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SYMMETRIC $*$-POLYNOMIALS ON $\mathbb{C}^n$

$*$-Polynomials are natural generalizations of usual polynomials between complex vector spaces. A $*$-polynomial is a function between complex vector spaces X and Y, which is a sum of so-called (p,q)-polynomials. In turn, for nonnegative integers p and q, a (p,q)-polynomial is a function between X and Y, which is the restriction to the diagonal of some mapping, acting from the Cartesian power X^{p+q} to Y, which is linear with respect to every of its first p arguments, antilinear with respect to every of its last q arguments and invariant with respect to permutations of its first p arguments and last q arguments separately.

In this work we construct formulas for recovering of (p,q)-polynomial components of $*$-polynomials, acting between complex vector spaces X and Y, by the values of $*$-polynomials. We use these formulas for investigations of $*$-polynomials, acting from the n-dimensional complex vector space $\mathbb{C}^n$ to $\mathbb{C}$, which are symmetric, that is, invariant with respect to permutations of coordinates of its argument. We show that every symmetric $*$-polynomial, acting from $\mathbb{C}^n$ to $\mathbb{C}$, can be represented as an algebraic combination of some "elementary" symmetric $*$-polynomials.

Results of the paper can be used for investigations of algebras, generated by symmetric $*$-polynomials, acting from $\mathbb{C}^n$ to $\mathbb{C}$.

Key words and phrases: (p,q)-polynomial, $*$-polynomial, symmetric $*$-polynomial.

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INTRODUCTION AND PRELIMINARIES

$*$-Polynomials (see definition below), acting between complex vector spaces X and Y, were studied in [4–6]. If X has a symmetric structure, like a symmetric basis, it is natural to consider $*$-polynomials, which are invariant (symmetric) with respect to a group of operators, acting on X, which preserve this structure.

Symmetric (invariant) analytic functions of several complex variables with respect to a group of operators on the n-dimensional complex vector space $\mathbb{C}^n$ were investigated by many authors (see, e. g., [1–3]).

In this work we consider symmetric (see definition below) $*$-polynomials, acting from $\mathbb{C}^n$ to $\mathbb{C}$. We investigate the structure of such $*$-polynomials and show that every symmetric $*$-polynomial, acting from $\mathbb{C}^n$ to $\mathbb{C}$, can be represented as an algebraic combination of some "elementary" symmetric $*$-polynomials. Also we establish the general result, which gives us the method of recovering of components of a $*$-polynomial by the values of this $*$-polynomial.

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{Z}_+$ be the set of all nonnegative integers. Let X and Y be complex vector spaces. A mapping $A : X^{p+q} \to Y$, where $p, q \in \mathbb{Z}_+$ are such that $p \neq 0$ or $q \neq 0$, is called a (p,q)*-linear mapping*, if A is linear with respect to every

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of first p arguments and it is antilinear with respect to every of last q arguments. A (p,q)-linear mapping, which is invariant with respect to permutations of its first p arguments and last q arguments separately, is called (p,q)-*symmetric*. A mapping $P : X \to Y$ is called a (p,q)-*polynomial* if there exists a (p,q)-symmetric (p,q)-linear mapping $A_P : X^{p+q} \to Y$ such that P is the restriction to the diagonal of A_P, i.e.

$$P(x) = A_P(\underbrace{x,\ldots,x}_{p+q})$$

for every $x \in X$. The mapping A_P is called the (p,q)-symmetric (p,q)-linear mapping, *associated* with P. Note that

$$P(x_1+\ldots+x_m) = \sum_{\substack{\mu_1+\ldots+\mu_m=p\\ \mu_1,\ldots,\mu_m\in\mathbb{Z}_+}} \sum_{\substack{\nu_1+\ldots+\nu_m=p\\ \nu_1,\ldots,\nu_m\in\mathbb{Z}_+}} \frac{p!}{\mu_1!\ldots\mu_m!}\frac{q!}{\nu_1!\ldots\nu_m!} \times A_P(\underbrace{x_1,\ldots,x_1}_{\mu_1},\ldots,\underbrace{x_m,\ldots,x_m}_{\mu_m},\underbrace{x_1,\ldots,x_1}_{\nu_1},\ldots,\underbrace{x_m,\ldots,x_m}_{\nu_m}), \quad (1)$$

for every $x_1,\ldots,x_m \in X$. Also note that

$$P(\lambda x) = \lambda^p \bar{\lambda}^q P(x) \quad (2)$$

for every $x \in X$ and $\lambda \in \mathbb{C}$.

