ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 379–398
doi:10.15330/cmp.11.2.379-398

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.379–398

MULYAVA O.M.[1], SHEREMETA M.M.[2], TRUKHAN YU.S.[2]

PROPERTIES OF SOLUTIONS OF A HETEROGENEOUS DIFFERENTIAL EQUATION OF THE SECOND ORDER

Suppose that a power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $R[A] \in [1, +\infty]$. For a heterogeneous differential equation

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z) w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2) w = A(z)$$

with complex parameters geometrical properties of its solutions (convexity, starlikeness and close-to-convexity) in the unit disk are investigated. Two cases are considered: if $\gamma_2 \neq 0$ and $\gamma_2 = 0$. We also consider cases when parameters of the equation are real numbers. Also we prove that for a solution f of this equation the radius of convergence $R[f]$ equals to $R[A]$ and the recurrent formulas for the coefficients of the power series of $f(z)$ are found. For entire solutions it is proved that the order of a solution f is not less then the order of A ($\varrho[f] \geq \varrho[A]$) and the estimate is sharp. The same inequality holds for generalized orders ($\varrho_{\alpha\beta}[f] \geq \varrho_{\alpha\beta}[A]$). For entire solutions of this equation the belonging to convergence classes is studied. Finally, we consider a linear differential equation of the endless order $\sum\limits_{n=0}^{\infty} \frac{a_n}{n!} w^{(n)} = \Phi(z)$, and study a possible growth of its solutions.

Key words and phrases: differential equation, convexity, starlikeness, close-to-convexity, generalized order, convergence class.

[1] National University of Food Technologies, 68 Volodymyrska str., 01601, Kyiv, Ukraine
[2] Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: info@nuft.edu.ua (Mulyava O.M.), m_m_sheremeta@gmail.com (Sheremeta M.M.), yurkotrukhan@gmail.com (Trukhan Yu.S.)

INTRODUCTION

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [4, p.203] that the condition $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0 (z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan [7] the function f is said to be close-to-convex in $\mathbb{D}$ (see also [4, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\operatorname{Re}(f'(z)/\Phi'(z)) > 0 \, (z \in \mathbb{D})$. A close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that the function f is close-to-convex

УДК 517.925.4
2010 *Mathematics Subject Classification:* 30B10, 30C45,35B08.

in $\mathbb{D}$ if and only if the function $(f(z)-f(0))/f'(0)$ is close-to-convex in $\mathbb{D}$. Therefore, f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z)=z+\sum_{n=2}^{\infty} g_n z^n \tag{2}$$

is close-to-convex in $\mathbb{D}$, where $g_n=f_n/f_1$. We remark that a function defined by (2) is said to be starlike in $\mathbb{D}$, if $g(\mathbb{D})$ is a starlike domain with respect to the origin and the condition $\mathrm{Re}\,\{zg'(z)/g(z)\}>0\,(z\in\mathbb{D})$ is necessary and sufficient for the starlikeness of g. It is clear that every starlike function is close-to-convex. We remark also that if the function g is starlike, then the function cg is starlike, where $c=const$.

S.M. Shah [9] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2w''+(\beta_0z^2+\beta_1z)w'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0$$

under which there exits a transcendental solution given by (1) such that either all its derivatives or even derivatives or odd derivatives are close-to-convex functions in $\mathbb{D}$. The investigations of Shah are continued in the papers [12–15].

Here we consider a heterogeneous differential equation

$$z^2w''+(\beta_0z^2+\beta_1z)w'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=\sum_{n=0}^{\infty} a_nz^n, \tag{3}$$

where parameters β_0, β_1, γ_0, γ_1, γ_2 are complex and the power series $A(z)=\sum_{n=0}^{\infty} a_nz^n$ has the radius of convergence $R[A]\in(0,+\infty]$. We will investigate conditions such that equation (3) has convex or close-to-convex solutions, and in the case if a solution is entire function we will study its possible growth and belonging to convergence classes.

1 PRELIMINARY LEMMAS

At first we remark that an analytic in some neighborhood of the origin of coordinates function given by (1) is a solution of equation (3) if and only if

$$\sum_{n=2}^{\infty} n(n-1)f_nz^n+\beta_0\sum_{n=2}^{\infty}(n-1)f_{n-1}z^n+\gamma_0\sum_{n=2}^{\infty} f_{n-2}z^n$$
$$+\beta_1\sum_{n=1}^{\infty} nf_nz^n+\gamma_1\sum_{n=1}^{\infty} f_{n-1}z^n+\gamma_2\sum_{n=0}^{\infty} f_nz^n\equiv\sum_{n=0}^{\infty} a_nz^n,$$

i. e.

$$\gamma_2f_0=a_0,\quad (\beta_1+\gamma_2)f_1+\gamma_1f_0=a_1 \tag{4}$$

and for $n\geq 2$

$$(n(n+\beta_1-1)+\gamma_2)f_n+(\beta_0(n-1)+\gamma_1)f_{n-1}+\gamma_0f_{n-2}=a_n. \tag{5}$$

Lemma 1. *If a function defined by* (1) *is a solution of equation* (3) *and* $n(n+\beta_1-1)+\gamma_2\neq 0$ *for all* $n\geq 2$, *then* $R[f]=R[A]$.

Proof. Suppose at first that $R[A] < +\infty$. From (5) for $n \geq 2$ we have

$$f_n = -\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)+\gamma_2} f_{n-1} - \frac{\gamma_0}{n(n+\beta_1-1)+\gamma_2} f_{n-2} + \frac{a_n}{n(n+\beta_1-1)+\gamma_2}. \tag{6}$$

Let $n_0 = n_0(R[A])$ is such that for all $n \geq n_0$

$$R[A]\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| \leq \frac{1}{4}, \quad R[A]^2\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right| \leq \frac{1}{4}. \tag{7}$$

Then for each $r < R[A]$

$$\begin{aligned}
\sum_{n=n_0}^{\infty} |f_n| r^n &\leq \sum_{n=n_0}^{\infty} r\left|\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)+\gamma_2}\right| |f_{n-1}| r^{n-1} \\
&+ \sum_{n=n_0}^{\infty} r^2\left|\frac{\gamma_0}{n(n+\beta_1-1)+\gamma_2}\right| |f_{n-2}| r^{n-2} + \sum_{n=n_0}^{\infty} \frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|} \\
&= r\sum_{n=n_0-1}^{\infty}\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| |f_n| r^n \\
&+ r^2\sum_{n=n_0-2}^{\infty}\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right| |f_n| r^n + \sum_{n=n_0}^{\infty}\frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|} \\
&= r\sum_{n=n_0}^{\infty}\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| |f_n| r^n + r\left|\frac{\beta_0(n_0-1)+\gamma_1}{n_0(n_0-1+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0-1} \\
&+ r^2\sum_{n=n_0}^{\infty}\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right| |f_n| r^n + r^2\left|\frac{\gamma_0}{n_0(n_0+\beta_1-1)+\gamma_2}\right| |f_{n_0-2}| r^{n_0-2} \\
&+ r^2\left|\frac{\gamma_0}{(n_0+1)(n_0+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0-1} + \sum_{n=n_0}^{\infty}\frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|},
\end{aligned}$$

whence

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\left(1 - r\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| - r^2\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right|\right) |f_n| r^n \\
&\leq \left|\frac{\beta_0(n_0-1)+\gamma_1}{n_0(n_0-1+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0} + \left|\frac{\gamma_0}{n_0(n_0+\beta_1-1)+\gamma_2}\right| |f_{n_0-2}| r^{n_0} \\
&+ \left|\frac{\gamma_0}{(n_0+1)(n_0+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0+1} + \sum_{n=n_0}^{\infty}\frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|}.
\end{aligned}$$

In view of (7) hence we obtain

$$\begin{aligned}
\frac{1}{2}\sum_{n=n_0}^{\infty} |f_n| r^n &\leq \left|\frac{\beta_0(n_0-1)+\gamma_1}{n_0(n_0-1+\beta_1)+\gamma_2}\right| |f_{n_0-1}| R[A]^{n_0} + \left|\frac{\gamma_0}{n_0(n_0+\beta_1-1)+\gamma_2}\right| |f_{n_0-2}| R[A]^{n_0} \\
&+ \left|\frac{\gamma_0}{(n_0+1)(n_0+\beta_1)+\gamma_2}\right| |f_{n_0-1}| R[A]^{n_0+1} + \sum_{n=n_0}^{\infty}\frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|} < +\infty,
\end{aligned}$$

i. e. $R[f] \geq R[A]$. On the other hand, from (5) we get

$$\begin{aligned}
\sum_{n=2}^{\infty} |a_n| r^n &\leq \sum_{n=2}^{\infty} |(n(n+\beta_1-1)+\gamma_2)| |f_n| r^n \\
&+ r\sum_{n=2}^{\infty} |\beta_0(n-1)+\gamma_1| |f_{n-1}| r^{n-1} + r^2\sum_{n=2}^{\infty} |\gamma_0| |f_{n-2}| r^{n-2},
\end{aligned}$$

and, since the convergence of the series $\sum\limits_{n=n_0}^{\infty} |f_n|r^n$ implies the convergence of each series in right-hand side of the last inequality, we have $R[A] \geq R[f]$. In the case if $R[A] < +\infty$ the equality $R[A] = R[f]$ is proved.

