$ such that $(n,n)\in V(\overline{e})\cap E(T)$ for all integers $n \geqslant n_0$. Then for $n = 2n_0 - a$ and $k = 2n_0 - b$ we get that $(n,n),(k,k)\in V(\overline{e})\cap E(T)$. But we have

$$(a,b)\cdot(n,n) = (a,b)\cdot(2n_0-a,2n_0-a) = (2n_0-a-b+a,2n_0-a) = (2n_0-b,2n_0-a)$$

and

$$(k,k)\cdot(a,b) = (2n_0-b,2n_0-b)\cdot(a,b) = (2n_0-b,2n_0-b-a+b) = (2n_0-b,2n_0-a),$$

which contradicts $U((a,b)\cdot\overline{e})\cap U(\overline{e}\cdot(a,b)) = \varnothing$. The obtained contradiction implies that $(a,b)\cdot\overline{e} = \overline{e}\cdot(a,b)$ for every $(a,b)\in\mathscr{C}_{\mathbb{Z}}$.

Next we show that $x\cdot\overline{e} = \overline{e}\cdot x$ for every $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$. Suppose contrary: there exists $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$ such that $x\cdot\overline{e}\neq\overline{e}\cdot x$. Then the Hausdorffness of the space T implies that there exist open neighbourhoods $U(x\cdot\overline{e})$ and $U(\overline{e}\cdot x)$ of the points $x\cdot\overline{e}$ and $\overline{e}\cdot x$ in T such that $U(x\cdot\overline{e})\cap U(\overline{e}\cdot x) = \varnothing$. The continuity of the semigroup operation of T implies that there exists an open neighbourhood $V(x)$ of x in T such that the following conditions hold:

$$V(x)\cdot\{\overline{e}\} \subseteq U(x\cdot\overline{e}) \qquad \text{and} \qquad \{\overline{e}\}\cdot V(x) \subseteq U(\overline{e}\cdot x).$$

Since $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T we conclude that there exists $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ such that $(a,b)\in V(x)$. Then we get that $(a,b)\cdot\overline{e} = \overline{e}\cdot(a,b)$, which contradicts $U(x\cdot\overline{e})\cap U(\overline{e}\cdot x) = \varnothing$. The obtained contradiction implies that $x\cdot\overline{e} = \overline{e}\cdot x$ for every $x\in T$.

We define a map $\mathfrak{h}\colon T\to I$ by the formula $(x)\mathfrak{h} = x\cdot\overline{e}$. Since $x\cdot\overline{e} = \overline{e}\cdot x$ for every $x\in T$ we get that $\mathfrak{h}$ is a homomorphism. Since $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T, Proposition 2.2 and assertion (iii) of Theorem 2 imply that the topological group I contains a dense cyclic subgroup. Since I is a locally compact topological group, Pontryagin-Weil Theorem (see [15, p. 71, Theorem 19]) implies that either I is compact or I is discrete. If I is compact, then by Proposition 3.2$(viii)$ we get that

$$S = T \setminus \bigcup_{(a,b)\notin\mathscr{C}_{\mathbb{N}}[0]} {\uparrow}(a,b)$$

is a closed subset in T. Then by Theorem 3.3.9 from [9] S is a locally compact space. Obviously, $S = \mathscr{C}_{\mathbb{N}}[0] \cup I$. Since I is a locally compact ideal in T, Proposition 2.1($viii$) and Proposition II.4 from [8] imply that the Rees quotient semigroup S/I with the quotient topology is locally compact topological inverse semigroup which is isomorphic to the bicyclic semigroup with an adjoined zero. This contradicts Proposition V.3 from [8]. The obtained contradiction implies that the group I is discrete and hence I is a discrete additive group of integers.

(iii) Let $(a, b), (c, d) \in \mathscr{C}_{\mathbb{Z}}$ such that $(a, b)\mathfrak{C}_{mg}(c, d)$. Then there exists an idempotent $(n, n) \in \mathscr{C}_{\mathbb{Z}}$ such that $(a, b) \cdot (n, n) = (c, d) \cdot (n, n)$. Since $(i, i) \cdot \overline{e} = \overline{e}$ for every idempotent $(i, i) \in \mathscr{C}_{\mathbb{Z}}$ we get that $((a, b))\mathfrak{h} = ((c, d))\mathfrak{h}$.

