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ESTIMATES OF APPROXIMATIVE CHARACTERISTICS OF THE CLASSES $B^{\Omega}_{p,\theta}$ OF PERIODIC FUNCTIONS OF SEVERAL VARIABLES WITH GIVEN MAJORANT OF MIXED MODULI OF CONTINUITY IN THE SPACE L_q

In this paper, we continue the study of approximative characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables whose majorant of the mixed moduli of continuity contains both exponential and logarithmic multipliers. We obtain the exact-order estimates of the orthoprojective widths of the classes $B^{\Omega}_{p,\theta}$ in the space L_q, $1 \le p < q < \infty$, and also establish the exact-order estimates of approximation for these classes of functions in the space L_q by using linear operators satisfying certain conditions.

Key words and phrases: orthoprojective width, mixed modulus of continuity, linear operator, Vallée-Poussin kernel, Fejér kernel.

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INTRODUCTION

Let $\mathbb{R}^d, d \ge 1$ denote d-dimensional space with elements

$$x = (x_1, \dots, x_d),\ (x, y) = x_1 y_1 + \dots + x_d y_d$$

and let $L_p(\pi_d)$, $1 \le p < \infty$, be the space of functions $f(x) = f(x_1, \dots, x_d)$, which are 2π-periodic in each variable and summable in degree p on the cube $\pi_d = \prod\limits_{j=1}^{d} [0; 2\pi]$ for which the norm is defined as follows:

$$\|f\|_{L_p(\pi_d)} = \|f\|_p = \left((2\pi)^{-d} \int\limits_{\pi_d} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Respectively, $L_\infty(\pi_d)$ is the space of essentially bounded functions $f(x) = f(x_1, \dots, x_d)$, which are 2π- periodic in each variable, with the norm

$$\|f\|_{L_\infty(\pi_d)} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \pi_d} |f(x)|.$$

Further, we assume that, for functions $f \in L_p(\pi_d)$, the following additional condition holds:

$$\int_0^{2\pi} f(x) dx_j = 0 \quad j = \overline{1, d}.$$

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For $f \in L_p(\pi_d), 1 \le p \le \infty$, and $t = (t_1, \dots, t_d)$, $t_j \ge 0$, $j = \overline{1,d}$, we consider the mixed modulus of continuity of the order l

$$\Omega_l(f,t)_p = \sup_{\substack{|h_j| \le t_j \\ j=\overline{1,d}}} \|\Delta_h^l f(\cdot)\|_p,$$

where $l \in \mathbb{N}$, $\Delta_h^l f(x) = \Delta_{h_1}^l \dots \Delta_{h_d}^l f(x) = \Delta_{h_d}^l(\dots(\Delta_{h_1}^l f(x)))$ is a mixed difference of the order l with a vector step $h = (h_1, \dots, h_d)$, and the difference of the l^{th} order with a step h_j in the variable x_j is defined as follows:

$$\Delta_{h_j}^l f(x) = \sum_{n=0}^{l} (-1)^{l-n} C_l^n f(x_1, \dots, x_{j-1}, x_j + nh_j, x_{j+1}, \dots, x_d).$$

Let $\Omega(t) = \Omega(t_1, \dots, t_d)$ be a given function of the type of a mixed modulus of continuity of the order l, which satisfies the following conditions:

1) $\Omega(t) > 0$, $t_j > 0$, $j = \overline{1,d}$; $\Omega(t) = 0$, $\prod\limits_{j=1}^{d} t_j = 0$;

2) $\Omega(t)$ is nondecreasing in each variable;

3) $\Omega(m_1 t_1, \dots, m_d t_d) \le \left(\prod\limits_{j=1}^{d} m_j \right)^l \Omega(t)$, $m_j \in \mathbb{N}$, $j = \overline{1,d}$;

4) $\Omega(t)$ is continuous for $t_j \ge 0$, $j = \overline{1,d}$.

We assume that $\Omega(t)$ satisfies also the conditions (S) and (S_l), which are called the Bari-Stechkin conditions [1]. This means the following.

A function of one variable $\varphi(\tau) \ge 0$ satisfies the condition (S) if $\varphi(\tau)/\tau^\alpha$ almost increases for some $\alpha > 0$, i.e., there exists a constant $C_1 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\varphi(\tau_1)}{\tau_1^\alpha} \le C_1 \frac{\varphi(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \le \tau_2 \le 1.$$

A function $\varphi(\tau) \ge 0$ satisfies the condition (S_l) if $\varphi(\tau)/\tau^\gamma$ almost decreases for some $0 < \gamma < l$, i.e., there exists a constant $C_2 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\varphi(\tau_1)}{\tau_1^\gamma} \ge C_2 \frac{\varphi(\tau_2)}{\tau_2^\gamma}, \quad 0 < \tau_1 \le \tau_2 \le 1.$$

We say that $\Omega(t)$ satisfies the conditions (S) and (S_l) if $\Omega(t)$ satisfies these conditions in each variable t_j for fixed t_i, $i \ne j$.

