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MULTIPLICATIVE POLYNOMIAL MAPPINGS ON COMMUTATIVE BANACH ALGEBRAS

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We consider the multiplicative polynomial mappings on commutative algebras in this work. We call a multiplicative polynomial trivial, if it can be represented as a product of characters. In the paper we investigate the following question: does there exists a nontrivial multiplicative polynomial functional on a commutative algebra?

Introduction and definitions

Let X and Y be linear spaces. A map $P_n : X \to Y$ is an n *degree homogeneous polynomial* (n-homogeneous polynomial) if there is an n-linear mapping $B_n : \underbrace{X \times X \times \ldots \times X}_{n} \to Y$ such that $P_n(x) = B_n(x, x, \ldots, x)$.

Note that according to the polarization formula (see [5]), for every n-homogeneous polynomial P_n there exists a unique symmetric n-linear mapping B_n, associated with P_n.

The polynomial mapping P between algebras X and Y is called *multiplicative,* if $P(x_1 \cdot x_2) = P(x_1) \cdot P(x_2)$ for every $x_1, x_2 \in X$. It is known that every multiplicative polynomial functional is homogeneous ([7]).

Let us denote by $M(X)$ the set of characters (linear multiplicative functionals) on the algebra X. It is clear that the product of characters $\varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_n$, $\varphi_k \in M(X)$, $k = 1, \ldots, n$ is a multiplicative polynomial functional.

We call a multiplicative polynomial *trivial*, if it can be represented as a product of characters.

There is an example of nontrivial multiplicative polynomial on noncommutative algebra. Let us consider the algebra of square $n \times n$ matrixes M_n. The mapping $d(A) = \det(A)$ is a multiplicative polynomial functional, but $d(A)$ is not a product of characters because M_n has no nonzero characters. So, it is interesting to know: *Does there exists a nontrivial multiplicative polynomial functional on a commutative algebra?*

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1 Main results

Proposition 1.1. *If a commutative semi-simple Banach algebra A without nilpotent elements admits a nontrivial multiplicative polynomial, then the algebra $P(\mathbb{C}^m)$ of polynomials of m variables admits a nontrivial multiplicative polynomial for some m.*

Proof. Let P be a nontrivial multiplicative polynomial map. Without loss of the generality we can assume that P is irreducible. Then there exists a finite dimensional subspace V in A such that the restriction of P onto V is irreducible [8].

Let us consider a subalgebra A_0 generated by elements in V and the unity of A. This subalgebra is finitely generated and has no nilpotent elements. It is well known in Algebraic Geometry that such algebra is isomorphic to a ring of all polynomials on an algebraic variety. That is, A_0 is isomorphic to $P(\mathbb{C}^m)/I$ for some m where I is an ideal in $P(\mathbb{C}^m)$. Let $T : P(\mathbb{C}^m) \to P(\mathbb{C}^m)/I$ be the factor map and P_0 be the restriction of P onto A_0. Then $P_0 \circ T$ is a required irreducible nontrivial multiplicative polynomial on $P(\mathbb{C}^m)$. ☐

So, it can be useful to investigate the algebra $P(\mathbb{C}^m)$ of polynomials of m variables.

Theorem 1. *Every multiplicative n degree polynomial functional can be represented as a product of characters on the algebra $P(\mathbb{C})$ of polynomials of one complex variable.*

Proof. Let $D_n : P(\mathbb{C}) \to \mathbb{C}$ be a multiplicative n degree polynomial functional. D_n is a homogeneous polynomial map. According to [7] there is a character d_n on the symmetric tensor product $\underbrace{P(\mathbb{C}) \otimes_s \ldots \otimes_s P(\mathbb{C})}_{n} = \otimes_s^n P(\mathbb{C})$ such that $D_n(p) = d_n(\otimes^n p)$ for every $p \in P(\mathbb{C})$.

Let us describe the algebra $\otimes_s^n P(\mathbb{C})$. Every element of this algebra can be presented by

$$\sum_i \underbrace{p_i \otimes \ldots \otimes p_i}_{n},$$

where $p_i \in P(\mathbb{C})$. Every element like $\otimes^n p_i$ is a polynomial of n variables $p_i(x_1) \cdot \ldots \cdot p_i(x_n)$. So, algebra $\otimes_s^n P(\mathbb{C})$ is isomorphic to the algebra of polynomials of n variables, that are symmetric about the permutation of this variables. Let us denote it by $P_s(\mathbb{C}^n)$.