Let $(a, b), (c, d) \in \mathscr{C}_{\mathbb{Z}}$ such that $((a, b))\mathfrak{h} = ((c, d))\mathfrak{h}$. Suppose the contrary: $(a, b)\cdot(n, n) \neq (c, d) \cdot (n, n)$ for any idempotent $(n, n) \in \mathscr{C}_{\mathbb{Z}}$. If $(a, b) \cdot (n_1, n_1) = (c, d) \cdot (n_2, n_2)$ for some idempotents $(n_1, n_1), (n_2, n_2) \in \mathscr{C}_{\mathbb{Z}}$, then we have that

$$
\begin{aligned}
(a, b) \cdot (n_1, n_1) \cdot (n_2, n_2) &= (a, b) \cdot (n_1, n_1) \cdot (n_1, n_1) \cdot (n_2, n_2) \\
&= (c, d) \cdot (n_2, n_2) \cdot (n_1, n_1) \cdot (n_2, n_2) \\
&= (c, d) \cdot (n_1, n_1) \cdot (n_2, n_2).
\end{aligned}
$$

Therefore we get that $(a, b) \cdot (n_1, n_1) \neq (c, d) \cdot (n_2, n_2)$ for all idempotents $(n_1, n_1), (n_2, n_2) \in \mathscr{C}_{\mathbb{Z}}$. Then Proposition 2.1($vi$) implies that $b - a \neq d - c$, and hence by the proof of Proposition 2.2 we get that the congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$ which is generated by the homomorphism $\mathfrak{h}$ distincts from the minimal group congruence $\mathfrak{C}_{mg}$ on $\mathscr{C}_{\mathbb{Z}}$. Then the ideal I is not isomorphic to the additive group of integers $\mathbb{Z}$ and hence by Proposition 2.2 we have that the ideal I contains a finite cyclic group. This contradicts assertion (ii). The obtained contradiction implies our assertion.

(iv) Assertions (ii) and (iii) imply that the subsemigroup $S = \mathscr{C}_{\mathbb{Z}} \cup I$ of T is algebraically isomorphic to the inverse semigroup S_3 from Example 3. We identify the group I with G_0 and put $\overline{e} = 0 \in G_0$.

By τ we denote the topology of the topological inverse semigroup T. Since G_0 is a discrete subgroup of T, assertion (i) implies that there exists a compact open neighbourhood $U(0)$ of 0 in T with the following property:

> $U(0) \subseteq E(T)$ and there is a positive integer n_0 such that $n_0 = \max\{(n, n) \in E(\mathscr{C}_{\mathbb{Z}}) \mid (n, n) \in U(0)\}$ and $(i, i) \in U(0)$ for all integers $i \geqslant n_0$.

Hence, we get that $\mathscr{B}_3(0) = \{U_n(0) \mid n \in \mathbb{N}\}$ is a base of the topology of the space T at the point $0 \in G_0 \subseteq T$, where $U_n(0) = \{0\} \cup \{(n + i, n + i) \mid i \in \mathbb{N}\}$.

We fix an arbitrary element $k \in G_0$. Without loss of generality we can assume that $k \geqslant 0$. Then $k^{-1} = -k \in \mathbb{Z} = G_0$. Since G_0 is a discrete subgroup of T, the continuity of the homomorphism $\mathfrak{h} \colon T \to G_0 \colon x \mapsto x \cdot \overline{e} = x \cdot 0$ implies that $(k)\mathfrak{h}^{-1}$ is an open subset in T. We observe that, since the homomorphism $\mathfrak{h}$ generates the minimal group congruence on $\mathscr{C}_{\mathbb{Z}}$ (see assertion (iii)) we get that $(k)\mathfrak{h}^{-1} \cap \mathscr{C}_{\mathbb{Z}} = \{(a, b) \in \mathscr{C}_{\mathbb{Z}} \mid b - a = k\}$. Also, since

$$
{\uparrow}(a, b) = \{(x, y) \in \mathscr{C}_{\mathbb{Z}} \mid (x, y) \cdot (b, b) = (a, b)\},
$$

for every $(a,b)\in\mathscr{C}_{\mathbb{Z}}$, Proposition 3.2($viii$) implies that ${\uparrow}(a,b)$ is a closed-and-open subset in T for every $(a,b)\in\mathscr{C}_{\mathbb{Z}}$. Hence we get that $\{k\}\cup\{(i,i+k)\in\mathscr{C}_{\mathbb{Z}}\mid i=1,2,3,\ldots\}$ is an open subset in T.

