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ON FELLER SEMIGROUP GENERATED BY SOLUTION OF NONLOCAL PARABOLIC CONJUGATION PROBLEM

The paper deals with the problem of construction of Feller semigroup for one-dimensional inhomogeneous diffusion processes with membrane placed at a point whose position on the real line is determined by a given function that depends on the time variable. It is assumed that in the inner points of the half-lines separated by a membrane the desired process must coincide with the ordinary diffusion processes given there, and its behavior on the common boundary of these regions is determined by the nonlocal conjugation condition of Feller-Wentzell's type. This problem is often called a problem of pasting together two diffusion processes on a line.

In order to study the described problem we use analytical methods. Such an approach allows us to determine the desired operator family using the solution of the corresponding problem of conjugation for a linear parabolic equation of the second order (the Kolmogorov backward equation) with discontinuous coefficients. This solution is constructed by the boundary integral equations method under the assumption that the coefficients of the equation satisfy the Hölder condition with a nonzero exponent, the initial function is bounded and continuous on the whole real line, and the parameters characterizing the Feller-Wentzell conjugation condition and the curve defining the common boundary of the domains, where the equation is given, satisfies the Hölder condition with exponent greater than $\frac{1}{2}$.

Key words and phrases: Feller semigroup, diffusion process, parabolic problem of conjugation.

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INTRODUCTION

Consider on a plane (s, x) the set

$$S_t = \{(s,x): \quad 0 \le s < t \le T, \ -\infty < x < \infty\},$$

and denote by $\overline{S}_t$ the closure of S_t. Suppose that $\overline{S}_t$ contains a continuous curve $x = h(s)$, $0 \le s \le T$, which separates S_t into two domains:

$$S_t^{(1)} = \{(s,x): \quad 0 \le s < t \le T, \ -\infty < x < h(s)\}$$

and

$$S_t^{(2)} = \{(s,x): \quad 0 \le s < t \le T, \ h(s) < x < \infty\}.$$

Put $D_{1s} = (-\infty, h(s))$ and $D_{2s} = (h(s), \infty)$.

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Consider in S_T two uniformly parabolic operators with bounded coefficients

$$\frac{\partial}{\partial s} + L_s^{(i)} \equiv \frac{\partial}{\partial s} + \frac{1}{2} b_i(s,x) \frac{\partial^2}{\partial x^2} + a_i(s,x) \frac{\partial}{\partial x}, \ i = 1,2. \tag{1}$$

The problem is to find a solution $u(s,x,t)$ of the equation

$$\frac{\partial u}{\partial s} + L_s^{(i)} u = 0, \quad (s,x) \in S_t^{(i)}, \ i = 1,2, \tag{2}$$

which satisfies the 'initial' condition

$$\lim_{s \uparrow t} u(s,x,t) = \varphi(x), \quad x \in \mathbb{R}, \tag{3}$$

two conjugation conditions

$$u(s,h(s)-0),t) = u(s,h(s)+0,t), \quad 0 \le s \le t \le T, \tag{4}$$

$$\gamma(s)u(s,h(s),t) + \int\limits_{D_{1s} \cup D_{2s}} [u(s,h(s),t) - u(s,y,t)]\mu(s,dy) = 0, \quad 0 \le s \le t \le T, \tag{5}$$

and two fitting conditions

$$\varphi(h(t)-0) = \varphi(h(t)+0), \tag{6}$$

$$\gamma(t)\varphi(h(t)) + \int\limits_{D_{1t} \cup D_{2t}} [\varphi(h(t)) - \varphi(y)]\mu(t,dy) = 0. \tag{7}$$

The initial function $\varphi(x)$ in (3) is assumed to be bounded and continuous on $\mathbb{R}$ (in this case condition (6) holds automatically), the function $\gamma(s)$ and the Borel measure $\mu(s,\cdot)$ in (5) are nonnegative and such that $\gamma(s) + \mu(s, D_{1s} \cup D_{2s}) > 0$ for all $s \in [0,T]$.