If $R[A] = +\infty$, then the proof of the equality $R[A] = R[f]$ is similar. Now it is enough to choose $n_0 = n_0(R)$ for every $R \in (0, +\infty)$ so that inequality (7) holds with R instead of $R[A]$. Then instead of the inequality $R[f] \geq R[A]$ we obtain the inequality $R[f] \geq R$, whence in view of the arbitrariness of R we get the equality $R[f] = +\infty$. Lemma 1 is proved. □

For the investigation of the convexity and the starlikeness of solutions of differential equation (3) we will use the following lemma ([1,5,6]).

Lemma 2. *If* $\sum\limits_{n=2}^{\infty} n|g_n| \leq 1$, *then function* (2) *is starlike, and if* $\sum\limits_{n=2}^{\infty} n^2|g_n| \leq 1$, *then it is convex in* $\mathbb{D}$.

From Lemma 2 the following lemma follows.

Lemma 3. *If* $\sum\limits_{n=2}^{\infty} n|f_n| \leq |f_1|$, *then function* (1) *is close-to-convex, and if* $\sum\limits_{n=2}^{\infty} n^2|f_n| \leq |f_1|$, *then it is convex in* $\mathbb{D}$.

From the first equality (4) it is clear that the choice of coefficients f_n of solution (1) of equation (3) depends on the equality of the parameter γ_2 to zero.

2 Close-to-convexity and Convexity in the Case $\gamma_2 \neq 0$

From (4) we get $f_0 = a_0/\gamma_2$ and $(\beta_1 + \gamma_2)f_1 = a_1 - \gamma_1 f_0$. Since we find univalent solutions, f_1 must be not equal to zero. In view of (4) two cases are possible:

2a) $a_1 - \gamma_1 f_0 \neq 0$ and $\beta_1 + \gamma_2 \neq 0$;

2b) $a_1 - \gamma_1 f_0 = \beta_1 + \gamma_2 = 0$.

By the conditions 2a) from (4) we get $f_1 = \dfrac{a_1 - \gamma_1 f_0}{\beta_1 + \gamma_2} = \dfrac{\gamma_2 a_1 - \gamma_1 a_0}{\gamma_2(\beta_1 + \gamma_2)}$, and thus the solution is of the form

$$f(z) = \frac{a_0}{\gamma_2} + \frac{\gamma_2 a_1 - \gamma_1 a_0}{\gamma_2(\beta_1 + \gamma_2)} z + \sum_{n=2}^{\infty} f_n z^n, \tag{8}$$

where the coefficients f_n are defined by the recurrent formula (5). Supposing that $n(n + \beta_1 - 1) + \gamma_2 \neq 0$ for all $n \geq 2$, this formula can be rewritten in the form (6).

Suppose that $|\beta_1| < 1$ and $|\gamma_2|/2 < (1 - |\beta_1|)$. Then $|n(n + \beta_1 - 1) + \gamma_2| \geq$ $\geq n(n - 1 - |\beta_1|) - |\gamma_2|$ and, since the function $x^2 - (1 + |\beta_1|)x - |\gamma_2|$ is increasing on $[2, +\infty)$, we have $n(n - 1 - |\beta_1|) - |\gamma_2| \geq 2(1 - |\beta_1|) - |\gamma_2| > 0$ for all $n \geq 2$. Therefore, (6) implies

$$|f_n| \leq \frac{|\beta_0|(n-1) + |\gamma_1|}{n(n - 1 - |\beta_1|) - |\gamma_2|} |f_{n-1}| + \frac{|\gamma_0|}{n(n - 1 - |\beta_1|) - |\gamma_2|} |f_{n-2}| + \frac{|a_n|}{n(n - 1 - |\beta_1|) - |\gamma_2|}. \tag{9}$$

Hence it follows that

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n|f_n| \le \sum_{n=2}^{\infty} \frac{n}{n-1}\frac{|\beta_0|(n-1)+|\gamma_1|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-1)|f_{n-1}| \\
&+\sum_{n=2}^{\infty} \frac{n}{n-2}\frac{|\gamma_0|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-2)|f_{n-2}| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|} \\
&=\sum_{n=1}^{\infty}\frac{n+1}{n}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n|f_n| + \sum_{n=0}^{\infty}\frac{n+2}{n}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n|f_n| \\
&+\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|} = \sum_{n=2}^{\infty}\frac{n+1}{n}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n|f_n| \\
&+2\frac{|\beta_0|+|\gamma_1|}{2(1-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty}\frac{n+2}{n}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n|f_n| \\
&+\frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}.
\end{aligned}
\tag{10}
$$

Since for $n \ge 2$

$$
\frac{n+1}{n}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|} = \frac{|\beta_0|+|\gamma_1|/n}{(n-|\beta_1|)-|\gamma_2|/(n+1)} \le \frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}
$$

and

$$
\frac{n+2}{n}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|} = \frac{|\gamma_0|/n}{(n+1-|\beta_1|)-|\gamma_2|/(n+2)} \le \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4},
$$

from (10) it follows that

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n|f_n| \le \sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}n|f_n| + \sum_{n=2}^{\infty}\frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}n|f_n| + \frac{2(|\beta_0|+|\gamma_1|)|f_1|}{2(1-|\beta_1|)-|\gamma_2|} \\
&+\frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}
\end{aligned}
$$

and by the condition

$$
\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3} + \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4} < 1 \tag{11}
$$

we obtain

$$
\begin{aligned}
&\left(1-\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3} - \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}\right)\sum_{n=2}^{\infty} n|f_n| \le 2\frac{|\beta_0|+|\gamma_1|}{2(1-|\beta_1|)-|\gamma_2|}|f_1| \\
&+\frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|},
\end{aligned}
$$

whence

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n|f_n| \le \left(\left(\frac{2(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|} + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)|f_1| + \frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| \right. \\
&\left. +\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\right)\left(1-\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3} - \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}\right)^{-1}.
\end{aligned}
\tag{12}
$$

By Lemma 3 solution (1) of equation (3) is close-to-convex if the right-hand side of (12) is less than $|f_1|$, i. e.

$$\left(\frac{2(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}+\frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)|f_1|+\frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\le\left(1-\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}\right)|f_1|. \tag{13}$$

Thus, the following proposition is proved.

Proposition 1. *Let $\gamma_2\neq 0$, $a_1\gamma_2-a_0\gamma_1\neq 0$, $\beta_1+\gamma_2\neq 0$, $|\beta_1|<1$, $|\gamma_2|/2<(1-|\beta_1|)$ and $R[A]\ge 1$. If*

$$\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\le\left(1-\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}-\frac{2(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}-\frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)\frac{|\gamma_2a_1-\gamma_1a_0|}{|\gamma_2(\beta_1+\gamma_2)|}-\frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}\frac{|a_0|}{|\gamma_2|}, \tag{14}$$

then there exists a solution given by (8) *of differential equation* (3) *with $R[f]=R[A]$, which is close-to-convex in $\mathbb{D}$. If moreover $a_0=0$ it is starlike.*

Indeed, the condition (14) is equivalent to condition (13), and (13) implies (11).

We will pass to the convexity. From (9) we get

$$\begin{aligned}
\sum_{n=2}^{\infty}n^2|f_n| &\le \sum_{n=2}^{\infty}\frac{n^2}{(n-1)^2}\frac{|\beta_0|(n-1)+|\gamma_1|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-1)^2|f_{n-1}| \\
&+\sum_{n=2}^{\infty}\frac{n^2}{(n-2)^2}\frac{|\gamma_0|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-2)^2|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|} \\
&=\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^2}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n^2|f_n| \\
&+\sum_{n=0}^{\infty}\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n^2|f_n|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|} \\
&=\sum_{n=2}^{\infty}\frac{(n+1)^2}{n^2}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n^2|f_n|+4\frac{|\beta_0|+|\gamma_1|}{2(1-|\beta_1|)-|\gamma_2|}|f_1| \\
&+\sum_{n=2}^{\infty}\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n^2|f_n|+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| \\
&+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}.
\end{aligned}$$

Since now for $n\ge 2$

$$\frac{(n+1)^2}{n^2}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}\le\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}$$

and

$$\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}\le 2\frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4},$$

by the condition

$$\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}+\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}<1,$$

as above we obtain

$$\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right)\sum_{n=2}^{\infty}n^2|f_n|\leq\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}|f_1|$$
$$+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|},$$

i. e.

$$\sum_{n=2}^{\infty}n^2|f_n|\leq\left(\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}|f_1|+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|\right.$$
$$\left.+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\right)\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right)^{-1}, \tag{15}$$

By Lemma 3 a solution given by (1) of equation (3) is convex if the right-hand side of (15) is less than $|f_1|$, i. e.

$$\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}|f_1|+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|$$
$$+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\leq\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right)|f_1|.$$

Thus, the following proposition is proved.