We fix an arbitrary positive integer i. Since $(i+k,i)\cdot k=0\in G_0$, the continuity of the semigroup operation in T implies that for every $U_i(0)\in\mathscr{B}_3(0)$ there exists an open neighbourhood

$$V(k)\subseteq\{k\}\cup\{(i,i+k)\in\mathscr{C}_{\mathbb{Z}}\mid i=1,2,3,\ldots\}$$

of k in T such that $(i+k,i)\cdot V(k)\subseteq U_i(0)$. Then the semigroup operation of $\mathscr{C}_{\mathbb{Z}}$ implies that $V(k)\subseteq U_i(k)$ for $U_i(k)\in\mathscr{B}_3(k)$.

We observe that for every $k\in G_0$ and for every positive integer i we have that

$$0\cdot(i,k+i)=k\qquad\text{and}\qquad U_i(0)\cdot\{(i,i+k)\}=U_i(k),$$

where $U_i(0)\in\mathscr{B}_3(0)$ and $U_i(k)\in\mathscr{B}_3(k)$. Then the continuity of the semigroup operation in T implies that for every open neighbourhood $W(k)$ of k in T there exists $U_i(0)\in\mathscr{B}_3(0)$ such that

$$U_i(0)\cdot\{(i,i+k)\}=U_i(k)\subseteq W(k).$$

This implies that the bases of topologies τ and τ_3 at the point $k\in T$ coincide.

In the case when $k<0$ the proof is similar. This completes the proof of our assertion. □

Theorem 3 implies the following:

Corollary 4.1. *Let T be a Hausdorff locally compact topological inverse semigroup. If T contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup such that $I=T\setminus{\uparrow}\mathscr{C}_{\mathbb{Z}}\neq\varnothing$ and ${\uparrow}\mathscr{C}_{\mathbb{Z}}=\mathscr{C}_{\mathbb{Z}}$, then T is topologically isomorphic to the topological inverse semigroup (S_3,τ_3) from Example 3.*

Theorem 4. *Let (T,τ) be a Hausdorff locally compact topological inverse monoid with unit 1_T. If $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that ${\uparrow}\mathscr{C}_{\mathbb{Z}}=T$ and the group of units of T is singleton, then there exists a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that (T,τ) is topologically isomorphic to the semigroup (S_1,τ_1) from Example 1.*

Proof. By the assumption of the theorem we have that $T\setminus\mathscr{C}_{\mathbb{Z}}=\{1_T\}$. Then Lemma 3.2($i$) implies that there exists a base $\mathscr{B}(1_T)$ of the topology τ at the unit 1_T such that $U(1_T)\subseteq E(\mathscr{C}_{\mathbb{Z}})$ for any $U(1_T)\in\mathscr{B}(1_T)$. Also statements ($c$) and ($d$) of Theorem 1.7 from [6, Vol. 1] imply that we can assume that $(n,n)\in U(1_T)$ if and only if n is a negative integer. Since by Corollary 3.1 every element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is an isolated point of T, without loss of generality we can assume that all elements of the base $\mathscr{B}(1_T)$ are closed-and-open subsets of T. Also, the local compactness of T implies that without loss of generality we can assume that the base $\mathscr{B}(1_T)$ consists of compact subsets, and Corollary 3.3.6 from [9] implies that the base $\mathscr{B}(1_T)$ is countable.

We suppose that $\mathscr{B}(1_T)=\{U_n(1_T)\mid n=1,2,3,\ldots\}$. We put

$$W_1(1_T)=U_1(1_T)\qquad\text{and}\qquad W_i(1_T)=W_{i-1}(1_T)\cap U_i(1_T),$$

for all $i=2,3,4,\ldots$. We observe that $\widetilde{\mathscr{B}}(1_T)=\{W_n(1_T)\mid n=1,2,3,\ldots\}$ is a base of the topology τ at the unit 1_T of T such that $W_{n+1}(1_T)\subsetneqq W_n(1_T)$ for every positive integer

n. Then the compactness of $U_i(1_T)$, $i = 1, 2, 3, \ldots$, and the discreteness of the space $\mathscr{C}_{\mathbb{Z}}$ imply that the family $\widetilde{\mathscr{B}}(1_T)$ consists of compact-and-open subsets of T. Let $\{m_i\}_{i\in\mathbb{N}}$ be a decreasing sequence of negative integers such that $\bigcup\limits_{i=1}^{\infty}\{(m_i, m_i)\} = W_1(1_T) \setminus \{1_T\}$. We put $V_n = \{1_T\} \cup \{(m_i, m_i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant n\}$ for every positive integer n. Since every element of the family $\widetilde{\mathscr{B}}(1_T)$ is a compact subset of T, Corollary 3.1 implies that the family

$$\overline{\mathscr{B}}(1_T) = \{V_n \mid n = 1, 2, 3, \ldots\}$$

is a base of the topology τ at 1_T of T and this completes the proof of our theorem. $\square$