The problem (2)–(7) arises, in particular, in the theory of diffusion processes in the construction of a one-dimensional model of the diffusion phenomenon with a membrane, or, what is the same, in solving using the analytical methods the so-called problem of pasting together two diffusion processes on a line [3,4,8,9]. In the considered case, the membrane is supposed to be moving, and it is placed at the point $x = h(s)$, which is at the same time the point of pasting together two given diffusion processes. If we assume that the solution $u(s,x,t) \equiv T_{st}\varphi(x)$ of (2)–(7) is a two-parameter Feller semigroup associated with some inhomogeneous Markov process on a line, then the validity for it of equation (2) implies that this process coincides in D_{is} with the diffusion processes given there by the differential operators $L_s^{(i)}$, $i = 1,2$, and initial condition (3) is in agreement with the equality $T_{ss} = I$, where I is the identity operator. Next, conjugation condition (4) is the reflection of the Feller property of the process and equality (5) is the Feller-Wentzell conjugation condition which has two terms. The local term is responsible for disappearance of the diffusing particle and the nonlocal one for the jump-like nature of the exit of process from the boundary of the region. Recall that in the general case the Feller-Wentzell conjugation condition contains also the derivatives of the unknown function in both variables, which correspond to the properties of the partial reflection at the common boundary of the regions and the phenomenon of 'viscosity' [1,6,11].

The classical solvability of problem (2)–(7) is proved under the assumption that the coefficients of equation (2) satisfy the Hölder condition with a nonzero exponent, the initial function

φ in (3) is bounded and continuous on the whole real line, and the parameters γ, μ characterizing the Feller-Wentzell conjugation condition (5) and the curve $x = h(s)$ defining the common boundary of the domains $S_t^{(1)}$ and $S_t^{(2)}$ satisfy the Hölder condition with exponent greater than $\frac{1}{2}$. In the investigations we use the fundamental solutions of the parabolic equations and the heat potentials generated by them [2, 5, 8]. As a result of their application, problem (2)–(7) is reduced to a system of two singular Volterra integral equations of the second kind which solution is obtained by the method of successive approximations.

Note that a similar problem was considered earlier in [9] for the case where the membrane is placed at a fixed point of the line. We also mention works [7, 10], which present the results concerning the construction of diffusion processes with jumps at the points of the boundary of the region by the methods of stochastic [7] and functional analysis [10].

Assume that the following conditions I–V are satisfied.

I. Equation (2) is a parabolic equation in the domain $\overline{S}_T$, i.e., there exist positive constants b and B such that

$$0 < b \leq b_i(s,x) \leq B < \infty, \; i = 1,2, \; (s,x) \in \overline{S}_T.$$

II. The coefficients $b_i(s,x)$ and $a_i(s,x)$, $i = 1,2$, are continuous in (s,x) and belong to the Hölder class $H^{\frac{\alpha}{2},\alpha}(\overline{S}_T)$, $0 < \alpha < 1$ (to recall the definitions of Hölder classes see [5]).

III. The initial function $\varphi(x)$ belongs to the space of bounded continuous functions, which we will denote by $C_b(\mathbb{R})$. The norm in this space is defined by the equality $\|\varphi\| = \sup\limits_{x\in\mathbb{R}} |\varphi(x)|$.

IV. In condition (5) the measure $\mu(s,\cdot)$ is nonnegative, $\mu(s, D_{1s} \cup D_{2s}) = 1$, $s \in [0,T]$ and for all $f \in C_b(\mathbb{R})$ the integrals

$$G_f^{(i)}(s) = \int\limits_{D_{is}} f(y)\mu(s,dy), \quad i = 1,2,$$

belong to the Hölder class $H^{\frac{1+\alpha}{2}}([0,T])$.

V. The functions $\gamma(s)$ and $h(s)$ are continuous and belong to $H^{\frac{1+\alpha}{2}}([0,T])$.

In view of IV condition (5) can be rewritten as follows

$$(\gamma(s)+1)u(s,h(s),t) = \int\limits_{D_{1s}\cup D_{2s}} u(s,y,t)\mu(s,dy). \tag{8}$$

Conditions I, II provide the existence of a fundamental solution for each of the equations in (2) (see [5,8]), i.e., the existence of a function $G_i(s,x,t,y)$, $i = 1,2$ $(0 \leq s < t \leq T;\; x,y \in \mathbb{R})$, which satisfies equation (2) for fixed $t \in (0,T]$, $y \in \mathbb{R}$ as a function of $(s,x) \in [0,t) \times \mathbb{R}$ and has the form

$$G_i(s,x,t,y) = Z_{i0}(s,x,t,y) + Z_{i1}(s,x,t,y), \quad i = 1,2, \tag{9}$$

where

$$Z_{i0}(s,x,t,y) = [2\pi b_i(t,y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y-x)^2}{2b_i(t,y)(t-s)}\right\}, \tag{10}$$

$$Z_{i1}(s,x,t,y) = \int\limits_s^t d\tau \int\limits_{\mathbb{R}} Z_{i0}(s,x,\tau,z)Q_i(\tau,z,t,y)dz, \tag{11}$$

and the function $Q_i(s,x,t,y)$ is a solution of some singular Volterra integral equation of the second kind.