For convenience, we define $(0,0)$-polynomials from X to Y as constant mappings.

A mapping $P : X \to Y$ is called a $*$-*polynomial* if it can be represented in the form

$$P = \sum_{k=0}^{K}\sum_{j=0}^{k} P_{j,k-j}, \quad (3)$$

where $K \in \mathbb{Z}_+$ and $P_{j,k-j}$ is a $(j,k-j)$-polynomial for every $k \in \{0,\ldots,K\}$ and $j \in \{0,\ldots,k\}$. Let $\deg P$ be the maximal number $k \in \mathbb{Z}_+$, for which there exists $j \in \{0,\ldots,k\}$ such that $P_{j,k-j} \not\equiv 0$.

A $*$-polynomial $P : \mathbb{C}^n \to \mathbb{C}$, where $n \in \mathbb{N}$, is called *symmetric* if

$$P((z_1,\ldots,z_n)) = P((z_{\sigma(1)},\ldots,z_{\sigma(n)}))$$

for every $(z_1,\ldots,z_n) \in \mathbb{C}^n$ and for every bijection $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$.

For every $\gamma = (\gamma_1,\gamma_2) \in \mathbb{Z}_+^2$ let us define a (γ_1,γ_2)-polynomial $H_\gamma^{(n)} : \mathbb{C}^n \to \mathbb{C}$ by

$$H_\gamma^{(n)}(z) = \sum_{m=1}^{n} z_m^{\gamma_1}\bar{z}_m^{\gamma_2}, \quad (4)$$

where $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$. Note that $H_\gamma^{(n)}$ is symmetric.

A mapping $f : S \to \mathbb{C}$, where S is an arbitrary set, is called an *algebraic combination* of mappings $f_1,\ldots,f_k : S \to \mathbb{C}$ if there exists a polynomial $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$f(x) = Q(f_1(x),\ldots,f_k(x))$$

for every $x \in S$.

In this work we show that every symmetric $*$-polynomial, acting from $\mathbb{C}^n$ to $\mathbb{C}$, can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$, defined by (4).

1 THE MAIN RESULT

Let us prove formulas for recovering of (p,q)-polynomials by the values of a $*$-polynomial. For complex numbers $t_1,\ldots,t_m$, let $V_{t_1,\ldots,t_m}$ be the Vandermonde matrix:

$$V_{t_1,\ldots,t_m} := \begin{pmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{m-1} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \ldots & t_m^{m-1} \end{pmatrix}.$$

It is well-known that

$$\det(V_{t_1,\ldots,t_m}) = \prod_{1\le j<s\le m} (t_s - t_j).$$

If all the numbers $t_1,\ldots,t_m$ are distinct, then $\det(V_{t_1,\ldots,t_m}) \neq 0$.

Proposition 1. *Let $P : X \to Y$ be a $*$-polynomial of the form* (3)*, where X and Y are complex vector spaces. Let $\lambda_0,\ldots,\lambda_K$ be distinct real numbers. Then*

$$\sum_{j=0}^{k} P_{j,k-j}(x) = \sum_{s=0}^{K} w_{ks} P(\lambda_s x)$$

for every $k \in \{0,\ldots,K\}$ and $x \in X$, where w_{ks} are elements of the matrix $W = (w_{ks})_{k,s=\overline{0,K}}$, which is the inverse matrix of the Vandermonde matrix $V_{\lambda_0,\ldots,\lambda_K}$.