Theorems 3 and 4 imply the following:

Corollary 4.2. *Let (T, τ) be a Hausdorff locally compact topological inverse semigroup. If $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that the group of units of T is singleton, then there exists a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that (T, τ) is topologically isomorphic either to the semigroup (S_1, τ_1) from Example 1 or to the semigroup (S_4, τ_4) from Example 4.*

Theorem 5. *Let (T, τ) be a Hausdorff locally compact topological inverse monoid with unit 1_T. Suppose that $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that the following conditions hold:*

(i) $\uparrow\mathscr{C}_{\mathbb{Z}} = T$;

(ii) *the group of units $H(1_T)$ of T is non-singleton; and*

(iii) *there exists an integer j such that $K = \{1_T\} \cup \{(i, i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant j\}$ is a compact subset of T.*

Then there exists a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer i and (T, τ) is topologically isomorphic to the semigroup (S_2, τ_2) for $n = 1$ from Example 2.

Proof. As in the proof of Theorem 4 we construct a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that the family

$$\mathscr{B}(1_T) = \{U_i(1_T) \mid i = 1, 2, 3, \ldots\}$$

determines a base of the topology τ at the point 1_T of T, where

$$U_j(1_T) = \{1_T\} \cup \{(m_i, m_i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant j\}.$$

The compactness of the set K implies that we can construct a sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer i.

Then for every element x of the group of units $H(1_T)$ left and right translations $\lambda_x \colon T \to T : s \mapsto x \cdot s$ and $\rho_x \colon T \to T : s \mapsto s \cdot x$ are homeomorphisms of the topological space T (see [6, Vol. 1, P. 19]), and hence the following families

$$\mathscr{B}_l(x) = \{x \cdot U_i(1_T) \mid U_i(1_T) \in \mathscr{B}(1_T)\}$$

and

$$\mathscr{B}_r(x) = \{U_i(1_T) \cdot x \mid U_i(1_T) \in \mathscr{B}(1_T)\}$$

are bases of the topology τ at the point 1_T of T. Also, we observe that the family

$$\mathscr{B}(x) = \{U \cap V \mid U \in \mathscr{B}_l(x) \text{ and } V \in \mathscr{B}_r(x)\}$$

is a base of the topology τ at the point 1_T of T.

Then Lemma 3.2 and Proposition 3.2 imply that the group of units $H(1_T)$ of T is topologically isomorphic to the discrete additive group of integers $\mathbb{Z}_+$. Let g be a generator of $\mathbb{Z}_+$. Then by Lemma 3.2(iii) there exist an open neighbourhood $U(g)$ of the point g in T and an integer k such that $a - b = k$ for all $(a, b) \in U(g) \cap \mathscr{C}_{\mathbb{Z}}$. Without loss of generality we can assume that g is a positive integer and $k < 0$. Then we have that

$$g \cdot U_i(1_T) = \{(m_i + k, m_i) \mid (m_i, m_i) \in U_i(1_T)\} \cup \{g\} \tag{3}$$

and

$$U_i(1_T) \cdot g = \{(m_i, m_i - k) \mid (m_i, m_i) \in U_i(1_T)\} \cup \{g\} \tag{4}$$

We shall show that equality (4) holds. Let be $(m_i, m_i) \in U_i(1_T)$. Then we get

$$((m_i, m_i) \cdot g) \cdot ((m_i, m_i) \cdot g)^{-1} = (m_i, m_i) \cdot g \cdot g^{-1} \cdot (m_i, m_i)^{-1} = (m_i, m_i) \cdot 1_T \cdot (m_i, m_i) = (m_i, m_i).$$

Since $(m_i, m_i) \cdot g \in \mathscr{C}_{\mathbb{Z}}$ and $\mathscr{C}_{\mathbb{Z}}$ is an inverse semigroup we conclude that $(m_i, m_i) \cdot g = (m_i, a)$ for some integer a, and by Lemma 3.2(vi) we have that $(m_i, m_i) \cdot g = (m_i, m_i - k)$. This completes the proof of equality (4). The proof of equality (3) is similar. Then Lemma 3.2(vi), equalities (3) and (4) imply that T is topologically isomorphic to the semigroup (S_2, τ_2) for $n = 1$ from Example 2. This completes the proof of the theorem. □

Theorems 3 and 5 imply the following:

Corollary 4.3. *Let (T, τ) be a Hausdorff locally compact topological inverse monoid with unit 1_T. Suppose that $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that the following conditions hold:*

(i) the group of units $H(1_T)$ of T is non-singleton; and

(ii) there exists an integer j such that $K = \{1_T\} \cup \{(i, i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant j\}$ is a compact subset of T.