Proposition 2. *Let* $\gamma_2\neq0$, $a_1\gamma_2-a_0\gamma_1\neq0$, $\beta_1+\gamma_2\neq0$, $|\beta_1|<1$, $|\gamma_2|/2<(1-|\beta_1|)$ *and* $R[A]\geq1$. *If*

$$\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\leq\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right.$$
$$\left.-\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}-\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)\frac{|\gamma_2a_1-\gamma_1a_0|}{|\gamma_2(\beta_1+\gamma_2)|}-\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}\frac{|a_0|}{|\gamma_2|}, \tag{16}$$

then there exists a solution defined by (8) *of differential equation* (3) *with* $R[f]=R[A]$, *which is convex in* $\mathbb{D}$.

Uniting Propositions 1 and 2 we get such theorem.

Theorem 1. *Let* $\gamma_2\neq0$, $a_1\gamma_2-a_0\gamma_1\neq0$, $\beta_1+\gamma_2\neq0$, $|\beta_1|<1$, $|\gamma_2|/2<(1-|\beta_1|)$ *and* $R[A]\geq1$. *Then there exists a solution given by* (8) *of differential equation* (3) *with* $R[f]=R[A]$, *which by the condition* (14) *is close-to-convex and by the condition* (16) *is convex in* $\mathbb{D}$. *If* $a_0=0$ *and* (14) *holds then* (8) *is starlike.*

The conditions $|\beta_1|<1$ and $|\gamma_2|/2<(1-|\beta_1|)$ in Theorem 1 can be weakened if β_1 and γ_2 are real numbers. We will consider a simple case, when $\gamma_2>0$, $\beta_1>-1$ and $\gamma_2+\beta_1>0$.

Suppose also that $\gamma_2 a_1 - \gamma_1 a_0 \neq 0$. Then from recurrent formula (6) we have

$$\sum_{n=2}^{\infty} n|f_n| \le \sum_{n=2}^{\infty} \frac{n}{n-1}\frac{|\beta_0|(n-1)+|\gamma_1|}{n(n+\beta_1-1)+\gamma_2}(n-1)|f_{n-1}|$$
$$+\sum_{n=2}^{\infty}\frac{n}{n-2}\frac{|\gamma_0|}{n(n+\beta_1-1)+\gamma_2}(n-2)|f_{n-2}| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2}$$
$$\le \sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/(n-1)}{(n+\beta_1-1)+\gamma_2/n}(n-1)|f_{n-1}| + \sum_{n=2}^{\infty}\frac{|\gamma_0|/(n-2)}{(n+\beta_1-1)+\gamma_2/n}(n-2)|f_{n-2}|$$
$$+\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2} \le \sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/(n-1)}{(n+\beta_1-1)}(n-1)|f_{n-1}|$$
$$+\sum_{n=2}^{\infty}\frac{|\gamma_0|/(n-2)}{(n+\beta_1-1)}(n-2)|f_{n-2}| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2}$$
$$=\sum_{n=1}^{\infty}\frac{|\beta_0|+|\gamma_1|/n}{n+\beta_1}n|f_n| + \sum_{n=0}^{\infty}\frac{|\gamma_0|/n}{(n+\beta_1+1)}n|f_n| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2}$$
$$\le \frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1| + \sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}n|f_n| + \frac{|\gamma_0|}{\beta_1+1}|f_0| + \frac{|\gamma_0|}{2+\beta_1}|f_1|$$
$$+\sum_{n=2}^{\infty}\frac{|\gamma_0|/2}{3+\beta_1}n|f_n| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2},$$

whence by the condition

$$\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1} + \frac{|\gamma_0|/2}{3+\beta_1} < 1$$

we obtain

$$\left(1-\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}-\frac{|\gamma_0|/2}{3+\beta_1}\right)\sum_{n=2}^{\infty} n|f_n| \le \frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1|$$
$$+\frac{|\gamma_0|}{\beta_1+1}|f_0| + \frac{|\gamma_0|}{2+\beta_1}|f_1| + \sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2}. \tag{17}$$

Similarly we get

$$\sum_{n=2}^{\infty} n^2|f_n| \le \sum_{n=2}^{\infty}\frac{n}{n-1}\frac{|\beta_0|+|\gamma_1|/(n-1)}{n+\beta_1-1}(n-1)^2|f_{n-1}|$$
$$+\sum_{n=2}^{\infty}\frac{n|\gamma_0|/(n-2)^2}{(n+\beta_1-1)}(n-2)^2|f_{n-2}| + \sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2}$$
$$=\sum_{n=1}^{\infty}\frac{n+1}{n}\frac{|\beta_0|+|\gamma_1|/n}{n+\beta_1}n^2|f_n| + \sum_{n=0}^{\infty}\frac{(n+2)|\gamma_0|/n^2}{(n+\beta_1+1)}n^2|f_n| + \sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2}$$
$$\le 2\frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1| + \sum_{n=2}^{\infty}\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}n^2|f_n| + \frac{2|\gamma_0|}{\beta_1+1}|f_0| + \frac{3|\gamma_0|}{2+\beta_1}|f_1|$$
$$+\sum_{n=2}^{\infty}\frac{|\gamma_0|}{3+\beta_1}n^2|f_n| + \sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2},$$

whence by the condition

$$\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1} + \frac{|\gamma_0|}{3+\beta_1} < 1$$

we get

$$\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}-\frac{|\gamma_0|}{3+\beta_1}\right)\sum_{n=2}^{\infty}n^2|f_n|\leq 2\frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1|$$
$$+\frac{2|\gamma_0|}{\beta_1+1}|f_0|+\frac{3|\gamma_0|}{2+\beta_1}|f_1|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2}. \tag{18}$$

From (17) and (18) we obtain the following proposition.

Proposition 3. *Let $\gamma_2>0$, $\beta_1>-1$, $\gamma_2+\beta_1>0$, $\gamma_2a_1-\gamma_1a_0\neq 0$ and $R[A]\geq 1$. Then there exists a solution* (8) *of differential equation* (3) *with $R[f]=R[A]$, which by the condition*

$$\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2}\leq\left(1-\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}-\frac{|\gamma_0|/2}{3+\beta_1}-\frac{|\beta_0|+|\gamma_1|}{1+\beta_1}\right.$$
$$\left.-\frac{|\gamma_0|}{2+\beta_1}\right)\frac{|\gamma_2a_1-\gamma_1a_0|}{\gamma_2(\beta_1+\gamma_2)}-\frac{|\gamma_0|}{\beta_1+1}\frac{|a_0|}{|\gamma_2|}$$

is close-to-convex (starlike if $a_0=0$) and by the condition

$$\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2}\leq\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}-\frac{|\gamma_0|}{3+\beta_1}-2\frac{|\beta_0|+|\gamma_1|}{1+\beta_1}\right.$$
$$\left.-\frac{3|\gamma_0|}{2+\beta_1}\right)\frac{|\gamma_2a_1-\gamma_1a_0|}{\gamma_2(\beta_1+\gamma_2)}-\frac{2|\gamma_0|}{1+\beta_1}\frac{|a_0|}{|\gamma_2|}$$

is a convex function in $\mathbb{D}$.