Note that

$$\left|D_s^r D_x^p Z_{i0}(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{12}$$

$$\left|D_s^r D_x^p Z_{i1}(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{13}$$

where $i=1,2,\ 0\le s<t\le T,\ x,y\in\mathbb{R},\ C$ i c are positive constants; in the sequel, various positive constants will be denoted by symbols C or c; r and p are nonnegative integers satisfying $2r+p\le 2,\ D_s^r$ is the partial derivative with respect to s of order $r,\ D_x^p$ is the partial derivative with respect to x of order p.

Given a fundamental solution $G_i(s,x,t,y),\ i=1,2,$ and a function $h(s)$, we define the integrals

$$u_{i0}(s,x,t) = \int\limits_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy, \quad i=1,2, \tag{14}$$

$$u_{i1}(s,x,t) = \int\limits_s^t G_i(s,x,\tau,h(\tau))V_i(\tau,t)d\tau, \quad i=1,2. \tag{15}$$

Here φ and $V_i,\ i=1,2$ are given functions, $0\le s<t\le T,\ x\in\mathbb{R}$. In the theory of parabolic equations the function $u_{i0}(s,x,t)$ is called the Poisson potential, and the function $u_{i1}(s,x,t)$ the parabolic simple-layer potential.

We recall some properties of functions $u_{i0}(s,x,t)$ and $u_{i1}(s,x,t),\ i=1,2$. Let $\varphi\in C_b(\mathbb{R})$. Then from the properties of the fundamental solution $G_i(s,x,t,y),\ i=1,2,$ it follows that the potential u_{i0} exists and satisfies equation (2) and the 'initial' condition

$$\lim_{s\uparrow t} u_{i0}(s,x,t) = \varphi(x), \quad x\in\mathbb{R},\ i=1,2, \tag{16}$$

in the domain $(s,x)\in[0,t)\times\mathbb{R}$ for a fixed $t\in(0,T]$ as a function of arguments (s,x).

In addition, for the function $u_{i0}(s,x,t),\ i=1,2,$ the inequality

$$\left|D_s^r D_x^p u_{i0}(s,x,t)\right| \le C(t-s)^{-\frac{2r+p}{2}}\|\varphi\|, \tag{17}$$

(where r and p are positive integers for which $2r+p\le 2$) holds in each of the domains $0\le s<t\le T,\ x\in\mathbb{R}$.

Consider integral (15). If we assume that the density $V(\tau,t)$ is continuous for $\tau\in[s,t)$ and has a weak singularity with exponent $\ge -\frac{1}{2}$ when $\tau=t$, then the function $u_{i1}(s,x,t),\ i=1,2,$

is bounded and continuous in $0 \le s \le t \le T,\ x \in \mathbb{R}$, it satisfies equation (2) in the domain $(s,x) \in [0,t) \times (\mathbb{R} \setminus h(s))$ and the initial condition

$$\lim_{s\uparrow t} u_{i1}(s,x,t) = 0, \quad x \in \mathbb{R},\ i = 1,2. \tag{18}$$

An important property of the function u_{i1} is reflected in the so-called theorem on the jump of the co-normal derivative of the parabolic simple-layer potential (see, for instance, [5,8]). In the present paper this assertion is not used, and therefore we do not provide it.