It is well known that for every symmetric polynomial $p(x_1, \ldots, x_n)$ there exists a polynomial q, such that

$$p(x_1, \ldots, x_n) = q(G_1(x_1, \ldots, x_n), \ldots, G_n(x_1, \ldots, x_n)),$$

where

$$G_1(x_1, \ldots, x_n) = x_1 + \ldots + x_n,$$

$$G_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n,$$

$$\ldots\ldots\ldots$$

$$G_n = x_1x_2 \cdot \ldots \cdot x_n,$$

that is G_i, $i = 1, \ldots, n$ are the elementary symmetric polynomials. The mapping $p \mapsto q$ is an isomorphism from the algebra $P_s(\mathbb{C}^n)$ of symmetric polynomials onto the algebra of all polynomials $P(\mathbb{C}^n)$. Every character of algebra $P(\mathbb{C}^n)$ is an evaluation of some point $\mathbb{C}^n$. So, there is a correspondence between characters d_n on $\otimes^n P(\mathbb{C})$ and some characters φ on $P(\mathbb{C}^n)$, which is an evaluation of polynomial q at some point $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. That is,

$$d_n(\otimes^n p) = d_n(p(x_1) \cdot \ldots \cdot p(x_n)) = \varphi(q(x_1, \dots, x_n)) = q(\alpha_1, \dots, \alpha_n),$$

where $p(x_1) \cdot \ldots \cdot p(x_n) = q(G_1(x_1, \dots, x_n), \dots, G_n(x_1, \dots, x_n))$. Let $x_1^0, \dots, x_n^0$ be the solutions of system

$$\begin{cases} x_1 + x_2 + \ldots + x_n = \alpha_1; \\ x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n = \alpha_2; \\ \ldots\ldots\ldots\ldots \\ x_1x_2 \cdot \ldots \cdot x_n = \alpha_n \end{cases}.$$

According to the Viet theorem, $x_1^0, \dots, x_n^0$ are the solutions of the equation

$$x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \ldots + (-1)^n \alpha_n = 0.$$

Then $D_n(p) = q(\alpha_1, \dots, \alpha_n) = p(x_1^0) \cdot \ldots \cdot p(x_n^0)$, that was needed to show. □

To prove of this theorem we use the existence of a homomorphism from $\otimes^n P(\mathbb{C}) = P({}^n\mathbb{C})$ onto $\otimes_s^n P(\mathbb{C})$. It is easy to show that this condition is sufficient in the general case.

Theorem 2. *If there exists a surjective homomorphism $\varphi : \otimes^n A \to \otimes_s^n A$, then every n-homogeneous multiplicative polynomial on algebra A can be represented as a product of characters.*

Proof. Suppose that there exists the surjective homomorphism φ from $\otimes^n A$ onto $\otimes_s^n A$. For a given n-homogeneous multiplicative polynomial $D(x), x \in A$, it is defined a character d on $\otimes_s^n A$ such that

$$d(\otimes^n p) = D(x).$$

Thus

$$\otimes^n A \to \otimes_s^n A \to \mathbb{C}$$

Let $p \in \otimes_s^n A$ and $u \in \varphi^{-1}(p)$, $u \in \otimes^n A$. Then $d \circ \varphi$ is a character on $\otimes^n A$ and $d \circ \varphi(u) = d(p)$. According to [4], every character of $\otimes^n A$ is of the form

$$d \circ \varphi(a \otimes \ldots \otimes a) = \psi_1(a) \ldots \psi_n(a),$$

$a \in A$ for some characters $\psi_1, \dots, \psi_n$ on A. So if $p = a \otimes \ldots \otimes a$, then

$$D(a) = d(p) = d \circ \varphi(u) = \psi_1(a) \ldots \psi_n(a),$$

that is, D is trivial. □

Corollary 1.1. *Let A be the completion of algebra $P(\mathbb{C})$ in some locally convex metrizabled topology τ, such that (A, τ) is a topology algebra. Then every multiplicative polynomial by n degree on A can be presented as a product of characters.*