Now we suppose that the condition 2b) holds, that is, $\gamma_2\neq 0$ and $a_1-\gamma_1f_0=\beta_1+\gamma_2=0$. Then $f_0=a_0/\gamma_2$ and f_1 can be arbitrary number, in particular we can choose $f_1=1$. Thus, the solution will have a form

$$f(z)=\frac{a_0}{\gamma_2}+z+\sum_{n=2}^{\infty}f_nz^n, \tag{19}$$

where the coefficients f_n are defined by the recurrent formula

$$(n-1)(n+\beta_1)f_n+(\beta_0(n-1)+\gamma_1)f_{n-1}+\gamma_0f_{n-2}=a_n.$$

Supposing that $n+\beta_1\neq 0$ for all $n\geq 2$, this formula can be rewritten in the form

$$f_n=-\frac{\beta_0(n-1)+\gamma_1}{(n-1)(n+\beta_1)}f_{n-1}-\frac{\gamma_0}{(n-1)(n+\beta_1)}f_{n-2}+\frac{a_n}{(n-1)(n+\beta_1)},$$

whence by the condition $|\beta_1|<2$ we have

$$\sum_{n=2}^{\infty}n|f_n|\leq\sum_{n=2}^{\infty}\frac{n}{n-1}\frac{(n-1)|\beta_0|+|\gamma_1|}{(n-1)(n-|\beta_1|)}(n-1)|f_{n-1}|$$
$$+\sum_{n=2}^{\infty}\frac{n}{n-2}\frac{|\gamma_0|}{(n-1)(n-|\beta_1|)}(n-2)|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)}$$
$$=2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\sum_{n=2}^{\infty}\frac{n+1}{n}\frac{|\beta_0|+|\gamma_1|/n}{n+1-|\beta_1|}n|f_n|+\frac{2|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{3|\gamma_0|}{2(3-|\beta_1|)}$$
$$+\sum_{n=2}^{\infty}\frac{n+2}{n}\frac{|\gamma_0|}{(n+1)(n+2-|\beta_1|)}n|f_n|+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)}$$
$$\leq\sum_{n=2}^{\infty}\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{3-|\beta_1|}n|f_n|+\sum_{n=2}^{\infty}\frac{2|\gamma_0|}{3(4-|\beta_1|)}n|f_n|+2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}$$
$$+\frac{2|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{3|\gamma_0|}{2(3-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)},$$

i. e. by the condition

$$\frac{3(2|\beta_0|+|\gamma_1|)}{4(3-|\beta_1|)}+\frac{2|\gamma_0|}{3(4-|\beta_1|)}<1$$

we get

$$\begin{gathered}\left(1-\frac{3(2|\beta_0|+|\gamma_1|)}{4(3-|\beta_1|)}-\frac{2|\gamma_0|}{3(4-|\beta_1|)}\right)\sum_{n=2}^{\infty}n|f_n| \\ \le 2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{2|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{3|\gamma_0|}{2(3-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)},\end{gathered}\tag{20}$$

Similarly,

$$\begin{aligned}&\sum_{n=2}^{\infty}n^2|f_n|\le\sum_{n=2}^{\infty}\frac{n^2}{(n-1)^2}\frac{(n-1)|\beta_0|+|\gamma_1|}{(n-1)(n-|\beta_1|)}(n-1)^2|f_{n-1}| \\ &+\sum_{n=2}^{\infty}\frac{n^2}{(n-2)^2}\frac{|\gamma_0|}{(n-1)(n-|\beta_1|)}(n-2)^2|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)} \\ &=4\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\sum_{n=2}^{\infty}\frac{(n+1)^2}{n^2}\frac{|\beta_0|+|\gamma_1|/n}{n+1-|\beta_1|}n^2|f_n|+\frac{4|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{9|\gamma_0|}{2(3-|\beta_1|)} \\ &+\sum_{n=2}^{\infty}\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+1)(n+2-|\beta_1|)}n^2|f_n|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)} \\ &\le\sum_{n=2}^{\infty}\frac{9}{4}\frac{|\beta_0|+|\gamma_1|/2}{3-|\beta_1|}n^2|f_n|+\sum_{n=2}^{\infty}\frac{16}{4}\frac{|\gamma_0|}{3(4-|\beta_1|)|}n^2|f_n| \\ &+4\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{4|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{9|\gamma_0|}{2(3-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)},\end{aligned}$$

i. e. by the condition

$$\frac{9(2|\beta_0|+|\gamma_1|)}{8(3-|\beta_1|)}+\frac{4|\gamma_0|}{3(4-|\beta_1)|}<1$$

we get

$$\begin{gathered}\left(1-\frac{9(2|\beta_0|+|\gamma_1|)}{8(3-|\beta_1|)}-\frac{4|\gamma_0|}{3(4-|\beta_1|)}\right)\sum_{n=2}^{\infty}n^2|f_n| \\ \le 4\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{9|\gamma_0|}{2(3-|\beta_1|)}+\frac{4|\gamma_0|}{2-|\beta_1|}\frac{|a_0|}{|\gamma_2|}+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)}.\end{gathered}\tag{21}$$

In view of Lemma 3 from (20) and (21), as in the proof of Proposition 1, we obtain the following theorem.

Theorem 2. *Let* $\gamma_2\neq0$, $a_1\gamma_2-a_0\gamma_1=\beta_1+\gamma_2=0$, $|\beta_1|<2$ *and* $R[A]\ge1$. *Then there exists a solution given by* (19) *of differential equation* (3) *with* $R[f]=R[A]$, *which by the condition*

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)}\le 1&-\frac{3(2|\beta_0|+|\gamma_1|)}{4(3-|\beta_1|)}-\frac{2|\gamma_0|}{3(4-|\beta_1|)} \\ &-2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}-\frac{3|\gamma_0|}{2(3-|\beta_1|)}-\frac{2|\gamma_0|}{2-|\beta_1|}\frac{|a_0|}{|\gamma_2|}\end{aligned}$$

is close-to-convex (if $a_0 = 0$ then starlike) and by the condition

$$\sum_{n=2}^{\infty} \frac{n^2|a_n|}{(n-1)(n-|\beta_1|)} \le 1 - \frac{9(2|\beta_0|+|\gamma_1|)}{8(3-|\beta_1|)} - \frac{4|\gamma_0|}{3(4-|\beta_1|)} - 4\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|} - \frac{9|\gamma_0|}{2(3-|\beta_1|)} - \frac{4|\gamma_0|}{2-|\beta_1|}\frac{|a_0|}{|\gamma_2|}$$

is a convex function in $\mathbb{D}$.

In the case of real parameters γ_2 and β_1 as above it is easy to obtain following statement.

Proposition 4. *Let $\gamma_2 > 0$, $a_1\gamma_2 - a_0\gamma_1 = \beta_1 + \gamma_2 = 0$, $\beta_1 > -2$ and $R[A] \ge 1$. Then there exists a solution given by* (19) *of differential equation* (3) *with $R[f] = R[A]$, which by the condition*

$$\sum_{n=2}^{\infty} \frac{n|a_n|}{(n-1)(n+\beta_1)} \le 1 - \frac{3}{4}\frac{2|\beta_0|+|\gamma_1|}{4(3+\beta_1)} - \frac{2|\gamma_0|}{3(4+\beta_1)} - 2\frac{|\beta_0|+|\gamma_1|}{2+\beta_1} - \frac{3|\gamma_0|}{2(3+\beta_1)} - \frac{2|\gamma_0||a_0|}{(2+\beta_1)\gamma_2}$$

is close-to-convex (starlike if $a_0 = 0$) and by the condition

$$\sum_{n=2}^{\infty} \frac{n^2|a_n|}{(n-1)(n+\beta_1)} \le 1 - \frac{9}{8}\frac{2|\beta_0|+|\gamma_1|}{3+\beta_1} - \frac{(4/3)|\gamma_0|}{4+\beta_1} - 4\frac{|\beta_0|+|\gamma_1|}{2+\beta_1} - \frac{(9/2)|\gamma_0|}{3+\beta_1} - \frac{4|\gamma_0||a_0|}{(2+\beta_1)\gamma_2}$$

is a convex function in $\mathbb{D}$.

3 CLOSE-TO-CONVEXITY AND CONVEXITY IN THE CASE $\gamma_2 = 0$

In this case from (4) it follows that $a_0 = 0$, i. e. f_0 can be arbitrary number, and we choose $f_0 = 0$. Then $\beta_1 f_1 = a_1$. Since we are finding univalent solutions $f_1 \ne 0$. Therefore, two cases are possible:

3a) $a_1 \ne 0$ and $\beta_1 \ne 0$;

3b) $a_1 = \beta_1 = 0$.