Proof. Let $x \in X$. For every $s \in \{0,\ldots,K\}$, by (3),

$$P(\lambda_s x) = \sum_{k=0}^{K}\sum_{j=0}^{k} P_{j,k-j}(\lambda_s x).$$

By (2), taking into account that λ_s is real,

$$P_{j,k-j}(\lambda_s x) = \lambda_s^j \bar{\lambda}_s^{k-j} P_{j,k-j}(x) = \lambda_s^j \lambda_s^{k-j} P_{j,k-j}(x) = \lambda_s^k P_{j,k-j}(x).$$

Therefore, for every $s \in \{0,\ldots,K\}$,

$$P(\lambda_s x) = \sum_{k=0}^{K} \lambda_s^k \sum_{j=0}^{k} P_{j,k-j}(x).$$

Thus, we have the vector equality

$$\big(P(\lambda_0 x),\ldots,P(\lambda_K x)\big)^T = V_{\lambda_0,\ldots,\lambda_K}\big(P_{0,0}(x), \textstyle\sum_{j=0}^{1} P_{j,1-j}(x),\ldots,\sum_{j=0}^{K} P_{j,K-j}(x)\big)^T.$$

Since $\lambda_0,\ldots,\lambda_K$ are distinct, it follows that $\det(V_{\lambda_0,\ldots,\lambda_K}) \neq 0$. Consequently, $V_{\lambda_0,\ldots,\lambda_K}$ is invertible. Let

$$W = (w_{ks})_{k,s=\overline{0,K}} := V^{-1}_{\lambda_0,\ldots,\lambda_K}.$$

Then

$$\big(P_{0,0}(x), \textstyle\sum_{j=0}^{1} P_{j,1-j}(x),\ldots,\sum_{j=0}^{K} P_{j,K-j}(x)\big)^T = W\big(P(\lambda_0 x),\ldots,P(\lambda_K x)\big)^T.$$

Therefore,

$$\sum_{j=0}^{k} P_{j,k-j}(x) = \sum_{s=0}^{K} w_{ks} P(\lambda_s x)$$

for every $k \in \{0,\ldots,K\}$. □

Proposition 2. *Let $k \in \mathbb{Z}_+$ and $P_{j,k-j} : X \to Y$ be a $(j, k-j)$-polynomial for every $j \in \{0, \dots, k\}$, where X and Y are complex vector spaces. Let $\varepsilon_0, \dots, \varepsilon_k$ be complex numbers such that $\varepsilon_0^2, \dots, \varepsilon_k^2$ are distinct and $|\varepsilon_0| = \dots = |\varepsilon_k| = 1$. Then*

$$P_{j,k-j}(x) = \sum_{l=0}^{k} u_{jl}\varepsilon_l^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_l x)$$

for every $j \in \{0, \dots, k\}$ and $x \in X$, where u_{jl} are elements of the matrix $U = (u_{jl})_{j,l=\overline{0,K}}$, which is the inverse matrix of the Vandermonde matrix $V_{\varepsilon_0^2,\dots,\varepsilon_K^2}$.

Proof. Let $x \in X$. For every $j, l \in \{0, \dots, k\}$, by (2), $P_{j,k-j}(\varepsilon_l x) = \varepsilon_l^j \bar{\varepsilon}_l^{k-j} P_{j,k-j}(x)$. Since $|\varepsilon_l| = 1$, it follows that $\bar{\varepsilon}_l^{k-j} = \varepsilon_l^{j-k}$. Therefore,$P_{j,k-j}(\varepsilon_l x) = \varepsilon_l^{2j-k} P_{j,k-j}(x)$.