Theorem 3. *Every multiplicative polynomial of second degree is trivial on the algebra $P(\mathbb{C}^2)$ of polynomials of two variables.*

Proof. We use a similar idea that in proof of Theorem 1. Let $D_2 : P(\mathbb{C}^2) \to \mathbb{C}$ be a homogeneous multiplicative polynomial by second degree. Due to [7], there exists the character d on the symmetric tensor product $P(\mathbb{C}^2) \otimes_s P(\mathbb{C}^2)$ such that $D_2(p) = d_2(p \otimes p)$ for any $p \in P(\mathbb{C}^2)$.

The algebra $P(\mathbb{C}^2) \otimes_s P(\mathbb{C}^2)$ is isomorphic to an algebra of four variables polynomials generated by $p(x_1, y_1)p(x_2, y_2)$, which are symmetric about the permutations of pairs x_1, y_1 and x_2, y_2 simultaneously. These polynomials are called block-symmetric polynomials.

A polynomial wich is symmetric with respect to the permutation of pairs x_1, y_1 and x_2, y_2, can be presented by polynomials $R_1 = x_1 + x_2$, $R_2 = x_1x_2$, $R_3 = y_1 + y_2$, $R_4 = y_1y_2$, $R_5 = x_1y_2 + x_2y_1$, that is $p(x_1, y_1)p(x_2, y_2) = q(R_1, R_2, R_3, R_4, R_5)$. But the polynomials R_1, R_2, R_3, R_4, R_5 are algebraically depending and this dependence can be wrote by the following formula:

$$R_5^2 - R_1R_3R_5 + R_1^2R_4 - 4R_2R_4 + R_3^2R_2 = 0. \tag{1}$$

Thus the polynomial q is an element of factor-algebra which is generated by polynomial (1) and it is determined by this polynomial zeros. The character d on $P(\mathbb{C}^2) \otimes_s P(\mathbb{C}^2)$ is a value of polynomial q at the point $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, which is the solution of the equation (1), that is

$$d(p \otimes p) = d(p(x_1, y_1)p(x_2, y_2)) = q(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5),$$

where $p(x_1, y_1)p(x_2, y_2) = q(R_1, R_2, R_3, R_4, R_5)$. Let $x_1^0, x_2^0, y_1^0, y_2^0$ be the solution of the equations system

$$\begin{cases} x_1 + x_2 = \alpha_1; \\ x_1x_2 = \alpha_2; \\ y_1 + y_2 = \alpha_3; \\ y_1y_2 = \alpha_4; \\ x_1y_2 + x_2y_1 = \alpha_5 \end{cases} .$$

According to the Viet theorem the pairs x_1^0, x_2^0 and y_1^0, y_2^0 are the solution of the equations

$$x^2 - \alpha_1 x + \alpha_2 = 0,$$

$$y^2 - \alpha_3 x + \alpha_4 = 0$$

respectively. Now we need to put $\alpha_5 = x_1^0y_2^0 + x_2^0y_1^0$. Then $D_2(p) = q(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = p(x_1^0, y_1^0)p(x_2^0, y_2^0)$, that was needed to show. □

Using similar methods and recent results on block-symmetric polynomials [1] it is possible to prove an analogue of Theorem 3 for n degree multiplicative polynomials on $P(\mathbb{C}^2)$.

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У цій роботі ми розглядаємо мультиплікативні поліноміальні відображення на комутативних алгебрах. Мультиплікативний поліном, що розкладається в добуток характерів, називатимемо тривіальним. Досліджуємо питання: чи існує нетривіальний мультиплікативний поліном на комутативній алгебрі?

Лабачук О.В., Загороднюк А.В. *Мультипликативные полиномиальные отображения на коммутативных банаховых алгебрах* // Карпатские математические публикации. — 2012. — Т.4, №2. — С. 284–288.

В этой работе мы рассматриваем мультипликативные полиномиальные отображение на коммутативных алгебрах. Мультипликативный полином, которой можна записать произведением характеров, назовем тривиальным. Исследуем вопрос: существует ли нетривиальный мультипликативный полином на коммутативной алгебре?