By the condition 3a) a solution of equation (3) has the form

$$f(z) = \frac{a_1}{\beta_1} z + \sum_{n=2}^{\infty} f_n z^n, \tag{22}$$

where the coefficients f_n are defined by recurrent formula

$$n(n+\beta_1-1)f_n + (\beta_0(n-1)+\gamma_1)f_{n-1} + \gamma_0 f_{n-2} = a_n,$$

from which by the condition $n + \beta_1 - 1 \ne 0$ for all $n \ge 2$ it follows that

$$f_n = -\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)} f_{n-1} - \frac{\gamma_0}{n(n+\beta_1-1)} f_{n-2} + \frac{a_n}{n(n+\beta_1-1)},$$

whence by the condition $|\beta_1|<1$ we get

$$\begin{aligned}\sum_{n=2}^{\infty} n|f_n| &\le \sum_{n=2}^{\infty}\frac{n}{n-1}\frac{(n-1)|\beta_0|+|\gamma_1|}{n(n-|\beta_1|-1)}(n-1)|f_{n-1}|\\ &+\sum_{n=2}^{\infty}\frac{n}{n-2}\frac{|\gamma_0|}{n(n-|\beta_1|-1)}(n-2)|f_{n-2}|+\sum_{n=2}^{\infty}\frac{|a_n|}{n-|\beta_1|-1}\\ &=\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}|f_1|+\sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/n}{n-|\beta_1|}n|f_n|+\frac{|\gamma_0|}{2-|\beta_1|}|f_1|+\sum_{n=2}^{\infty}\frac{|\gamma_0|/n}{n+1-|\beta_1|}n|f_n|\\ &+\sum_{n=2}^{\infty}\frac{|a_n|}{n-|\beta_1|-1}\le\sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/2}{2-|\beta_1|}n|f_n|+\sum_{n=2}^{\infty}\frac{|\gamma_0|/2}{3-|\beta_1|}n|f_n|\\ &+\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{|a_n|}{(n-|\beta_1|-1)},\end{aligned}$$

i. e. by the condition

$$\frac{|\beta_0|+|\gamma_1|/2}{2-|\beta_1|}+\frac{|\gamma_0|/2}{3-|\beta_1|}<1$$

we obtain

$$\begin{aligned}&\left(1-\frac{|\beta_0|+|\gamma_1|/2}{2-|\beta_1|}-\frac{|\gamma_0|/2}{3-|\beta_1|}\right)\sum_{n=2}^{\infty} n|f_n|\\ &\le\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{|a_n|}{(n-|\beta_1|-1)}.\end{aligned}\tag{23}$$

Similarly,

$$\begin{aligned}\sum_{n=2}^{\infty} n^2|f_n| &\le\sum_{n=2}^{\infty}\frac{n^2}{(n-1)^2}\frac{(n-1)|\beta_0|+|\gamma_1|}{n(n-|\beta_1|-1)}(n-1)^2|f_{n-1}|\\ &+\sum_{n=2}^{\infty}\frac{n^2}{(n-2)^2}\frac{|\gamma_0|}{n(n-|\beta_1|-1)}(n-2)^2|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n|a_n|}{n-|\beta_1|-1}\\ &=2\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\sum_{n=2}^{\infty}\frac{n+1}{n}\frac{|\beta_0|+|\gamma_1|/n}{n-|\beta_1|}n^2|f_n|+\frac{3|\gamma_0|}{2-|\beta_1|}|f_1|\\ &+\sum_{n=2}^{\infty}\frac{n+2}{n^2}\frac{|\gamma_0|}{n+1-|\beta_1|}n^2|f_n|+\sum_{n=2}^{\infty}\frac{n|a_n|}{n-|\beta_1|-1}\le\sum_{n=2}^{\infty}\frac{3}{4}\frac{2|\beta_0|+|\gamma_1|}{2-|\beta_1|}n^2|f_n|\\ &+\sum_{n=2}^{\infty}\frac{|\gamma_0|}{3-|\beta_1|}n^2|f_n|+2\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{3|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-|\beta_1|-1)},\end{aligned}$$

i. e. by the condition

$$\frac{3}{4}\frac{2|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{|\gamma_0|}{3-|\beta_1|}<1$$

we get

$$\begin{aligned}&\left(1-\frac{3}{4}\frac{2|\beta_0|+|\gamma_1|}{2-|\beta_1|}-\frac{|\gamma_0|}{3-|\beta_1|}\right)\sum_{n=2}^{\infty} n^2|f_n|\\ &\le 2\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{3|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-|\beta_1|-1)}.\end{aligned}\tag{24}$$

In view of Lemma 3 from (23) and (24) in the usual way we obtain the following theorem.

Theorem 3. *Let* $\gamma_2 = 0$, $a_1 \neq 0$, $\beta_1 \neq 0$, $|\beta_1| < 1$ *and* $R[A] \geq 1$. *Then there exists a solution given by* (22) *of differential equation* (3) *with* $R[f] = R[A]$, *which by the condition*

$$\sum_{n=2}^{\infty} \frac{|a_n|}{(n - |\beta_1| - 1)} \leq \left(1 - \frac{|\beta_0| + |\gamma_1|/2 + |\gamma_0|}{2 - |\beta_1|} - \frac{|\gamma_0|/2}{3 - |\beta_1|} - \frac{|\beta_0| + |\gamma_1|}{1 - |\beta_1|}\right) \frac{|a_1|}{|\beta_1|}$$

is starlike, and by the condition

$$\sum_{n=2}^{\infty} \frac{n|a_n|}{(n - |\beta_1| - 1)} \leq \left(1 - \frac{3}{4}\frac{2|\beta_0| + |\gamma_1| + 4|\gamma_0|}{2 - |\beta_1|} - \frac{|\gamma_0|}{3 - |\beta_1|} - 2\frac{|\beta_0| + |\gamma_1|}{1 - |\beta_1|}\right) \frac{|a_1|}{|\beta_1|}$$

is a convex function in $\mathbb{D}$.

For a real parameter β_1 in the usual way we obtain the following proposition.

Proposition 5. *Let* $\gamma_2 = 0$, $a_1 \neq 0$, $\beta_1 \neq 0$, $\beta_1 > -1$ *and* $R[A] \geq 1$. *Then there exists a solution given by* (22) *of differential equation* (3) *with* $R[f] = R[A]$, *which by the condition*

$$\sum_{n=2}^{\infty} \frac{|a_n|}{(n + \beta_1 - 1)} \leq \left(1 - \frac{|\beta_0| + |\gamma_1|/2 + |\gamma_0|}{2 + \beta_1} - \frac{|\gamma_0|/2}{3 + \beta_1} - \frac{|\beta_0| + |\gamma_1|}{1 + \beta_1}\right) \frac{|a_1|}{|\beta_1|}$$

is starlike, and by the condition

$$\sum_{n=2}^{\infty} \frac{n|a_n|}{(n + \beta_1 - 1)} \leq \left(1 - \frac{3}{4}\frac{2|\beta_0| + |\gamma_1| + 4|\gamma_0|}{2 + \beta_1} - \frac{|\gamma_0|}{3 + \beta_1} - 2\frac{|\beta_0| + |\gamma_1|}{1 + \beta_1}\right) \frac{|a_1|}{|\beta_1|}$$

is a convex function in $\mathbb{D}$.

If the condition 3b) holds then we can choose $f_1 = 1$ and search a solution in a form

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n, \tag{25}$$

where the coefficients f_n are defined by recurrent formula

$$f_n = -\frac{\beta_0(n-1) + \gamma_1}{n(n-1)} f_{n-1} - \frac{\gamma_0}{n(n-1)} f_{n-2} + \frac{a_n}{n(n-1)}. \tag{26}$$

Then

$$\begin{aligned}\sum_{n=2}^{\infty} n|f_n| &\leq |\beta_0| + |\gamma_1| + \sum_{n=2}^{\infty} \frac{n|\beta_0| + |\gamma_1|}{n^2} n|f_n| + \frac{|\gamma_0|}{2} + \sum_{n=2}^{\infty} \frac{|\gamma_0|}{(n+1)n} n|f_n| + \sum_{n=2}^{\infty} \frac{|a_n|}{n-1} \\ &\leq \sum_{n=2}^{\infty} \frac{2|\beta_0| + |\gamma_1|}{4} n|f_n| + \sum_{n=2}^{\infty} \frac{|\gamma_0|}{6} n|f_n| + |\beta_0| + |\gamma_1| + \frac{|\gamma_0|}{2} + \sum_{n=2}^{\infty} \frac{|a_n|}{n-1}\end{aligned}$$

and by the condition $(2|\beta_0| + |\gamma_1|)/4 + |\gamma_0|/6 < 1$ we get

$$(1 - (2|\beta_0| + |\gamma_1|)/4 - |\gamma_0|/6) \sum_{n=2}^{\infty} n|f_n| \leq |\beta_0| + |\gamma_1| + |\gamma_0|/2 + \sum_{n=2}^{\infty} \frac{|a_n|}{n-1}. \tag{27}$$

Similarly,

$$\begin{aligned}\sum_{n=2}^{\infty} n^2|f_n| &\leq \sum_{n=2}^{\infty} n^2 \frac{|\beta_0|(n-1) + |\gamma_1|}{n(n-1)} |f_{n-1}| + \sum_{n=2}^{\infty} n^2 \frac{|\gamma_0|}{n(n-1)} |f_{n-2}| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n-1} \\ &\leq 2(|\beta_0| + |\gamma_1|) + \sum_{n=2}^{\infty} \frac{3}{8}(2|\beta_0| + |\gamma_1|) n^2|f_n| + 3|\gamma_0|/2 + \sum_{n=2}^{\infty} \frac{|\gamma_0|}{3} n^2|f_n| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n-1},\end{aligned}$$

i. e. by the condition $3(2|\beta_0|+|\gamma_1|)/8+|\gamma_0|/3<1$

$$(1-3(2|\beta_0|+|\gamma_1|)/8-|\gamma_0|/3)\sum_{n=2}^{\infty}n^2|f_n|\le 2(|\beta_0|+|\gamma_1|)+3|\gamma_0|/2+\sum_{n=2}^{\infty}\frac{n|a_n|}{n-1}. \tag{28}$$

In view of Lemma 2 from (27) and (28) in the usual way we obtain the following theorem.