1 Existence and uniqueness

We find a solution of (2)–(7) in the form of sum of potentials u_{i0} and u_{i1} with unknown densities $V_i(s,t),\ i = 1,2$:

$$\begin{aligned} u(s,x,t) = &\int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy \\ &+ \int_s^t G_i(s,x,\tau,h(\tau))V_i(\tau,t)d\tau, \quad (s,x) \in \overline{S}_t^{(i)},\ i = 1,2. \end{aligned} \tag{19}$$

Using conjugation conditions (4), (5) and (8), we get the following system of Volterra integral equations of the first kind for $V_i(s,t)$:

$$\begin{aligned} &(\gamma(s)+1)\int_s^t G_i(s,h(s),\tau,h(\tau))V_i(\tau,t)d\tau \\ &-\sum_{j=1}^{2}\int_s^t V_j(\tau,t)d\tau \int_{D_{js}} G_j(s,y,\tau,h(\tau))\mu(s,dy) = \Phi_i(s,t), \quad i = 1,2, \end{aligned} \tag{20}$$

where

$$\Phi_i(s,t) = \sum_{j=1}^{2}\int_{D_{js}} u_{j0}(s,y,t)\mu(s,dy) - (\gamma(s)+1)u_{i0}(s,h(s),t), \quad i = 1,2.$$

Consider the function $\Phi_i(s,t)$ in (20). Let us prove that

$$\lim_{s\uparrow t} \Phi_i(s,t) = 0, \quad i = 1,2; \tag{21}$$

$$|\Phi_i(s,t) - \Phi_i(\widetilde{s},t)| \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad \widetilde{s} < s. \tag{22}$$

Assertion (21) can be easily verified using property (16) of the Poisson potential u_{i0} and fitting condition (7):

$$\begin{aligned} \lim_{s\uparrow t} \Phi_i(s,t) &= \sum_{j=1}^{2}\int_{D_{jt}} \varphi(y)\mu(t,dy) - (\gamma(t)+1)\varphi(h(t)) \\ &= \int_{D_{1t}\cup D_{2t}} [\varphi(y) - \varphi(h(t))]\mu(t,dy) - \gamma(t)\varphi(h(t)) = 0. \end{aligned}$$

To prove inequality (22), we write the difference $\Phi_i(s,t) - \Phi_i(\widetilde{s},t)$ as a sum $I_1 + I_2 + I_3$, where

$$I_1 = \sum_{j=1}^{2} \int_{D_{js}} [u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)]\mu(s,dy),$$
$$I_2 = (\gamma(\widetilde{s}) + 1)u_{i0}(\widetilde{s},h(\widetilde{s}),t) - (\gamma(s)+1)u_{i0}(s,h(s),t),$$
$$I_3 = \sum_{j=1}^{2} \left(\int_{D_{js}} u_{j0}(\widetilde{s},y,t)\mu(s,dy) - \int_{D_{j\widetilde{s}}} u_{j0}(\widetilde{s},y,t)\mu(\widetilde{s},dy) \right),$$

and study separately each term of this sum.

Since for $\widetilde{s} < s$

$$\begin{aligned}
&|u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)| \\
&= |u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)|^{\frac{1+\alpha}{2}} |u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)|^{\frac{1-\alpha}{2}} \\
&\le \left| \frac{\partial u_{j0}(\hat{s},y,t)}{\partial \hat{s}} \bigg|_{\hat{s}=\widetilde{s}+\theta(s-\widetilde{s})} \cdot (s-\widetilde{s}) \right|^{\frac{1+\alpha}{2}} (|u_{j0}(s,y,t)| + |u_{j0}(\widetilde{s},y,t)|)^{\frac{1-\alpha}{2}} \\
&\le C\|\varphi\| \left[(t-\widetilde{s}-\theta(s-\widetilde{s}))^{-1}(s-\widetilde{s}) \right]^{\frac{1+\alpha}{2}} \le C\|\varphi\| [((t-s) \\
&+ (s-\widetilde{s})(1-\theta))^{-1}(s-\widetilde{s})]^{\frac{1+\alpha}{2}} \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad 0<\theta<1,
\end{aligned}$$

inequality (22) holds for the term I_1. Recalling that the functions γ and h are Hölder continuous (see assumption V) and using previous considerations, we arrive at inequality (22) for I_2. For I_3 we have the estimate

$$|I_3| \le C\|\varphi\|(s-\widetilde{s})^{\frac{1+\alpha}{2}},$$

which is an obvious consequence of assumption IV. Thus,

$$|I_1 + I_2 + I_3| \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad \widetilde{s} < s,$$

what had to be proved.

In order to regularize system of Volterra integral equations of the first kind (20), we apply to both sides of each of its equations the integro-differential operator $\mathcal{E}$, which acts by the rule

$$\mathcal{E}(s,t)\Phi_i = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t (\rho - s)^{-\frac{1}{2}} \Phi_i(\rho,t)d\rho, \quad 0 \le s < t \le T,\ i = 1,2. \tag{23}$$

Consider first the action of the operator $\mathcal{E}$ on the right hand side of the i-th equation of system (20), $i = 1,2$.