Thus, let $1 \le p \le \infty$, $1 \le \theta \le \infty$, and let $\Omega(t)$ be a given function of the type of a mixed modulus of continuity of the order l. Then the classes $B_{p,\theta}^\Omega$ are defined in the following way [21]:

$$B_{p,\theta}^\Omega = \{ f \in L_p(\pi_d) : \|f\|_{B_{p,\theta}^\Omega} \le 1 \},$$

where

$$\|f\|_{B_{p,\theta}^\Omega} = \left\{ \int\limits_{\pi_d} \left(\frac{\Omega_l(f,t)_p}{\Omega(t)} \right)^\theta \prod_{j=1}^{d} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}}, \quad 1 \le \theta < \infty,$$

$$\|f\|_{B^{\Omega}_{p,\infty}} = \sup_{t>0} \frac{\Omega_l(f,t)_p}{\Omega(t)},$$

(the expression $t > 0$ for $t = (t_1, \dots, t_d)$ is equivalent to $t_j > 0, j = \overline{1,d}$).

We note that, for $\theta = \infty$, the classes $B^{\Omega}_{p,\theta}$ coincide with the classes H^{Ω}_p, which were considered by N.N. Pustovoitov in [13].

In the subsequent, it will be convenient to use the equivalent (to within absolute constants) definition of the classes $B^{\Omega}_{p,\theta}$. For this purpose, we need the corresponding notations.

To every vector $s = (s_1, \dots, s_d)$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$, we put the set

$$\rho(s) = \left\{k = (k_1, \dots, k_d) : 2^{s_j - 1} \le |k_j| < 2^{s_j}, k_j \in \mathbb{Z}, j = \overline{1,d}\right\}$$

in correspondence, and, for $f \in L_p(\pi_d), 1 < p < \infty$, we denote

$$\delta_s(f) := \delta_s(f,x) = \sum_{k \in \rho(s)} \widehat{f}(k) e^{i(k,x)},$$

where

$$\widehat{f}(k) = (2\pi)^{-d} \int_{\pi_d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f.

Let $1 < p < \infty$, $1 \le \theta \le \infty$ and let $\Omega(t)$ be a given function of the type of a mixed modulus of continuity of the order l that satisfies the conditions 1 – 4, (S) and (S_l). Then, to within absolute constants, the classes $B^{\Omega}_{p,\theta}$ can be defined as follows [21]:

$$B^{\Omega}_{p,\theta} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\theta}} = \left(\sum_s \Omega^{-\theta}(2^{-s}) \|\delta_s(f)\|_p^{\theta} \right)^{\frac{1}{\theta}} \le 1 \right\} \tag{1}$$

for $1 \le \theta < \infty$ and

$$B^{\Omega}_{p,\infty} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\infty}} = \sup_s \frac{\|\delta_s(f)\|_p}{\Omega(2^{-s})} \le 1 \right\}. \tag{2}$$

Here and below, $\Omega(2^{-s}) = \Omega(2^{-s_1}, \dots, 2^{-s_d})$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$.

The given definitions of the classes $B^{\Omega}_{p,\theta}$ can be extended also to the extreme values $p = 1$ and $p = \infty$, by modifying the "blocks" $\delta_s(f)$ in (1) and (2). Let $V_n(t)$ stand for a Vallée-Poussin kernel of the order $2n - 1$, i.e.,

$$V_n(t) = 1 + 2\sum_{k=1}^{n} \cos kt + 2 \sum_{k=n+1}^{2n-1} \left(1 - \frac{k-n}{n}\right) \cos kt.$$

To every vector $s = (s_1, \dots, s_d), s_j \in \mathbb{N}, j = \overline{1,d}$, we put the polynomial

$$A_s(x) = \prod_{j=1}^{d} \left(V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j) \right)$$

in correspondence. For $f \in L_p(\pi_d), 1 \le p \le \infty$, by $A_s(f)$ we denote the convolution

$$A_s(f) := A_s(f,x) = (f * A_s)(x).$$

Then, to within absolute constants, the classes $B^{\Omega}_{p,\theta}$, $1 \le p \le \infty$, can be defined as follows:

$$B^{\Omega}_{p,\theta} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\theta}} = \left(\sum_s \Omega^{-\theta}(2^{-s}) \|A_s(f)\|_p^{\theta} \right)^{\frac{1}{\theta}} \le 1 \right\} \tag{3}$$

for $1 \le \theta < \infty$ and

$$B^{\Omega}_{p,\infty} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\infty}} = \sup_s \frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \le 1 \right\}. \tag{4}$$

We note that relations (3) and (4) were obtained in works [18] and [13], respectively.

We note also that, for $\Omega(t) = \prod\limits_{j=1}^{d} t_j^{r_j}$, $0 < r_j < l$, the classes $B^{\Omega}_{p,\theta}$ are analogs of the well-known Besov $B^r_{p,\theta}$, $1 \le \theta < \infty$, and Nikol'skii $B^r_{p,\infty} = H^r_p$ classes (see, e.g., [8]).

In what follows, we study the classes $B^{\Omega}_{p,\theta}$ that are defined by the function $\Omega(t)$:

$$\Omega(t) = \Omega(t_1, \dots, t_d) = \begin{cases} \prod\limits_{j=1}^{d} \dfrac{t_j^r}{\left(\log \frac{1}{t_j}\right)_+^{b_j}}, & \text{if } t_j > 0,\ j = \overline{1,d}; \\ 0, & \text{if } \prod\limits_{j=1}^{d} t_j = 0. \end{cases} \tag{5}$$

Here and below, we consider the logarithms with base 2, and

$$\left(\log \frac{1}{t_j} \right)_+ = \max \left\{ 1, \log \frac{1}{t_j} \right\}.$$

In addition, we assume that $b_j \in \mathbb{R}$, $j = \overline{1,d}$, and $0 < r < l$. Hence, properties 1–4 and the conditions (S) and (S_l) are satisfied for the function $\Omega(t)$ of the form (5).