Theorem 4. *Let $\gamma_2=a_0=\beta_1=a_1=0$ and $R[A]\ge 1$. Then there exists a solution given by* (25) *of differential equation* (3) *with $R[f]=R[A]$, which by the condition*

$$\sum_{n=2}^{\infty}\frac{|a_n|}{n-1}\le 1-\frac{3}{2}|\beta_0|-\frac{5}{4}|\gamma_1|-\frac{2}{3}|\gamma_0| \tag{29}$$

is starlike, and by the condition

$$\sum_{n=2}^{\infty}\frac{n|a_n|}{n-1}\le 1-\frac{11}{4}|\beta_0|-\frac{19}{8}|\gamma_1|-\frac{11}{6}|\gamma_0| \tag{30}$$

is a convex function in $\mathbb{D}$.

4 GROWTH OF ENTIRE SOLUTIONS

If $n(n+\beta_1-1)+\gamma_2\ne 0$ for all $n\ge 2$ by Lemma 1 a function given by (1) can be an entire solution of equation (3) only if the function A is entire.

For an entire function (1) let $M_f(r)=\max\{|f(z)|:|z|=r\}$, and for the characteristic of the growth of $M_f(r)$ we will use generalized orders. To give a definition of generalized order we denote, as in [11], by L a class of continuous nonnegative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge 0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each fixed $c\in(0,+\infty)$, i. e. α is slowly increasing function. Clearly, $L_{si}\subset L^0$. The value

$$\varrho_{\alpha\beta}[f]=\varlimsup_{r\to+\infty}\frac{\alpha(\ln M_f(r))}{\beta(\ln r)}\quad(\alpha\in L,\beta\in L)$$

is called [11] generalized order of f. The following lemma is true.

Lemma 4. *If $\alpha\in L_{si}$, $\beta\in L$, $\beta(x+O(1))=(1+o(1))\beta(x)$ as $x\to+\infty$ and f is an entire transcendental function then $\varrho_{\alpha\beta}[f']=\varrho_{\alpha\beta}[f]$.*

Proof. Indeed, from the integral formula of Cauchy it easily follows that $M_{f'}(r)\le M_f(r+1)$, whence we get $\varrho_{\alpha\beta}[f']\le\varrho_{\alpha\beta}[f]$. On the other hand, since $f(z)-f(0)=\int\limits_0^z f'(t)dt$, we have $M_f(r)\le rM_{f'}(r)+|f(0)|$ and, thus, $\ln M_f(r)\le\ln M_{f'}(r)+\ln r+o(1)=(1+o(1))\ln M_{f'}(r)$ as $r\to+\infty$, because the function f is transcendental. Hence we get $\varrho_{\alpha\beta}[f]\le\varrho_{\alpha\beta}[f']$. Lemma 4 is proved. □

We will use the theory of the value distribution of Nevanlinna. For an entire function f we put

$$T(r,f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\ln^+|f(re^{i\varphi})|d\varphi.$$

This function is said to be a characteristic function of Nevanlinna. It is known that

Lemma 5. *If $\alpha \in L_{si}$, $\beta \in L$, $\beta(x+O(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$ and f is an entire transcendental function then*

$$\varrho_{\alpha\beta}[f] = \varlimsup_{r\to+\infty} \frac{\alpha(T(r,f))}{\beta(\ln r)}. \tag{31}$$

Proof. Indeed, in [3, p. 54] it is proved that for $0 < r < r_1$

$$T(r,f) \le \ln^+ M_f(r) \le \frac{r_1+r}{r_1-r} T(r_1,f). \tag{32}$$

Choosing $r_1 = 2r$ and using (32), in view of the conditions $\alpha \in L_{si}$ and $\beta \in L^0$ hence we obtain

$$\begin{aligned} &\varlimsup_{r\to+\infty} \frac{\alpha(T(r,f))}{\beta(\ln r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(3T(2r,f))}{\beta(\ln r)} \\ &= \varlimsup_{r\to+\infty} \frac{\alpha(T(r,f))}{\beta(\ln r - \ln 2)} = \varlimsup_{r\to+\infty} \frac{\alpha(T(r,f))}{\beta(\ln r)}. \end{aligned}$$

Lemma 5 is proved. □

Now we prove the following theorem.

Theorem 5. *Let $\alpha \in L_{si}$, $\beta \in L$, $\alpha(\ln x) = o(\alpha(x))$, $\beta(x+O(1)) = (1+o(1))\beta(x)$, $\alpha(x) = o(\beta(x))$ as $x \to +\infty$ and f be an entire transcendental solution of the differential equation*

$$a_0(z)w + a_1(z)w' + \cdots + a_m(z)w^{(m)} = A(z), \tag{33}$$

where a_j are polynomials, $0 \le j \le m$, and A is an entire function. Then $\varrho_{\alpha\beta}[f] \ge \varrho_{\alpha\beta}[A]$.

Proof. If $\varrho_{\alpha\beta}[f] = +\infty$ then theorem is obvious.

So we consider the case $\varrho_{\alpha\beta}[f] < +\infty$. At first we remark that if P_m is a polynomial of degree $m \ge 1$ then [3, p.47] $T(r, P_m) = m \ln r + O(1)$ as $r \to +\infty$. Further we put

$$\Omega_m(z,f) = a_0(z)f(z) + a_1(z)f'(z) + \cdots + a_m(z)f^{(m)}(z),$$

where $a_j (1 \le j \le m)$ are polynomials and f is an entire functions. Using well-known [3, p.44] inequalities

$$T\left(r, \prod_{j=1}^{q} f_j\right) \le \sum_{j=1}^{q} T(r,f_j), \quad T\left(r, \sum_{j=1}^{q} f_j\right) \le \sum_{j=1}^{q} T(r,f_j) + \ln q$$

we have

$$T(r, \Omega_m(\cdot,f)) \le T(r,f) + T(r,f') + \cdots + T(r,f^{(m)}) + O(\ln r), \quad r \to +\infty. \tag{34}$$

By the lemma about a logarithmic derivative [3, p.122] $T(r,f'/f) = Q(r,f)$ for each entire function f, where $Q(r,f)$ is denoting [3, p.122] an arbitrary function such that:

1) if f has a finite order then $Q(r,f) = O(\ln r)$ as $r \to +\infty$;

2) if f has an infinite order then $Q(r,f) = O(\ln T(r,f) + \ln r)$ as $r \to +\infty$ outside, possibly, some set of finite measure.

Clearly, $Q(r,f)\pm Q(r,f)=Q(r,f)$ and $AQ(r,f)=Q(r,f)$ [3, p.122]. We remak also that since f has a finite generalized order then in view of (31) $T(r,f)\le\alpha^{-1}(\varrho\beta(\ln r))$ for $\varrho>\varrho_{\alpha\beta}[f]$ and $r\ge r_0$. Hence it follows that $Q(r,f)=O(\ln\alpha^{-1}(\varrho\beta(\ln r))+\ln r)$ as $r\to+\infty$ and by Lemma 4 $Q(r,f')=O(\ln\alpha^{-1}(\varrho\beta(\ln r))+\ln r)$ as $r\to+\infty$.