In view of (21) and (22), for the function $\widehat{\Phi}_i(s,t) \equiv \mathcal{E}(s,t)\Phi_i$ we easily get the following formula:

$$\begin{aligned}
\widehat{\Phi}_i(s,t) = \frac{1}{\sqrt{2\pi}} &\int_s^t (\rho-s)^{-\frac{3}{2}}[\Phi_i(\rho,t) - \Phi_i(s,t)]d\rho \\
&- \sqrt{\frac{2}{\pi}}(t-s)^{-\frac{1}{2}}\Phi_i(s,t), \quad i=1,2.
\end{aligned} \tag{24}$$

Besides, for the function $\widehat{\Phi}_i(s,t)$ in each domain of the form $0 \le s < t \le T$ the inequality

$$|\widehat{\Phi}_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}} \tag{25}$$

holds.

Now, we apply the operator $\mathcal{E}$ to the left hand side of the i-th equation of system (20), $i = 1,2$. As a result, we obtain the expression, which after changing the order of integration and using formulas (9), (10) can be represented in the form

$$-\frac{V_i(s,t)}{\sqrt{b_i(s,h(s))}} + \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\sum_{j=1}^{2}\int_s^t N_{ij}(s,\tau)V_j(\tau,t)d\tau, \quad i = 1,2, \tag{26}$$

where

$$\begin{aligned} N_{ii}(s,\tau) = \int_s^\tau (\rho-s)^{-\frac{1}{2}}\Big[&(Z_{i0}(\rho,h(\rho),\tau,h(\tau)) - Z_{i0}(\rho,0,\tau,0)) + \gamma(\rho)G_i(\rho,h(\rho),\tau,h(\tau)) \\ &+ Z_{i1}(\rho,h(\rho),\tau,h(\tau)) - \int_{D_{i\rho}} G_i(\rho,y,\tau,h(\tau))\mu(\rho,dy)\Big]d\rho, \quad i = j, \\ N_{ij}(s,\tau) = -\int_s^\tau (\rho-s)^{-\frac{1}{2}}&d\rho \int_{D_{j\rho}} G_j(\rho,y,\tau,h(\tau))\mu(\rho,dy), \quad i \ne j. \end{aligned}$$

To simplify the derivatives of integrals depending on parameters in expression (26), we show that

$$\lim_{s\uparrow\tau} N_{ij}(s,\tau) = 0. \tag{27}$$

In proving this fact, a certain complexity is only a study of the function

$$L_j(s,\tau) \equiv \int_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho \int_{D_{j\rho}} Z_{j0}(\rho,y,\tau,h(\tau))\mu(\rho,dy),$$

which appears in the expression for $N_{ij}(s,\tau)$ immediately after we rewrite G_j according to formula (9). For all other terms in formula for $N_{ij}(s,\tau)$ the relation (27) is easily established by using the inequalities (12), (13) and condition V.

Write the function $L_j(s,\tau)$ as follows

$$L_j(s,\tau) = L_{j1}(s,\tau) + L_{j2}(s,\tau), \tag{28}$$

where

$$L_{j1}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}d\rho$$
$$\times \Big[\int\limits_{D_{j\rho}} \exp\Big\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)} \Big\} \mu(\rho,dy)$$
$$- \int\limits_{D_{js}} \exp\Big\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)} \Big\} \mu(s,dy) \Big],$$
$$L_{j2}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}}d\rho$$
$$\times \int\limits_{D_{js}} \exp\Big\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)} \Big\} \mu(s,dy).$$

Since the functions $f_{\tau,\rho}(y) = \exp\left\{\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)}\right\}$ belong to $C_b(\mathbb{R})$ for all $0 \le s < \rho < \tau < t \le T$ and are bounded by 1 on this set, and since condition IV holds, we have

$$|L_{j1}(s,\tau)| \le C(\tau-s)^{\frac{1+\alpha}{2}}, \quad j=1,2. \tag{29}$$

Let us study the function $L_{j2}(s,\tau)$. Write it in the form

$$L_{j2}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_{D_{js}} \left[\exp\left\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-s)} \right\} \right.$$
$$\left. - \exp\left\{ -\frac{(y-h(s))^2}{2b_j(\tau,h(\tau))(\tau-s)} \right\} \right] R_j(s,\tau,y)\mu(s,dy)$$
$$+ \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_{D_{js}} \exp\left\{ -\frac{(y-h(s))^2}{2b_j(\tau,h(\tau))(\tau-s)} \right\} R_j(s,\tau,y)\mu(s,dy), \tag{30}$$