In the present paper we obtain the exact-order estimates of orthoprojective widths of the classes $B^{\Omega}_{p,\theta}$ in the space L_q, $1 \le p < q < \infty$. We recall that the notion of orthoprojective width was introduced by V. N. Temlyakov [23].

Let $\{u_i\}_{i=1}^M$ be an orthonormalized system of functions $u_i \in L_\infty(\pi_d)$, $f \in L_q(\pi_d)$, $1 \le q \le \infty$. We set

$$(f, u_i) = (2\pi)^{-d} \int_{\pi_d} f(x) \overline{u}_i(x) dx,$$

where $\overline{u}_i$ is the function complex conjugate to the function u_i.

To every function $f \in L_q(\pi_d)$, $1 \le q \le \infty$, we put an approximation of the form $\sum\limits_{i=1}^{M} (f, u_i) u_i$ in correspondence, i.e., the orthogonal projection of the function f onto the subspace generated by the system of functions $\{u_i\}_{i=1}^M$. Then, for the functional class $F \subset L_q(\pi_d)$, the quantity

$$d^{\perp}_M(F, L_q) = \inf_{\{u_i\}_{i=1}^M} \sup_{f \in F} \left\| f - \sum_{i=1}^{M} (f, u_i) u_i \right\|_q \tag{6}$$

is called the orthoprojective width (the Fourier-width) of this class in the space $L_q(\pi_d)$.

In addition to orthoprojective widths, we study the quantities $d^B_M(F, L_q)$ introduced by V.N. Temlyakov [22]). They are defined as follows:

$$d^{B}_{M}(F,L_q)=\inf_{G\in L_M(B)_q}\ \sup_{f\in F\cap D(G)}\|f-Gf\|_q. \tag{7}$$

Here, $L_M(B)_q$ stands for a set of linear operators satisfying the conditions:

a) the domain of definition $D(G)$ of these operators contains all trigonometric polynomials, and their domain of values is contained in a subspace with dimension M of the space $L_q(\pi_d)$;

b) there exists a number $B\geq 1$ such that, for all vectors $k=(k_1,\ldots,k_d)$, $k_j\in\mathbb{Z}$, $j=\overline{1,d}$, the inequality $\left\|Ge^{i(k,\cdot)}\right\|_2\leq B$ holds.

We note that $L_M(1)_2$ contains the operators of orthogonal projection onto the spaces with dimension M and the operators that are set on an orthonormalized system of functions with the help of the multiplier defined by a sequence $\{\lambda_m\}$ such that $|\lambda_m|\leq 1$ for all m.

From (6) and (7), it is easy to see that the quantities $d^{\perp}_{M}(F,L_q)$ and $d^{B}_{M}(F,L_q)$ are connected with each other by the inequality

$$d^{B}_{M}(F,L_q)\leq d^{\perp}_{M}(F,L_q). \tag{8}$$

At present, a lot of works are known, in which the quantities $d^{\perp}_{M}(F,L_q)$ and $d^{B}_{M}(F,L_q)$ were studied for various classes of functions. We mention works [14,16,17,22,24], where the quantities (6) and (7) were considered for the classes of functions of many variables $W^{r}_{p,\alpha}$, H^{r}_{p}, $B^{r}_{p,\theta}$, and H^{Ω}_{p} (see also numerous references therein). The quantities $d^{\perp}_{M}(B^{\Omega}_{p,\theta},L_q)$ and $d^{B}_{M}(B^{\Omega}_{p,\theta},L_q)$ for the classes of functions of many variables with a given function $\Omega(t)$ of the form (5) under the condition $b_j<r$, $j=\overline{1,d}$, were considered in works [4–7].

1 AUXILIARY ASSERTIONS

We now give several known assertions, which are used in the subsequent considerations.

As was noted above, $\Omega(t)$ is a function of the form (5). For a natural N, we set

$$\chi(N)=\Big\{s=(s_1,\ldots,s_d):\ s_j\in\mathbb{N},\ j=\overline{1,d},\ \Omega(2^{-s})\geq\frac{1}{N}\Big\},$$

$$Q(N)=\bigcup_{s\in\chi(N)}\rho(s).$$

We note that the approximation of certain classes of periodic functions of many variables with mixed generalized smoothness by trigonometric polynomials with "numbers" of harmonics from the sets that are analogs of $Q(N)$ was started in work [15]. Later, the approximations by trigonometric polynomials with "numbers" of harmonics from the sets $Q(N)$ were studied in works [4], [19], [20] and other ones.

The following proposition is true.

Lemma 1 ([14]). *For the number of elements of the set $Q(N)$, the following ordinal equalities hold:*

$$|Q(N)|\asymp N^{\frac{1}{r}}\left(\log N\right)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}+\nu-1},$$

if $b_1 \le \ldots \le b_\nu < r < b_{\nu+1} \le \ldots \le b_d$;

$$|Q(N)| \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}},$$

if $r \le b_1 \le \ldots \le b_d$, $b_2 > r$.

Here and below, the notation $\mu_1 \ll \mu_2$ for positive functions $\mu_1(N)$ and $\mu_2(N)$ means that there exists a constant $C > 0$ such that, $\forall N \in \mathbb{N}$, the inequality $\mu_1(N) \le C\mu_2(N)$ holds.