Therefore,

$$T(r,f')=T\left(r,f\frac{f'}{f}\right)\le T(r,f)+T\left(r,\frac{f'}{f}\right)=T(r,f)+Q(r,f)$$
$$=T(r,f)+O(\ln\alpha^{-1}(\varrho\beta(\ln r))+\ln r),\quad r\to+\infty.$$

Similarly,

$$T(r,f'')=T\left(r,f'\frac{f''}{f'}\right)\le T(r,f')+Q(r,f')=T(r,f)+O(\ln\alpha^{-1}(\varrho\beta(\ln r))+\ln r),\ r\to+\infty,$$

et cetera. As a result from (34) we will get

$$T(r,\Omega_m(\cdot,f))\le(m+1)T(r,f)+O(\ln\alpha^{-1}(\varrho\beta(\ln r))+\ln r),\quad r\to+\infty. \tag{35}$$

Since f is an entire solution of the differential equation (33), we have $\Omega_m(z,f)\equiv A(z)$. Therefore, since $\alpha\in L_{si}$, in view of (31) and (35) we obtain

$$\begin{aligned}\varrho_{\alpha\beta}[A]&=\varlimsup_{r\to+\infty}\frac{\alpha(T(r,A))}{\beta(\ln r)}\le\varlimsup_{r\to+\infty}\frac{\alpha((m+1)T(r,f)+K_1(\ln\alpha^{-1}(\varrho\beta(\ln r))+\ln r))}{\beta(\ln r)}\\
&\le\varlimsup_{r\to+\infty}\frac{\alpha(K_2\max\{T(r,f),\,\ln\alpha^{-1}(\varrho\beta(\ln r)),\,\ln r\})}{\beta(\ln r)}\\
&=\varlimsup_{r\to+\infty}\frac{\alpha(\max\{T(r,f),\,\ln\alpha^{-1}(\varrho\beta(\ln r)),\,\ln r\})}{\beta(\ln r)}\\
&=\varlimsup_{r\to+\infty}\frac{\max\{\alpha(T(r,f)),\,\alpha(\ln\alpha^{-1}(\varrho\beta(\ln r))),\,\alpha(\ln r)\}}{\beta(\ln r)}\\
&=\varlimsup_{r\to+\infty}\frac{\alpha(T(r,f))+\alpha(\ln\alpha^{-1}(\varrho\beta(\ln r)))+\alpha(\ln r)}{\beta(\ln r)}\\
&\le\varlimsup_{r\to+\infty}\frac{\alpha(T(r,f))}{\beta(\ln r)}+\varlimsup_{r\to+\infty}\frac{\alpha(\ln\alpha^{-1}(\varrho\beta(\ln r)))}{\beta(\ln r)}+\varlimsup_{r\to+\infty}\frac{\alpha(\ln r)}{\beta(\ln r)}.\end{aligned}$$

Since $\alpha(x)=o(\beta(x))$ as $x\to+\infty$ we have $\dfrac{\alpha(\ln r)}{\beta(\ln r)}\to 0$ as $r\to+\infty$. Simultaneously,

$$\varlimsup_{r\to+\infty}\frac{\alpha(\ln\alpha^{-1}(\varrho\beta(\ln r)))}{\beta(\ln r)}=\varlimsup_{x\to+\infty}\frac{\alpha(\ln\alpha^{-1}(\varrho x))}{x}=\varrho\varlimsup_{x\to+\infty}\frac{\alpha(\ln x)}{\alpha(x)}=0.$$

Therefore, $\varrho_{\alpha\beta}[A]\le\varrho_{\alpha\beta}[f]$ and Theorem 5 is proved. □

If we choose $\alpha(x)=\ln x$ and $\beta(x)=x$ for $x\ge x_0$ then we come to the following statement.

Corollary 1. *If function f be an entire transcendental solution of the differential equation* (3) *then $\varrho[f]\ge\varrho[A]$, where $\varrho[f]=\varlimsup\limits_{r\to+\infty}\dfrac{\ln\ln M(r,f)}{\ln r}$ is the order of f.*

We remark that the contrary inequality is not true in general. Indeed, if for example $\beta_0 = -1$, $\beta_1 = \gamma_1 = \gamma_2$, $-1 \le \gamma_0 < 0$ and all $a_n = 0$, then [13] there exists an entire solution f of equation (3) such that

$$\ln M(r,f) = \frac{1+o(1)}{2}\left(|\beta_0| + \sqrt{|\beta_0|^2 + 4|\gamma_0|^2}\right) r, \quad r \to +\infty.$$

Clearly, in this case $\varrho[A] = 0 < 1 = \varrho[f]$.

Suppose that $\gamma_2 = a_0 = \beta_1 = a_1 = \beta_0 = \gamma_1 = \gamma_0 = 0$ and $A(z) = \sum_{n=2}^{\infty} a_n z^n$ is an entire function. Then equation (3) has the form $w'' = \sum_{n=2}^{\infty} a_n z^{n-2}$ and the function $f(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{n(n-1)} z^n$ is a solution of this equation. Using the formula of Hadamard of the order it is easy to prove that $\varrho[A] = \varrho[f]$, i. e. the estimate $\varrho[A] \le \varrho[f]$ is sharp.

If $\varrho_{\alpha\beta}[f] = 0$ then for the characteristic of the growth of f it is used the belonging to generalized convergence classes. For $\alpha \in L$ and $\beta \in L$ we will say that an entire function f belongs to generalized convergence class if

$$\int_{r_0}^{\infty} \frac{\alpha(\ln M_f(r))}{r\beta(\ln r)} dr < +\infty, \tag{36}$$

Choosing $r_1 = 2r$ from (32) we get $T(r,f) \le \ln^+ M_f(r) \le 3T(2r,f)$. On the other hand, in [10] it is proved that if $\alpha \in L^0$ then α is RO-increasing [8], i. e. for every $h \in [1,a]$, $1 < a < +\infty$, and all $x \ge x_0$ the inequality $\alpha(hx)/\alpha(x) \le M(a) < +\infty$ is true. Therefore, if $\alpha \in L^0$, $\beta \in L$ and $\beta(x + O(1)) = O(\beta(x))$ as $x \to +\infty$ then (36) holds if and only if

$$\int_{r_0}^{\infty} \frac{\alpha(T(r,f))}{r\beta(\ln r)} dr < +\infty. \tag{37}$$

Using (35) we prove the following theorem.

Theorem 6. *Let $\alpha \in L^0$, $\beta \in L$, $\beta(x + O(1)) = O(\beta(x))$ as $x \to +\infty$ and*

$$\int_{x_0}^{\infty} \frac{\alpha(\ln \alpha^{-1}(\beta(x)))}{\beta(x)} dx < +\infty. \tag{38}$$

Suppose that f is an entire transcendental solution of the differential equation (33) where a_j are polynomials, $0 \le j \le m$, A is an entire function and $\varrho_{\alpha\beta}[f] = 0$. Then in order that f belongs to generalized convergence class, it is necessary that A belongs to this class.

Proof. Since $\varrho_{\alpha\beta}[f] = 0$, we have $Q(r,f) = O(\ln \alpha^{-1}(\beta(\ln r)) + \ln r)$ as $r \to +\infty$ and from (35) as above in view of the condition $\alpha \in L^0$ we obtain

$$\begin{aligned}
\int_{r_0}^{\infty} \frac{\alpha(T(r,A))}{r\beta(\ln r)} dr &= \int_{r_0}^{\infty} \frac{\alpha(T(r,\Omega_m(\cdot,f)))}{r\beta(\ln r)} dr \\
&\le \int_{r_0}^{\infty} \frac{\alpha((m+1)T(r,f) + K_1(\ln \alpha^{-1}(\beta(\ln r)) + \ln r))}{r\beta(\ln r)} dr \\
&\le \int_{r_0}^{\infty} \frac{\alpha(K_2 \max\{T(r,f),\, \ln \alpha^{-1}(\beta(\ln r)),\, \ln r\})}{\beta(\ln r)} \\
&\le M(K_2) \int_{r_0}^{\infty} \frac{\alpha(T(r,f)) + \alpha(\ln \alpha^{-1}(\beta(\ln r))) + \alpha(\ln r)}{r\beta(\ln r)} dr.
\end{aligned}$$

Since f is an entire function, from (36) it follows that $\int\limits_{r_0}^{\infty}\frac{\alpha(\ln r)}{r\beta(\ln r)}dr<+\infty$, and in view of (38)

$$\int\limits_{r_0}^{\infty}\frac{\alpha(\ln\alpha^{-1}(\beta(\ln r)))}{r\beta(\ln r)}dr=\int\limits_{x_0}^{\infty}\frac{\alpha(\ln\alpha^{-1}(\beta(x)))}{\beta(x)}dx<+\infty.$$

Therefore, (37) implies $\int\limits_{r_0}^{\infty}\frac{\alpha(T(r,A))}{r\beta(\ln r)}dr<+\infty$. Theorem 6 is proved. □

For entire functions of the minimal type of the order $\varrho\in(0,+\infty)$ G. Valiron [16, p.18] introduced the convergence class by the condition $\int\limits_{1}^{\infty}\frac{\ln M_f(r)}{r^{\varrho+1}}dr<+\infty$. If we choose $\alpha(x)=x$ and $\beta(x)=e^{\varrho x}$ for $x\ge x_0$, then from Theorem 6 we get the following statement.

Corollary 2. *If an entire function f is a solution of the differential equation* (3)*, then in order that f belongs to the convergence class of Valiron, it is necessary that A belongs to this class.*

Clearly, from the belonging of the function A to the convergence class of Valiron the belonging of the function f to this class does not follow. On the other hand, an entire solution of the differential equation $z^2w''=A(z)$ belongs to the convergence class of Valiron if and only if A belongs to this class.