where $R_j(s,\tau,y)$ denotes the integral

$$R_j(s,\tau,y) = \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} \exp\left\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-s)} \cdot \frac{\rho-s}{\tau-\rho} \right\} d\rho,$$

which after the substitution $z = \frac{\rho-s}{\tau-\rho}$ reduces to

$$R_j(s,\tau,y) = \int\limits_0^\infty z^{-\frac{1}{2}}(1+z)^{-1} \exp\left\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-s)} \cdot z \right\} dz,$$

and thus, satisfies the inequality

$$|R_j(s,\tau,y)| \le C. \tag{31}$$

Denote by $L_{j2}^{(1)}$ the first term in the right hand side of equality (30) and by $L_{j2}^{(2)}$ the second one.

If we express, using the Lagrange formula, the difference of exponents in the square brackets of the expression for $L_{j2}^{(1)}$ through the value of its derivative at the intermediate point $x = y - h(s) + \theta(h(s) - h(\tau))$, and then take this derivative, we get

$$L_{j2}^{(1)}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau, h(\tau))}} \int_{D_{js}} \frac{x}{b_j(\tau, h(\tau))(\tau - s)} \\ \times \exp\left\{-\frac{x^2}{2b_j(\tau, h(\tau))(\tau - s)}\right\} (h(\tau) - h(s)) R_j(s,\tau,y)\mu(s,dy).$$

From this equality and estimate (31) and condition V it follows that

$$|L_{j1}(s,\tau)| \leq C(\tau - s)^{\frac{\alpha}{2}}. \tag{32}$$

Then (31) implies

$$|L_{j2}(s,\tau)| \leq C\left(\mu\left(s, D_{js}^{\delta}\right) + \exp\left\{-\frac{\delta^2}{2B(\tau - s)}\right\}\right), \tag{33}$$

where $D_{js}^{\delta} = \{y \in D_{js} : |y - h(s)| < \delta\}$, δ is any positive number, B is the constant from I.

Combining (28)–(30), (32), (33), we conclude that

$$\lim_{s \uparrow \tau} L_j(s,\tau) = 0.$$

This completes the proof of (27).

With relation (27) in mind, we put the derivative under the integral sign in expression (26) and then equate this expression to (24). After elementary simplifications, we get the system of Volterra integral equations of the second kind, which is equivalent to (20)

$$V_i(s,t) = \sum_{j=1}^{2} \int_s^t K_{ij}(s,\tau) V_j(\tau,t) d\tau + \Psi_i(s,t), \quad i = 1,2, \tag{34}$$

where

$$\Psi_i(s,t) = -\sqrt{b_i(s,h(s))}\,\widehat{\Phi}_i(s,t),$$
$$K_{ij}(s,\tau) = \sqrt{\frac{2}{\pi}}\sqrt{b_i(s,h(s))} \cdot \frac{\partial}{\partial s} N_{ij}(s,\tau).$$

The function Ψ_i in (34) satisfies inequality (25), but kernels $K_{ij}(s,\tau)$ do not have the integrable singularity. For $K_{ij}(s,\tau)$ we can only get the estimate

$$K_{ij}(s,\tau) \leq C(\tau - s)^{-1}, \quad 0 \leq s < \tau < t \leq T. \tag{35}$$

Estimate (35) is caused by the integral

$$\int_{D_{js}^{\delta}} \frac{\partial Z_{j0}(s,y,\tau,h(\tau))}{\partial y} \mu(s,dy), \tag{36}$$

which is in the expression for the derivative of L_j

$$\frac{\partial}{\partial s}L_j(s,\tau)=\int\limits_s^\tau(\rho-s)^{-\frac{3}{2}}\Bigg[\int\limits_{D_{j\rho}}Z_{j0}(\rho,y,\tau,h(\tau))\mu(\rho,dy)$$
$$-\int\limits_{D_{js}}Z_{j0}(\rho,y,\tau,h(\tau))\mu(s,dy)\Bigg]d\rho$$
$$-\sqrt{\frac{\pi b_j(\tau,h(\tau))}{2}}\Bigg(\int\limits_{D_{js}^{(\delta)}}\frac{\partial Z_{j0}(s,y,\tau,h(\tau))}{\partial y}\mu(s,dy)$$
$$+\int\limits_{\mathbb{R}\setminus D_{js}^{(\delta)}}\frac{\partial Z_{j0}(s,y,\tau,h(\tau))}{\partial y}\mu(s,dy)\Bigg).$$

All other components of the expression for $K_{ij}(s,\tau)$ admit inequalities the right hand sides of which have the form $C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}$, where $C(\delta)$ is a positive constant depending on δ.