Then there exists a decreasing sequence of negative integers $\{m_i\}_{i \in \mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer i and (T, τ) is topologically isomorphic either to the semigroup (S_2, τ_2) from Example 2 or to the semigroup (S_5, τ_5) from Example 5.

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Received 19.10.2011

Фігель І.Р., Гутік О.В. *Про замикання розширеної біциклічної напівгрупи* // Карпатські математичні публікації. — 2011. — Т.3, №2. — С. 131–157.

У статті вивчається напівгрупа $\mathscr{C}_{\mathbb{Z}}$, яка є узагальненням біциклічної напівгрупи. Описано основні алгебраїчні властивості напівгрупи $\mathscr{C}_{\mathbb{Z}}$, зокрема доведено, що кожна нетривіальна конгруенція $\mathfrak{C}$ на напівгрупі $\mathscr{C}_{\mathbb{Z}}$ є груповою, і більше того, фактор-напівгрупа $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ ізоморфна циклічній групі. Показано, що на напівгрупі $\mathscr{C}_{\mathbb{Z}}$ не існує відмінних від дискретної гаусдорфових топологій τ таких, що $(\mathscr{C}_{\mathbb{Z}}, \tau)$ — напівтопологічна напівгрупа. Також вивчається замикання напівгрупи $\mathscr{C}_{\mathbb{Z}}$ у топологічній інверсній напівгрупі T. Показано, що непорожній наріст напівгрупи $\mathscr{C}_{\mathbb{Z}}$ у напівгрупі T складається з групи одиниць $H(1_T)$ напівгрупи T та двобічного ідеалу I в T, якщо $H(1_T) \neq \varnothing$ та $I \neq \varnothing$. У випадку, коли T є локально компактною топологічною інверсною напівгрупою та $I \neq \varnothing$, доведено, що ідеал I топологічно ізоморфний дискретній адитивній групі цілих чисел та описано топологію на піднапівгрупі $\mathscr{C}_{\mathbb{Z}} \cup I$. Також доведено, якщо група одиниць $H(1_T)$ в T є непорожньою, то або $H(1_T)$ є одноточковою множиною, або група $H(1_T)$ топологічно ізоморфна дискретній адитивній групі цілих чисел.

Фигель И.Р., Гутик О.В. *О замыкании расширенной бициклической полугруппы* // Карпатские математические публикации. — 2011. — Т.3, №2. — С. 131–157.

В работе изучается полугруппа $\mathscr{C}_{\mathbb{Z}}$, которая является обобщением бициклической полугруппы. Описаны основные алгебраические свойства полугруппы $\mathscr{C}_{\mathbb{Z}}$, в частности доказано, что каждая нетривиальная конгруэнция $\mathfrak{C}$ на $\mathscr{C}_{\mathbb{Z}}$ является групповой, и более того, фактор-полугруппа $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ изоморфна циклической группе. Показано, что на полугруппе $\mathscr{C}_{\mathbb{Z}}$ не существует отличных от дискретной топологий τ таких, что $(\mathscr{C}_{\mathbb{Z}}, \tau)$ — хаусдорфова полутопологическая полугруппа. Также изучается замыкание полугруппы $\mathscr{C}_{\mathbb{Z}}$ в топологической инверсной полугруппе T. Показано, что непустой нарост полугруппы $\mathscr{C}_{\mathbb{Z}}$ в полугруппе T состоит из группы единиц $H(1_T)$ полугруппы T и идеала I в T, когда $H(1_T) \neq \varnothing$ и $I \neq \varnothing$. В случае, когда T является локально компактной топологической инверсной полугруппой и $I \neq \varnothing$, доказано, что идеал I топологически изоморфен дискретной аддитивной группе целых чисел, и описано топологию на подполугруппе $\mathscr{C}_{\mathbb{Z}} \cup I$. Также показано, если группа единиц $H(1_T)$ в T непуста, то или $H(1_T)$ является одноточечным множеством, или группа $H(1_T)$ топологически изоморфна дискретной аддитивной группе целых чисел.