The relation $\mu_1 \asymp \mu_2$ holds if $\mu_1 \ll \mu_2$ and $\mu_1 \gg \mu_2$. We note also that all constants $C_i, i = 1, 2, \ldots$, which are used in what follows, can depend only on parameters that are contained in the definitions of a class and a dimension d of the space $\mathbb{R}^d$.

To formulate the following assertions, we note that, according to (5), the definition of a set $\chi(N)$ takes the form

$$\chi(N) = \Big\{ s = (s_1, \ldots, s_d) : \ s_j \in \mathbb{N},\ j = \overline{1,d},\ \prod_{j=1}^{d} 2^{rs_j} s_j^{b_j} \le N \Big\}.$$

Therefore,

$$\chi^{\perp}(N) = \mathbb{N}^d \setminus \chi(N).$$

Let

$$\Theta(N) = \Big\{ s = (s_1, \ldots, s_d) : \ s_j \in \mathbb{N}, j = \overline{1,d},\ \frac{1}{2^l N} \le \Omega(2^{-s}) < \frac{1}{N} \Big\}.$$

In work [11], it was established that the number of elements of the set $\Theta(N)$ satisfies the ordinal equality

$$|\Theta(N)| \asymp (\log N)^{d-1}.$$

Lemma 2 ([14]). *For the function* $\Omega(t)$ *defined by equality* (5) *for* $0 < \beta < r$, $0 < p < \infty$ *the relation*

$$\sum_{s \in \chi^{\perp}(N)} \big(\Omega(2^{-s}) 2^{\|s\|_1 \beta}\big)^p \ll \sum_{s \in \Theta(N)} \big(\Omega(2^{-s}) 2^{\|s\|_1 \beta}\big)^p$$

holds, where $\|s\|_1 = s_1 + \ldots + s_d$, $s_j \in \mathbb{N}$.

Lemma 3 ([14]). *If* $\gamma_1 \le \ldots \le \gamma_\nu < 1 < \gamma_{\nu+1} \le \ldots \le \gamma_d$, *then*

$$\sum_{s \in \Theta(N)} \prod_{j=1}^{d} s_j^{-\gamma_j} \asymp \big(\log N\big)^{-\gamma_1 - \ldots - \gamma_\nu + \nu - 1}.$$

If $1 \le \gamma_1 \le \ldots \le \gamma_d$, $\gamma_2 > 1$, *then*

$$\sum_{s \in \Theta(N)} \prod_{j=1}^{d} s_j^{-\gamma_j} \asymp \big(\log N\big)^{-\gamma_1}.$$

Lemma 4 ([22]). *Let* $1 \le p < q < \infty$ *and* $f \in L_p(\pi_d)$. *Then*

$$\|f\|_q^q \ll \sum_{s} \bigg(\|\delta_s(f)\|_p \ 2^{\|s\|_1 \left(\frac{1}{p} - \frac{1}{q}\right)} \bigg)^q.$$

Lemma 5 ([24])**.** *Let A be the linear operator given by the equality*

$$Ae^{i(k,x)} = \sum_{m=1}^{\overline{M}} a_m^k \psi_m(x),$$

where $\{\psi_m(x)\}_{m=1}^{\overline{M}}$ is the set of functions for which

$$\|\psi_m(\cdot)\|_2 \leq 1, \;\; m = 1, \dots, \overline{M}.$$

Then, for any trigonometric polynomial t, the following inequality holds:

$$\min_{y=x} Re\, At(x-y) \leq \left(\overline{M} \sum_{m=1}^{\overline{M}} \sum_{k} |a_m^k \widehat{t}(k)|^2 \right)^{\frac{1}{2}}.$$

Theorem 1 ([10])**.** *Let T_n be a trigonometric polynomial of the order $n = (n_1, \dots, n_d)$, i.e.,*

$$T_n(x) = \sum_{|k_1| \leq n_1} \dots \sum_{|k_d| \leq n_d} c_{k_1,\dots,k_d} e^{i(k,x)},$$

where n_j, $j = \overline{1,d}$ are natural numbers, and $c_{k_1,\dots,k_d}$ are any coefficients. Then, for $1 \leq p < q \leq \infty$ the inequality

$$\|T_n\|_q \leq 2^d \left(\prod_{j=1}^{d} n_j \right)^{\frac{1}{p} - \frac{1}{q}} \|T_n\|_p \tag{9}$$

holds.

Inequality (9) was established by S. M. Nikol'skii and is called the "inequality of different metrics". In the one-dimensional case for $p = \infty$, the corresponding inequality was proved by D. Jackson [3].

Theorem 2 (Littlewood-Paley theorem; see, e.g., [9], p. 65)**.** *Let $p \in (1, \infty)$. Then there exist positive numbers $C_3(p)$ and $C_4(p)$ such that, for every function $f \in L_p(\pi_d)$, the following relations are true:*

$$C_3(p) ||f||_p \leq \left\| \left(\sum_{s} |\delta_s(f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_4(p) \|f\|_p.$$

2 Main results

Passing to the statement of the propositions and their proof, we assume that $M = |Q(N)|$. First, we consider case $b_1 \leq \dots \leq b_\nu < r < b_{\nu+1} \leq \dots \leq b_d$. Then, according to Lemma 1, we have

$$M \asymp N^{\frac{1}{r}} \left(\log N \right)^{-\frac{b_1}{r} - \dots - \frac{b_\nu}{r} + \nu - 1},$$
$$\log M \asymp \log N, \;\; N \asymp M^r \left(\log M \right)^{b_1 + \dots + b_\nu - (\nu - 1)r}.$$

The following theorem is true.