Consequently,

$$\varepsilon_l^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_l x) = \sum_{j=0}^{k} \varepsilon_l^{2j} P_{j,k-j}(x)$$

for every $l \in \{0, \dots, k\}$. Thus, we have the vector equality

$$\big(\varepsilon_0^k \textstyle\sum_{j=0}^{k} P_{j,k-j}(\varepsilon_0 x), \dots, \varepsilon_k^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_k x)\big)^T = V_{\varepsilon_0^2,\dots,\varepsilon_k^2}\big(P_{0,k}(x), P_{1,k-1}(x), \dots, P_{k,0}(x)\big)^T.$$

Since $\varepsilon_0^2, \dots, \varepsilon_k^2$ are distinct, it follows that $\det(V_{\varepsilon_0^2,\dots,\varepsilon_k^2}) \neq 0$. Consequently, $V_{\varepsilon_0^2,\dots,\varepsilon_k^2}$ is invertible. Let

$$U = (u_{jl})_{j,l=\overline{0,k}} := V^{-1}_{\varepsilon_0^2,\dots,\varepsilon_k^2}.$$

Then

$$\big(P_{0,k}(x), P_{1,k-1}(x), \dots, P_{k,0}(x)\big)^T = U\big(\varepsilon_0^k \textstyle\sum_{j=0}^{k} P_{j,k-j}(\varepsilon_0 x), \dots, \varepsilon_k^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_k x)\big)^T.$$

Therefore,

$$P_{j,k-j}(x) = \sum_{l=0}^{k} u_{jl}\varepsilon_l^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_l x)$$

for every $j \in \{0, \dots, k\}$. ☐

Let us consider $*$-polynomials on $\mathbb{C}^n$.

Lemma 1. *Every $*$-polynomial $P : \mathbb{C}^n \to \mathbb{C}$ can be uniquely represented in the form*

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} \sum_{\substack{\mu_1+\dots+\mu_n=j \\ \mu_1,\dots,\mu_n \in \mathbb{Z}_+}} \sum_{\substack{\nu_1+\dots+\nu_n=k-j \\ \nu_1,\dots,\nu_n \in \mathbb{Z}_+}} \alpha_{\mu_1,\dots,\mu_n,\nu_1,\dots,\nu_n} z_1^{\mu_1} \dots z_n^{\mu_n} \bar{z}_1^{\nu_1} \dots \bar{z}_n^{\nu_n}, \tag{5}$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $K = \deg P$ and $\alpha_{\mu_1,\dots,\mu_n,\nu_1,\dots,\nu_n} \in \mathbb{C}$.

Proof. Let $P : \mathbb{C}^n \to \mathbb{C}$ be a $*$-polynomial of the form (3). If $K = 0$, then $P = P_{0,0}$, where $P_{0,0} \in \mathbb{C}$. Thus, in this case, we have the representation of P in the form (5). Consider the case $K > 0$. Every $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ can be represented as $z = \sum_{m=1}^{n} z_m e_m$, where

$$e_m = (\underbrace{0, \dots, 0}_{m-1}, 1, \underbrace{0, \dots, 0}_{n-m})$$

for every $m \in \{1, \ldots, n\}$. Therefore, by (1),

$$P(z) = P_{0,0} + \sum_{k=0}^{K} \sum_{j=0}^{k} \sum_{\substack{\mu_1+\ldots+\mu_n=j \\ \mu_1,\ldots,\mu_n \in \mathbb{Z}_+}} \sum_{\substack{\nu_1+\ldots+\nu_n=k-j \\ \nu_1,\ldots,\nu_n \in \mathbb{Z}_+}} \frac{j!}{\mu_1! \ldots \mu_n!} \frac{(k-j)!}{\nu_1! \ldots \nu_n!} z_1^{\mu_1} \ldots z_n^{\mu_n} \bar{z}_1^{\nu_1} \ldots \bar{z}_n^{\nu_n}$$
$$\times A_{P_{j,k-j}}(\underbrace{e_1, \ldots, e_1}_{\mu_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{\mu_n}, \underbrace{e_1, \ldots, e_1}_{\nu_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{\nu_n}),$$

where $A_{P_{j,k-j}}$ is the $(j, k-j)$-symmetric $(j, k-j)$-linear mapping, associated with the $(j, k-j)$-polynomial $P_{j,k-j}$. Let $\alpha_{0,\ldots,0} = P_{0,0}$ and

$$\alpha_{\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_n} = \frac{j!}{\mu_1! \ldots \mu_n!} \frac{(k-j)!}{\nu_1! \ldots \nu_n!}$$
$$\times A_{P_{j,k-j}}(\underbrace{e_1, \ldots, e_1}_{\mu_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{\mu_n}, \underbrace{e_1, \ldots, e_1}_{\nu_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{\nu_n})$$

for $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n \in \mathbb{Z}_+$ such that $1 \le \mu_1 + \ldots + \mu_n + \nu_1 + \ldots + \nu_n \le K$. Then