Finally we will consider a linear differential equation of the endless order

$$\sum_{n=0}^{\infty}\frac{a_n}{n!}w^{(n)}=\Phi(z), \tag{39}$$

where the characteristic function $\varphi(t)=\sum\limits_{n=0}^{\infty}\frac{a_n}{n!}t^n$ is entire and has a growth not higher than the normal type of the first order, and Φ is an entire function.

A.O. Gelfond [2] proved that equation (39) for every $\theta>1$ has an entire solution f such that

$$\ln\overline{M}_f(r)\le C(\theta)\ln\overline{M}_{\Phi}(\theta r),\quad r\ge r_0, \tag{40}$$

where $C(\theta)$ does not depend on r and $\ln\overline{M}_f(r)=r\max\left\{\frac{\ln M_f(t)}{t}:1\le t\le r\right\}$. Using this result we prove the following statement.

Proposition 6. *Equation* (39) *has an entire solution f such that:*

1) *if $\alpha(e^x)\in L_{si}$, $\beta\in L$, $\beta(x+O(1))\sim\beta(x)$ and $\alpha(x)=o(\beta(\ln x))$ as $x\to+\infty$, then $\varrho_{\alpha\beta}[f]\le\varrho_{\alpha\beta}[\Phi]$;*

2) *if $\alpha(e^x)\in L^0$, $\beta\in L$, $\beta(x+O(1))=O(\beta(x))$ as $x\to+\infty$ and $\int\limits_{r_0}^{\infty}\frac{\alpha(r)}{r\beta(\ln r)}dr<+\infty$, then the belonging of Φ to the generalized convergence class implies the belonging of f to this class.*

Proof. Clearly, $\ln M_f(r) \le \ln \overline{M}_f(r) \le r \ln M_f(r)$ for $r \ge 1$. Therefore, if $\alpha(e^x) \in L_{si}$ and $\beta(x + O(1)) \sim \beta(x)$ as $x \to +\infty$ then from (40) we have

$$\begin{aligned}
\varrho_{\alpha\beta}[f] &= \varlimsup_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(\ln \overline{M}_f(r))}{\beta(\ln r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(C(\theta) \ln \overline{M}_\Phi(\theta r))}{\beta(\ln(\theta r) - \ln\theta)} \\
&= \varlimsup_{r\to+\infty} \frac{\alpha(\ln \overline{M}_\Phi(r))}{\beta(\ln r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(r \ln M_\Phi(r))}{\beta(\ln r)} = \varlimsup_{r\to+\infty} \frac{\alpha(\exp\{\ln r + \ln\ln M_\Phi(r)\})}{\beta(\ln r)} \\
&\le \varlimsup_{r\to+\infty} \frac{\alpha(\exp\{2\max\{\ln r, \ln\ln M_\Phi(r)\}\})}{\beta(\ln r)} = \varlimsup_{r\to+\infty} \frac{\alpha(\exp\{\max\{\ln r, \ln\ln M_\Phi(r)\}\})}{\beta(\ln r)} \\
&= \varlimsup_{r\to+\infty} \frac{\max\{\alpha(r), \alpha(\ln M_\Phi(r))\}}{\beta(\ln r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(r) + \alpha(\ln M_\Phi(r))}{\beta(\ln r)} = \varrho_{\alpha\beta}[\Phi].
\end{aligned}$$

The firs part of Proposition 6 is proved.

Similarly, if $\alpha(e^x) \in L^0$, $\beta(x + O(1)) = O(\beta(x))$ as $x \to +\infty$ and $\int_{r_0}^{\infty} \frac{\alpha(\ln M_\Phi(r))}{r\beta(\ln r)} dr < +\infty$, then

$$\begin{aligned}
&\int_{r_0}^{\infty} \frac{\alpha(\ln M_f(r))}{r\beta(\ln r)} dr \le \int_{r_0}^{\infty} \frac{\alpha(C(\theta) \ln \overline{M}_\Phi(\theta r))}{r\beta(\ln r)} dr \le M_1 \int_{r_0}^{\infty} \frac{\alpha(r \ln M_\Phi(r))}{r\beta(\ln r)} dr \\
&\le M_1 \int_{r_0}^{\infty} \frac{\alpha(\exp\{2\max\{\ln r, \ln\ln M_\Phi(r)\}\})}{r\beta(\ln r)} dr \le M_1 M_2 \int_{r_0}^{\infty} \frac{\alpha(r) + \alpha(\ln M_\Phi(r))}{r\beta(\ln r)} dr < +\infty,
\end{aligned}$$

where $M_1 = M_1(\theta)$ and $M_2 = M_2(2)$. The proof of Proposition 6 is completed. □

REFERENCES

[1] Alexander J.F. *Functions which map the interior of the unit circle upon simple regions*. Ann. of Math. 1915, **17**, 12–22. doi:10.2307/2007212

[2] Gelfond A.O. *Linear differential equations of infinite order with constant coefficients and asymptotic periods of entire functions*. Tr. Mat. Inst. Steklova 1951, **38**, 42–67. (in Russian)

[3] Goldberg A.A., Ostrovsky I.V. Value distribution of meromorphic functions. Nauka, Moscow, 1970. (in Russian)

[4] Golusin G.M. Geometric Theory of Functions of a Complex Variable. Amer. Math. Soc., Providence, 1969.

[5] Goodman A.W. Univalent function. Vol. II. Mariner Pub. Co., 1983.

[6] Goodman A.W. *Univalent functions and nonanalytic curves*. Proc. Amer. Math. Soc. 1957, **8**, 597–601. doi: 10.1090/S0002-9939-1957-0086879-9

[7] Kaplan W. *Close-to-convex schlicht functions*. Michigan Math. J. 1952, **1** (2), 169–185. doi:10.1307/mmj/1028988895

[8] Seneta E. Regularly varying functions. Lecture Notes in Mathematics, 508, Springer-Verlag, Berlin, 1976.

[9] Shah S.M. *Univalence of a function f and its successive derivatives when f satisfies a differential equation, II*. J. Math. Anal. Appl. 1989, **142**, 422–430. doi:10.1016/0022-247X(89)90011-5

[10] Sheremeta M.M. *On two classes of positive functions and the belonging to them of main characteristics of entire functions*. Mat. Stud. 2003, **19** (1), 73–82.

[11] Sheremeta M.N. *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*. Izv. Vyssh. Uchebn. Zaved. Mat. 1967, (2), 100—108.(in Russian)

[12] Sheremeta Z.M. *Close-to-convexity of entire solutions of a differential equation*. Mat. Metodi Fiz.-Mekh. Polya 1999, **42** (3), 31–35. (in Ukrainian)

[13] Sheremeta Z.M. *On properties of entire solutions of a differential equation*. Diff. Equat. 2000, **36** (8), 1155–1161. doi:10.1007/BF02754183

[14] Sheremeta Z.M. *On entire solutions of a differential equation* . Mat. stud. 2000, **14** (1), 54–58.

[15] Sheremeta Z.M. *On the close-to-convexity of entire solutions of a differential equation*. Visnyk of the Lviv Univ. Ser. Mech. Math. 2000, **57**, 88–91. (in Ukrainian)

[16] Valiron G. Lectures on the general theory of integral functions. Edouard Privat, Toulouse, 1923.

Received 27.05.2019

Мулява О.М., Шеремета М.М., Трухан Ю.С. *Властивості розв'язків одного неоднорідного диференціального рівняння другого порядку* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 379–398.

Нехай степеневий ряд $A(z) = \sum_{n=0}^{\infty} a_n z^n$ має радіус збіжності $R[A] \in [1, +\infty]$. Для неоднорідного диференціального рівняння

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z) w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2) w = A(z)$$

з комплексними коефіцієнтами вивчаються геометричні властивості в одиничному крузі його розв'язків (опуклість, зірковість, близькість до опуклості). Розглядається два випадки: $\gamma_2 \neq 0$ і $\gamma_2 = 0$. Також ми розглядаємо випадки дійсних параметрів цього рівняння. Доведено, що для розв'язку f цього рівняння радіус збіжності $R[f]$ дорівнює $R[A]$ і знайдено рекурентні формули для знаходження коефіцієнтів степеневого розвинення $f(z)$. Для цілого розв'язку доведено, що порядок розв'язку f не менший ніж порядок функції A ($\varrho[f] \geq \varrho[A]$) і оцінка є точною. Аналогічна нерівність доведена для узагальнених порядків ($\varrho_{\alpha\beta}[f] \geq \varrho_{\alpha\beta}[A]$). Для цілого розв'язку цього рівняння вивчено належність до класу збіжності. Наприкінці розглядається лінійне диференціальне рівняння нескінченного порядку $\sum\limits_{n=0}^{\infty} \frac{a_n}{n!} w^{(n)} = \Phi(z)$, і вивчається можливе зростання його розв'язків.

Ключові слова і фрази: диференціальне рівняння, опуклість, зірковість, близькість до опуклості, узагальнений порядок, клас збіжності.