Despite the fact that the kernels $K_{ij}(s,\tau)$ do not have an integrable singularity, a solution of system of equations (34) exists and can be found by the ordinary method of successive approximations:

$$V_i(s,t)=\sum_{n=0}^{\infty}V_i^{(n)}(s,t),\quad 0\le s<t\le T,\ i=1,2, \tag{37}$$

where

$$V_i^{(0)}(s,t)=\Psi_i(s,t),$$
$$V_i^{(n)}(s,t)=\sum_{j=1}^{2}\int\limits_s^t K_{ij}(s,\tau)V_i^{(n-1)}(\tau,t)d\tau,\quad n=1,2,\ldots$$

The convergence of series (37) is the consequence of the following inequality, which is proved by induction according to the scheme applied in [9] in the study of system of equations (34) for the case when $h\equiv 0$:

$$\left|V_i^{(n)}(s,t)\right|\le C\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{n}C_n^k a^{(n-k)}(m(\delta))^k,\quad n=0,1,\ldots, \tag{38}$$

where

$$a^{(k)}=\frac{\left(2c(\delta)T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^k\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+k\alpha}{2}\right)},\quad k=0,1,\ldots,n,$$
$$m(\delta)=\max_{s\in[0,T]}\mu(s,D_{1s}^{\delta}\cup D_{2s}^{\delta})<1\ (\text{for sufficiently small }\delta).$$

From inequality (38) it also follows that the function $V_i(s,t),\ i=1,2$, admits the estimate

$$|V_i(s,t)|\le C\|\varphi\|(t-s)^{-\frac{1}{2}},\quad 0\le s<t\le T. \tag{39}$$

Thus, we have constructed the solution $u(s,x,t)$ of problem (2)–(7) of form (19), (37), which, in view of estimates (12), (13), (17), (39), belongs to the class $C^{1,2}(S_t^{(1)} \cup S_t^{(2)}) \cap C(\overline{S}_t)$ and satisfies the inequality

$$|u(s,x,t)| \leq C\|\varphi\|(t-s)^{-\frac{1}{2}}. \tag{40}$$

The assertion on the uniqueness of the constructed solution of problem (2)–(7) follows from the maximum principle [5].

The obtained result allows us to state the following theorem:

Theorem 1. *Let the conditions I–V hold. Then problem (2)–(7) has a unique solution belonging to $C^{1,2}(S_t^{(1)} \cup S_t^{(2)}) \cap C(\overline{S}_t)$. Besides, this solution admits representation (19), (37) and estimate (40).*

2 Construction of Feller semigroup

Denote by $C_0(\mathbb{R})$ the subspace of $C_b(\mathbb{R})$, which consists of all functions $\varphi \in C_b(\mathbb{R})$ for which the condition (7) holds. Since the subspace $C_0(\mathbb{R})$ is closed in $C_b(\mathbb{R})$, it is a Banach space.

We introduce the two-parameter family of linear operators $T_{st}: C_0(\mathbb{R}) \to C_0(\mathbb{R})$, $0 \leq s < t \leq T$, by the following rule:

$$T_{st}\varphi(x) = u(s,x,t,\varphi), \tag{41}$$

where $u(s,x,t,\varphi)$ is a solution of (2)–(7) with the function φ in (3).

Note that the operators T_{st} have the following properties in $C_0(\mathbb{R})$:

a) if a sequence of functions $\varphi_n \in C_0(\mathbb{R})$ is such that $\sup\limits_n \|\varphi_n\| < \infty$ and $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$, then $\lim\limits_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x)$ for all $0 \leq s < t \leq T,\ x \in \mathbb{R}$;

b) the operators T_{st} are positivity preserving $(0 \leq s < t \leq T)$, i.e, $T_{st}\varphi \geq 0$ for every $\varphi \in C_0(\mathbb{R})$ such that $\varphi \geq 0$;

c) the operators T_{st} are contractive $(0 \leq s < t \leq T)$, i.e., they do not increase the norm of the element;

d) $T_{st} = T_{s\tau}T_{\tau t}$, $0 \leq s < \tau < t \leq T$ (the semigroup propery).