Theorem 3. *Let $1 \le p < q < \infty$, $q < \theta < \infty$, and let $\Omega(t)$ be a function of the form* (5)*. Then, for $\frac{1}{p} - \frac{1}{q} < r < l$, $b_1 \le \ldots \le b_\nu < \frac{r}{\frac{q}{p}-1} < b_{\nu+1} \le \ldots \le b_d$, the relations*

$$d_M^{\perp}(B^{\Omega}_{p,\theta}, L_q) \asymp d_M^{B}(B^{\Omega}_{p,\theta}, L_q) \asymp M^{-r+\frac{1}{p}-\frac{1}{q}} \big(\log M\big)^{-b_1-\ldots-b_\nu+(\nu-1)\left(r-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)} \tag{10}$$

hold.

Proof. First, we establish the upper bounds in (10). According to (8), it is sufficient to obtain the upper bound for the orthoprojective width $d_M^{\perp}(B^{\Omega}_{p,\theta}, L_q)$.

For this purpose, we consider an approximation of the functions $f \in B^{\Omega}_{p,\theta}$ by trigonometric polynomials $t_{Q(N)}$ of the form

$$t_{Q(N)}(x) = \sum_{s\in\chi(N)} \delta_s(f,x).$$

Let q_0 be any number that satisfies the condition $p < q_0 < q$.

Then, using Lemma 4, and the relation

$$\|\delta_s(f)\|_{q_0} \asymp \|A_s(f)\|_{q_0},\ 1 < q_0 < \infty,$$

for $f \in B^{\Omega}_{p,\theta}$ we have

$$\|f - t_{Q(N)}\|_q = \Big\| f - \sum_{s\in\chi(N)} \delta_s(f) \Big\|_q = \Big\| \sum_{s\in\chi^{\perp}(N)} \delta_s(f) \Big\|_q$$

$$\ll \left(\sum_{s\in\chi^{\perp}(N)} \|\delta_s(f)\|_{q_0}^q 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q} \right)^{\frac{1}{q}} \asymp \left(\sum_{s\in\chi^{\perp}(N)} \|A_s(f)\|_{q_0}^q 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q} \right)^{\frac{1}{q}} = I_1.$$

Then, applying to $A_s(f)$ the Nikol'skii inequality of different metrics, we continue the estimate as follows:

$$I_1 \ll \left(\sum_{s\in\chi^{\perp}(N)} \|A_s(f)\|_p^q 2^{\|s\|_1(\frac{1}{p}-\frac{1}{q_0})q} 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q} \right)^{\frac{1}{q}} = \left(\sum_{s\in\chi^{\perp}(N)} \|A_s(f)\|_p^q 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)q} \right)^{\frac{1}{q}}$$

$$= \left(\sum_{s\in\chi^{\perp}(N)} \Omega^{-q}(2^{-s}) \|A_s(f)\|_p^q \Omega^q(2^{-s}) 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)q} \right)^{\frac{1}{q}} = I_2.$$

Using first the Hölder inequality with index $\frac{\theta}{q}$ and then Lemma 2, we get

$$I_2 \le \left(\sum_{s\in\chi^{\perp}(N)} \Omega^{-\theta}(2^{-s}) \|A_s(f)\|_p^{\theta} \right)^{\frac{1}{\theta}} \cdot \left(\sum_{s\in\chi^{\perp}(N)} \left(\Omega(2^{-s}) 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)} \right)^{\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}}$$

$$\ll \|f\|_{B^{\Omega}_{p,\theta}} \left(\sum_{s\in\chi^{\perp}(N)} \left(\Omega(2^{-s}) 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)} \right)^{\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}}$$

$$\ll \left(\sum_{s\in\Theta(N)} \left(\Omega(2^{-s}) 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)} \right)^{\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}} \leq N^{-1} \left(\sum_{s\in\Theta(N)} 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}} = I_3.$$

Taking into account that, for $s \in \Theta(N)$,

$$2^{\|s\|_1} \asymp N^{\frac{1}{r}} \prod_{j=1}^{d} s_j^{-\frac{b_j}{r}},$$

and using Lemma 3, we have

$$I_3 \asymp N^{-1} \left(\sum_{s\in\Theta(N)} N^{\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \prod_{j=1}^{d} s_j^{-\frac{b_j}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}}$$

$$= N^{-1+\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\sum_{s\in\Theta(N)} \prod_{j=1}^{d} s_j^{-\frac{b_j}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}}$$

$$\asymp N^{-1+\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)} \big(\log N\big)^{\left(-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)+(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)}$$

$$\asymp \left(M^r \big(\log M\big)^{b_1+\ldots+b_\nu-(\nu-1)r} \right)^{-1+\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)} \big(\log M\big)^{\left(-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)+(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)}$$

$$= M^{-r+\frac{1}{p}-\frac{1}{q}} \big(\log M\big)^{-b_1-\ldots-b_\nu+(\nu-1)\left(r-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}.$$

Thus, in view of the definition of orthoprojective width, the above reasoning gives the upper bound for $d^{\perp}_M(B^{\Omega}_{p,\theta}, L_q)$, and, respectively, for the quantity $d^B_M(B^{\Omega}_{p,\theta}, L_q)$.

Let us find the lower bounds in (10). Since inequality (8) holds, it is sufficient to obtain the lower bound for the quantity $d^B_M(B^{\Omega}_{p,\theta}, L_q)$.