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} \sum_{\substack{\mu_1+\ldots+\mu_n=j \\ \mu_1,\ldots,\mu_n \in \mathbb{Z}_+}} \sum_{\substack{\nu_1+\ldots+\nu_n=k-j \\ \nu_1,\ldots,\nu_n \in \mathbb{Z}_+}} \alpha_{\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_n} z_1^{\mu_1} \ldots z_n^{\mu_n} \bar{z}_1^{\nu_1} \ldots \bar{z}_n^{\nu_n}.$$

□

Theorem 1. *Every symmetric $*$-polynomial $P : \mathbb{C}^n \to \mathbb{C}$ can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$, where $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$ are such that $\gamma_1 + \gamma_2 \le \deg P$.*

Proof. We proceed by induction on n. In the case $n = 1$ for $z = z_1 \in \mathbb{C}$, by Lemma 1, we have

$$P(z) = \sum_{k=0}^{\deg P} \sum_{j=0}^{k} \alpha_{j,k-j} z_1^j \bar{z}_1^{k-j} = \sum_{k=0}^{\deg P} \sum_{j=0}^{k} \alpha_{j,k-j} H_{(j,k-j)}^{(1)}(z).$$

Suppose the statement holds for $n-1$ and prove it for n. Let $P : \mathbb{C}^n \to \mathbb{C}$ be a symmetric $*$-polynomial and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then $P(z)$ can be represented in the form

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \ldots, z_{n-1})),$$

where $K = \deg P$ and $r_{j,k-j} : \mathbb{C}^{n-1} \to \mathbb{C}$ are $*$-polynomials. Let us show that $*$-polynomials $r_{j,k-j}$ are symmetric. For fixed $z_1, \ldots, z_{n-1} \in \mathbb{C}$, the mapping $R : z_n \mapsto P((z_1, \ldots, z_n))$ is a $*$-polynomial, acting from $\mathbb{C}$ to $\mathbb{C}$. Let $\lambda_0, \ldots, \lambda_K$ be distinct real numbers. Then, by Proposition 1,

$$\sum_{j=0}^{k} z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \ldots, z_{n-1})) = \sum_{s=0}^{K} w_{ks} R(\lambda_s z_n) \tag{6}$$

for every $k \in \{0, \ldots, K\}$. For $k \in \{0, \ldots, K\}$, let $\varepsilon_0, \ldots, \varepsilon_k$ be complex numbers such that $\varepsilon_0^2, \ldots, \varepsilon_k^2$ are distinct and $|\varepsilon_0| = \ldots = |\varepsilon_k| = 1$. Then, by Proposition 2,

$$z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \ldots, z_{n-1})) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{j=0}^{k} (\varepsilon_l z_n)^j (\bar{\varepsilon}_l \bar{z}_n)^{k-j} r_{j,k-j}((z_1, \ldots, z_{n-1})) \tag{7}$$

for every $j \in \{0, \ldots, k\}$. By (6) and (7),

$$z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \ldots, z_{n-1})) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} R(\lambda_s \varepsilon_l z_n)$$

for every $k \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, k\}$. Let $z_n = 1$. Then

$$r_{j,k-j}((z_1, \ldots, z_{n-1})) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} R(\lambda_s \varepsilon_l) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} P((z_1, \ldots, z_{n-1}, \lambda_s \varepsilon_l)). \quad (8)$$