The proof of property a) is based on well known assertions of calculus on passage of the limit under the summation and integral signs (here this concerns series (37) and integrals on the right hand side of equality (19)). This property allows us to prove the next properties of the operator family T_{st}, without loss of generality, under the assumption that the function φ has a compact support.

Let us prove property b). Let $\varphi \in C_0(\mathbb{R})$ be a nonnegative function with a compact support. Denote by m the minimum of $T_{st}\varphi(x)$ in $(s,x) \in \overline{S}_t$. If we assume that $m < 0$, then from the minimum principle [5] it follows that the value m is attained only when $s \in (0,t)$ and $x = h(s)$. Fix $s_0 \in (0,t)$ for which $T_{s_0 t}\varphi(h(s_0)) = m$. Then

$$\gamma(s_0)T_{s_0 t}\varphi(h(s_0)) + \int\limits_{D_{1s_0} \cup D_{2s_0}} [T_{s_0 t}\varphi(h(s_0)) - T_{s_0 t}\varphi(y)]\mu(s_0, dy) < 0,$$

which contradicts (5). The contradiction we arrived at indicates that $m \geq 0$, what had to be proved.

The proof of property c) is similar to the proof of b).

The semigroup property of operators T_{st} is a consequence of the assertion on the uniqueness of the solution of problem (2)–(7). Indeed, to find $u(s,x,t) = T_{st}\varphi(x)$, provided $\lim_{s\uparrow t} u(s,x,t) = \varphi(x)$, one can solve the problem first in the time interval $[\tau, t]$, and then solve it in the time interval $[s, \tau]$ with that 'initial' function $u(\tau, x, t) = T_{\tau t}\varphi(x)$, which was obtained; in other words, $T_{st}\varphi(x) = T_{s\tau}(T_{\tau t}\varphi)(x)$, $\varphi \in C_0(\mathbb{R})$, or $T_{st} = T_{s\tau}T_{\tau t}$.

Properties a)–d) of operators T_{st} imply the following assertion.

Theorem 2. *Let the conditions of Theorem 1 hold. Then the two-parameter family of operators T_{st}, $0 \leq s < t \leq T$, defined by (41), describes the inhomogeneous Feller process on the line $\mathbb{R}$, which coincides in D_{1s} and D_{2s} with given diffusion processes generated by operators $L_s^{(1)}$ and $L_s^{(2)}$ respectively, and its behavior at point $x = h(s)$ is determined by conjugation condition (5).*

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Копитко Б.І., Шевчук Р.В. *Про напівгрупу Феллера, породжену розв'язком нелокальної параболічної задачі спряження* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 333–345.

У статті розглядається задача побудови напівгрупи Феллера для одновимірного неоднорідного дифузійного процесу з мембраною, розташованою в точці, положення якої на числовій прямій визначається за допомогою заданої функції, що залежить від часової змінної. При цьому припускається, що у внутрішніх точках півпрямих, розділених між собою мембраною, шуканий процес має збігатися із заданими там звичайними дифузійними процесами, а його поведінка на спільній межі цих областей визначається заданою нелокальною умовою спряження типу Феллера-Вентцеля. Дану задачу ще називають задачею про склеювання двох дифузійних процесів на прямій.

З метою вивчення сформульованої проблеми в роботі застосовано аналітичні методи. Такий підхід дозволяє визначити шукану сім'ю операторів з допомогою розв'язку відповідної задачі спряження для лінійного параболічного рівняння другого порядку (оберненого рівняння Колмогорова) з розривними коефіцієнтами. Цей розв'язок побудовано методом граничних інтегральних рівнянь за припущення, що коефіцієнти рівняння задовольняють умову Гельдера з ненульовим показником, початкова функція є обмеженою і неперервною на всій числовій прямій, а параметри, які характеризують умову спряження Феллера-Вентцеля та крива, що визначає спільну межу областей, де задане рівняння, задовольняють умову Гельдера з показником більшим, ніж $\frac{1}{2}$.

Ключові слова і фрази: напівгрупа Феллера, дифузійний процес, параболічна задача спряження.