With the help of the reasoning analogous to that in [12], we can prove the existence of a set $\Theta_1(N) \subset \Theta(N)$ such that, for $s = (s_1, \ldots, s_d) \in \Theta_1(N)$, the following relations are satisfied:

$$s_j \asymp \log N, \ \ j = \overline{1,d} \ \text{ and } \ |\Theta_1(N)| \asymp \big(\log N\big)^{d-1}.$$

Also we can assert that there exists a set

$$\Theta_1^{(\nu)}(N) = \{s \in \Theta(N) : s_j \asymp \log N, \ j = 1, \ldots, \nu, \ s_j = 1, \ j = \nu+1, \ldots, d\}$$

such that

$$|\Theta_1^{(\nu)}(N)| \asymp \big(\log N\big)^{\nu-1}.$$

Consider the set $\widetilde{Q}(N) = \bigcup_{s\in\Theta_1^{(\nu)}(N)} \rho(s)$. By $T(\widetilde{Q}(N))$ we denote the set of trigonometric polynomials with the "numbers" of harmonics from $\widetilde{Q}(N)$.

Let K_n be the Fejér kernel of the order n, i.e.,

$$K_n(t) = \sum_{|k|\leq n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}.$$

We set

$$g_1(x) = \sum_{s\in\Theta_1^{(\nu)}(N)} \mathcal{K}_s^{(\nu)}(x) \prod_{j=\nu+1}^{d} e^{ix_j},$$

where

$$\mathcal{K}_s^{(\nu)}(x) = \prod_{j=1}^{\nu} e^{ik_j^{s_j} x_j} K_{2^{s_j-2}}(x_j),$$

$$k_j^{s_j} = \begin{cases} 2^{s_j-1} + 2^{s_j-2}, & s_j \geq 2; \\ 1, & s_j = 1, j = \overline{1,\nu}. \end{cases}$$

Suppose that the operator G belongs to $L_M(B)_q, 1 < q < \infty$. Consider the operator $A = S_{\widetilde{Q}(N)}G$, where $S_{\widetilde{Q}(N)}$ is the operator of taking partial Fourier sum corresponding to the set $\widetilde{Q}(N)$. Then $A \in L_M(B)_q$ and the domain of values of the operator A is a subspace A_M of the space $T(\widetilde{Q}(N))$, whose dimension $\dim A_M = \overline{M} \leq M$. It follows from Theorem 2 that for $f \in T(\widetilde{Q}(N))$, the following relation is satisfied:

$$\|f - Af\|_q \ll \|f - Gf\|_q.$$

Consider the quantity

$$I = \sup_y \|g_1(x-y) - Ag_1(x-y)\|_\infty.$$

Obviously,

$$I \geq g_1(0) - \min_{y=x} ReAg_1(x-y).$$

Using Lemma 5, we obtain

$$\min_{y=x} ReAg_1(x-y) \leq M^{\frac{1}{2}} B \Big(\sum_k |\widehat{g}_1(k)|^2 \Big)^{\frac{1}{2}} \ll M^{\frac{1}{2}} B |\widetilde{Q}(N)|^{\frac{1}{2}}. \tag{11}$$

Further, taking into account the relation

$$|\Theta_1^{(\nu)}(N)| \asymp \big(\log N\big)^{\nu-1},$$

as well as

$$|\rho(s)| = 2^{||s||_1} \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r} - \ldots - \frac{b_\nu}{r}}, s \in \Theta_1^{(\nu)}(N),$$

we can write

$$|\widetilde{Q}(N)| \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r} - \ldots - \frac{b_\nu}{r} + \nu - 1}. \tag{12}$$

On the other hand,

$$g_1(0) \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r} - \ldots - \frac{b_\nu}{r} + \nu - 1} \asymp |\widetilde{Q}(N)|. \tag{13}$$

Using (11) and (12), we can chose a number N so that $|\widetilde{Q}(N)| \asymp M$ and the right-hand side of (13) will be at least twice as large as the right-hand side of (11).

For some $y^* = (y_1^*, \ldots, y_d^*)$, for this N we have

$$\|g_1(x-y^*) - Ag_1(x-y^*)\|_\infty \gg M. \tag{14}$$

Consider the function

$$g_2(x) = C_5 N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}} g_1(x),\ C_5 > 0.$$

We now show that, at the corresponding choice of the constant C_5, this function belongs to the class $B^{\Omega}_{p,\theta}$. Indeed, since

$$||K_n||_p \asymp n^{1-\frac{1}{p}}\, 1 \le p \le \infty,$$

for the Fejér kernel, we have

$$\left\|\mathcal{K}^{(\nu)}_s\right\|_p \asymp 2^{\|s\|_1\left(1-\frac{1}{p}\right)}\ 1 \le p \le \infty.$$

Thus, we can write

$$\begin{aligned}
\|g_2\|_{B^{\Omega}_{p,\theta}} &= \left(\sum_s \Omega^{-\theta}(2^{-s})\|A_s(g_2)\|_p^{\theta}\right)^{\frac{1}{\theta}} \\
&\ll N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\cdot\left(\sum_{s\in\Theta_1^{(\nu)}(N)} \Omega^{-\theta}(2^{-s})\|A_s(g_1)\|_p^{\theta}\right)^{\frac{1}{\theta}} \qquad (15)\\
&\ll \Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\cdot\left(\sum_{s\in\Theta_1^{(\nu)}(N)} 2^{\|s\|_1\left(1-\frac{1}{p}\right)\theta}\right)^{\frac{1}{\theta}} = I_4.
\end{aligned}$$