Let $\sigma : \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$ be a bijection. Then, by (8) and by the symmetry of P,

$$\begin{aligned} r_{j,k-j}((z_{\sigma(1)}, \ldots, z_{\sigma(n-1)})) &= \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} P((z_{\sigma(1)}, \ldots, z_{\sigma(n-1)}, \lambda_s \varepsilon_l)) \\ &= \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} P((z_1, \ldots, z_{n-1}, \lambda_s \varepsilon_l)) = r_{j,k-j}((z_1, \ldots, z_{n-1})). \end{aligned}$$

Thus, $r_{j,k-j}$ is symmetric for every $k \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, k\}$. By the induction hypothesis, every $*$-polynomial $r_{j,k-j}$ can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n-1)}$. Since

$$H_\gamma^{(n-1)}((z_1, \ldots, z_{n-1})) = H_\gamma^{(n)}((z_1, \ldots, z_n)) - z_n^{\gamma_1} \bar{z}_n^{\gamma_2}$$

for every $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$, it follows that P can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$ and $*$-polynomials, defined by $\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto z_n^{\gamma_1} \bar{z}_n^{\gamma_2} \in \mathbb{C}$, where $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$. Therefore,

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} z_n^j \bar{z}_n^{k-j} Q_{j,k-j}(z),$$

where $Q_{j,k-j}$ is an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$ for every $k \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, k\}$. Since $*$-polynomials $H_\gamma^{(n)}$ are symmetric, it follows that $*$-polynomials $Q_{j,k-j}$ are symmetric. Since $*$-polynomials P and $Q_{j,k-j}$ are symmetric, it follows that

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} z_m^j \bar{z}_m^{k-j} Q_{j,k-j}(z),$$

for every $m \in \{1, \ldots, n\}$. Therefore,

$$\sum_{m=1}^{n} P(z) = \sum_{m=1}^{n} \sum_{k=0}^{K} \sum_{j=0}^{k} z_m^j \bar{z}_m^{k-j} Q_{j,k-j}(z),$$

that is,

$$nP(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} \sum_{m=1}^{n} z_m^j \bar{z}_m^{k-j} Q_{j,k-j}(z).$$

Thus,

$$P(z) = \frac{1}{n} \sum_{k=0}^{K} \sum_{j=0}^{k} H_{(j,k-j)}^{(n)}(z) Q_{j,k-j}(z).$$

Hence, P is an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$. This completes the proof. $\square$

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Поняття $*$-полінома є природним узагальненням поняття полінома між комплексними векторними просторами. $*$-Поліном — це функція між комплексними векторними просторами X та Y, яка є сумою так званих (p, q)-поліномів. В свою чергу, для невід'ємних цілих чисел p і q, (p, q)-поліном — це функція між просторами X та Y, яка є звуженням на діагональ деякого відображення, що діє з декартового степеня X^{p+q} в Y, яке є лінійним відносно кожного зі своїх перших p аргументів, антилінійним відносно кожного зі своїх останніх q аргументів і інваріантним відносно перестановок окремо перших p аргументів і останніх q агрументів.

В даній роботі побудовано формули для знаходження (p, q)-поліноміальних компонентів $*$-поліномів, які діють між комплексними векторними просторами X та Y, за значеннями цих $*$-поліномів. Цей результат використано для дослідження $*$-поліномів, які діють з n-вимірного комплексного векторного простору $\mathbb{C}^n$ в $\mathbb{C}$, які є симетричними, тобто, інваріантними відносно перестановок координат їхнього аргумента. Показано, що кожен симетричний $*$-поліном, який діє з $\mathbb{C}^n$ в $\mathbb{C}$, можна подати у вигляді алгебраїчної комбінації деяких "елементарних" симетричних $*$-поліномів.

Результати даної роботи можуть бути використані для дослідження алгебр, породжених симетричними $*$-поліномами, які діють з $\mathbb{C}^n$ в $\mathbb{C}$.

Ключові слова і фрази: (p, q)-поліном, $*$-поліном, симетричний $*$-поліном.