Taking into account the fact that, for $s \in \Theta_1^{(\nu)}(N) \subset \Theta(N)$

$$2^{\|s\|_1} \asymp N^{\frac{1}{r}}\prod_{j=1}^{d} s_j^{-\frac{b_j}{r}},$$

and

$$s_j \asymp \log N,\ j = 1,\ldots,\nu,\ s_j = 1,\ j = \nu+1,\ldots d,\ |\Theta_1^{(\nu)}(N)| \asymp \big(\log N\big)^{\nu-1},$$

we get

$$\begin{aligned}
I_4 &\asymp \Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}} \\
&\times \Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{1-\frac{1}{p}}|\Theta_1^{(\nu)}(N)|^{\frac{1}{\theta}} \asymp \big(\log N\big)^{-\frac{\nu-1}{\theta}}\big(\log N\big)^{\frac{\nu-1}{\theta}} = 1.
\end{aligned} \qquad (16)$$

By comparing (15) and (16), we may conclude that $g_2 \in B^{\Omega}_{p,\theta}$ with the corresponding constant $C_5 > 0$.

It was established in work [14] that for $t \in T(\widetilde{Q}(N))$, the following estimate is satisfied:

$$\|t\|_\infty \ll \|t\|_q\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{q}}\big(\log N\big)^{(\nu-1)\left(1-\frac{1}{q}\right)}.$$

Taking into account the last relation and using estimate (14), we get

$$\|g_2(x-y^*)-Gg_2(x-y^*)\|_q$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\|g_1(x-y^*)-Gg_1(x-y^*)\|_q$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\|g_1(x-y^*)-Ag_1(x-y^*)\|_q$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}$$

$$\times\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{-\frac{1}{q}}\big(\log N\big)^{-(\nu-1)\left(1-\frac{1}{q}\right)}\|g_1(x-y^*)-Ag_1(x-y^*)\|_\infty$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_d}{r}+d-1}\Big)^{\frac{1}{p}-\frac{1}{q}-1}\big(\log N\big)^{(d-1)\left(-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}M$$

$$\asymp M^{-r}\big(\log M\big)^{-b_1-\ldots-b_\nu+(\nu-1)r}M^{\frac{1}{p}-\frac{1}{q}-1}\big(\log M\big)^{(\nu-1)\left(-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}M$$

$$=M^{-r+\frac{1}{p}-\frac{1}{q}}\big(\log M\big)^{-b_1-\ldots-b_\nu+(\nu-1)\left(r-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}.$$

The lower bounds in (10) are established. Theorem 3 is proved. □

In the following proposition, we consider other relations for the numbers $r, b_1, \ldots, b_d$. Let $r \le b_1 \le \ldots \le b_d$, $b_2 > r$. In this case, by Lemma 1, we obtain

$$M \asymp N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}},$$
$$\log M \asymp \log N,\ \ N \asymp M^r\big(\log M\big)^{b_1}.$$

Assume that

$$b_1 = \ldots = b_\nu < b_{\nu+1} \le \ldots \le b_d.$$

Then, for $\nu = 1$, the inequality $r \le b_1 < b_2$ holds. But $\nu \ge 2$, then $b_1 > r$.

Theorem 4. *Let $1 \le p < q < \infty$, $q < \theta < \infty$, and let $\Omega(t)$ be a function of the form* (5)*. Then, for $\frac{1}{p}-\frac{1}{q} < r < l$, $b_2 > \frac{r}{\frac{q}{p}-1}$, the order estimates*

$$d_M^{\perp}(B^{\Omega}_{p,\theta}, L_q) \asymp d_M^{B}(B^{\Omega}_{p,\theta}, L_q) \asymp M^{-r+\frac{1}{p}-\frac{1}{q}}\big(\log M\big)^{-b_1} \tag{17}$$

hold.

Proof. For $q < \theta < \infty$, the embedding $B^{\Omega}_{p,\theta} \subset H^{\Omega}_p$, is valid. Therefore, the upper bounds in (17) follow from the corresponding estimate $d_M^{\perp}(H^{\Omega}_p, L_q)$, proved in [14].

To get the lower bounds in (17), it is sufficient to get the corresponding lower bound for the quantity $d_M^B(B^{\Omega}_{p,\theta}, L_q)$.

We choose a vector $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_d) \in \Theta(N)$ so that

$$\tilde{s}_1 \asymp \log N, \quad \tilde{s}_2 = \ldots = \tilde{s}_d = 1,$$

and set

$$g_3(x) = \mathcal{K}_{\tilde{s}}(x) = e^{i(k^{\tilde{s}},x)}K_{2^{\tilde{s}_1-2}}(x_1),$$

where $k^{\tilde{s}} = (2^{\tilde{s}_1-1} + 2^{\tilde{s}_1-2}, 1, \dots, 1)$.

Suppose that the operator G belongs to $L_M(B)_q, 1 < q < \infty$. Consider the operator $A = S_{\rho(\tilde{s})}G$, where $S_{\rho(\tilde{s})}$ is the operator of taking partial Fourier sum corresponding to the set $\rho(\tilde{s})$.

Taking into account that

$$2^{\|\tilde{s}\|_1} \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}},$$

and using lemma 5, we get

$$\min_{y=x} ReAg_3(x-y) \le M^{\frac{1}{2}} B\bigg(\sum_k |\widehat{g}_3(k)|^2\bigg)^{\frac{1}{2}} \ll M^{\frac{1}{2}} \big(2^{\|\tilde{s}\|_1}\big)^{\frac{1}{2}} \asymp M^{\frac{1}{2}} N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}}. \tag{18}$$

On the other hand,

$$g_3(0) \asymp 2^{\|\tilde{s}\|_1} \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}}. \tag{19}$$

Therefore, we can chose a number N so that $|Q(N)| \asymp M$ and the right-hand side of (19) will be at least twice as large as the right-hand side of (18). For some $y^* = (y_1^*, \dots, y_d^*)$, for this N we have

$$\|g_3(x-y^*) - Ag_3(x-y^*)\|_\infty \gg M. \tag{20}$$

Consider the function

$$g_4(x) = C_6 N^{-1} 2^{||\tilde{s}||_1\left(\frac{1}{p}-1\right)} g_3(x),\ C_6 > 0.$$

We now show that, at the corresponding choice of the constant C_6, the function g_4 belongs to the class $B^{\Omega}_{p,\theta}$.

Indeed, in view of the properties of the Fejér kernel, we have

$$\|g_4\|_{B^{\Omega}_{p,\theta}} = \bigg(\sum_s \Omega^{-\theta}(2^{-s}) \|A_s(g_4)\|_p^\theta\bigg)^{\frac{1}{\theta}} \ll N^{-1} 2^{||\tilde{s}||_1\left(\frac{1}{p}-1\right)} \left(\Omega^{-\theta}(2^{-\tilde{s}}) \|A_{\tilde{s}}(g_3)\|_p^\theta\right)^{\frac{1}{\theta}}$$

$$\ll 2^{||\tilde{s}||_1\left(\frac{1}{p}-1\right)} \|A_{\tilde{s}}(g_3)\|_p \asymp 2^{||\tilde{s}||_1\left(\frac{1}{p}-1\right)} 2^{||\tilde{s}||_1\left(1-\frac{1}{p}\right)} = 1.$$

Hence, $g_4 \in B^{\Omega}_{p,\theta}$ with the corresponding constant $C_6 > 0$.

It was established in work [14] that for a trigonometric polynomial t with "numbers" of harmonics from the set $\rho(\tilde{s})$, the following relation is satisfied:

$$\|t\|_\infty \ll \|t\|_q 2^{\frac{\|\tilde{s}\|_1}{q}}.$$

Taking into account the last relation and using estimate (20), we get

$$\|g_4(x-y^*) - Gg_4(x-y^*)\|_q \gg N^{-1} 2^{||\tilde{s}||_1\left(\frac{1}{p}-1\right)} \|g_3(x-y^*) - Gg_3(x-y^*)\|_q$$

$$\gg N^{-1} 2^{||\tilde{s}||_1\left(\frac{1}{p}-1\right)} \|g_3(x-y^*) - Ag_3(x-y^*)\|_q$$

$$\gg N^{-1} 2^{||\tilde{s}||_1\left(\frac{1}{p}-1\right)} 2^{-\frac{\|\tilde{s}\|_1}{q}} \|g_3(x-y^*) - Ag_3(x-y^*)\|_\infty$$

$$\gg M^{-r} \big(\log M\big)^{-b_1} M^{\frac{1}{p}-\frac{1}{q}-1} M = M^{-r+\frac{1}{p}-\frac{1}{q}} \big(\log M\big)^{-b_1}.$$

The lower bounds in (17) are established. Theorem 4 is proved. □

Remark 1. *Results, corresponding to Theorems 3 and 4, but for the classes $B^{\Omega}_{p,\theta}$ in the space L_∞, are obtained in [2].*

Remark 2. *The analogues of Theorems 3 and 4 for the classes H^{Ω}_{p} are obtained by N.N. Pustovoitov in [14]. Moreover, if the conditions of Theorem 4 are satisfied, the ordinal relations*

$$d^{\perp}_{M}(B^{\Omega}_{p,\theta}, L_q) \asymp d^{B}_{M}(B^{\Omega}_{p,\theta}, L_q) \asymp d^{\perp}_{M}(H^{\Omega}_{p}, L_q) \asymp d^{B}_{M}(H^{\Omega}_{p}, L_q)$$

hold. In other words, the orders of the quantities $d^{\perp}_{M}(B^{\Omega}_{p,\theta}, L_q)$ and $d^{B}_{M}(B^{\Omega}_{p,\theta}, L_q)$ are independent on the parameter θ.

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Федуник-Яремчук О.В., Гембарська С.Б. *Оцінки апроксимативних характеристик класів $B^{\Omega}_{p,\theta}$ періодичних функцій багатьох змінних із заданою мажорантою мішаних модулів неперервності у просторі L_q* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 281–295.

В роботі продовжується вивчення апроксимативних характеристик класів $B^{\Omega}_{p,\theta}$ періодичних функцій багатьох змінних, мажоранта мішаних модулів неперервності яких містить як степеневі, так і логарифмічні множники. Одержано точні за порядком оцінки ортопроекційних поперечників класів $B^{\Omega}_{p,\theta}$ у просторі L_q, $1 \leq p < q < \infty$, а також встановлено точні за порядком оцінки наближення цих класів функцій у просторі L_q за допомогою лінійних операторів, які підпорядковані певним умовам.

Ключові слова і фрази: ортопроекційний поперечник, мішаний модуль неперервності, лінійний оператор, ядро Валле-Пуссена, ядро Фейєра.