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SOME ANALYTIC PROPERTIES OF THE WEYL FUNCTION OF A CLOSED LINEAR RELATION

Let L and L_0, where L is an expansion of L_0, be closed linear relations (multivalued operators) in a Hilbert space H. In terms of abstract boundary operators (i.e. in the form which in the case of differential operators leads immediately to boundary conditions) some analytic properties of the Weyl function $M(\lambda)$ corresponding to a certain boundary pair of the couple (L, L_0) are studied.

In particular, applying Hilbert resolvent identity for relations, the criterion of invertibility in the algebra of bounded linear operators in H for transformation $M(\lambda) - M(\lambda_0)$ in certain small punctured neighbourhood of λ_0 is established. It is proved that in this case λ_0 is a first-order pole for the operator-function $(M(\lambda) - M(\lambda_0))^{-1}$. The corresponding residue and Laurent series expansion are found.

Under some additional assumptions, the behaviour of so called γ-field Z_λ (being an operator-function closely connected to $M(\lambda)$) as $\lambda \to -\infty$ is investigated.

Key words and phrases: Hilbert space, relation, operator, extension, pole.

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INTRODUCTION

The theory of linear relations (multivalued operators) in Hilbert space was initiated by R. Arens [1]. Various aspects of the extension theory of linear relations (in particular, nondensely defined operators; first of all, Hermitian ones) were studied by a number of authors (see, e.g. [3, 15, 16], [5]– [8], [9], [10], [14]).

Let us explain that under (closed) linear relation in H, where H is a fixed complex Hilbert space equipped with inner product $(\cdot|\cdot)$, we understand a (closed) linear manifold in $H^2 \overset{def}{=} H \oplus H$ and that in the theory of linear relations every linear operator is identified with its graph. Each such relation T has the adjoint T^* which is defined as follows:

$$T^* = H^2 \ominus JT \left(= J(H^2 \ominus T) \right)$$

(here and below $\oplus$ and $\ominus$ are the symbols of orthogonal sum and orthogonal complement, respectively; for all $h_1, h_2 \in H$ $J(h_1, h_2) \overset{def}{=} (-ih_2, ih_1)$).

In this paper the role of initial object is played by two couples (L, L_0) and (M, M_0) of closed linear relations in H such that

$$L_0 \subset L, \quad M = L_0^*, \quad M_0 = L^*.$$

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Let us note that in [13, 15–17] the term "dual pair" was using instead of "couple" in the present paper. The authors of [2] were using the term *adjoint pair*.

The notion of Weyl function had been introduced at first in [4] under the assumption that L_0 is a nonnegative densely defined operator and $M = L$. Later on it was extended onto more wide varieties of operators and relations in some of papers, mentioned above (e.g. [5,7,15,16]). It turned out that this notion is very important in the extension theory, since certain classes of extensions of a given operator or relation may be described by using this notion.

In this article (which can be regarded as a continuation of investigations originated in [20, 21]) we study some analytic properties of the Weyl function of (L, L_0) corresponding to the certain its boundary pair (see Definitions 1, 2).

1 Notations and preliminary results

Through this paper we use the following notations:

$D(T),\ R(T),\ \ker T$ are, respectively, the domain, range, and kernel of a (linear) relation (in partial, operator) T:

$D(T) = \{y \in H |\, (\exists\, y' \in H) : (y, y') \in T\}$; $R(T) = \{y' \in H |\, (\exists\, y \in H) : (y, y') \in T\}$;

$\ker T = \{y \in H |\, (y, 0) \in T\}$;

if $\lambda \in \mathbb{C}$ then $T - \lambda = \{(y, y' - \lambda y) \,|\, (y, y') \in T\}$, sequently

$\ker(T - \lambda) = \{y \in H |\, (y, 0) \in T - \lambda\}$ $(= \{y \in H |\, (y, \lambda y) \in T\})$;

$T^{-1} = \{(y', y) \in H^2 |\, (y, y') \in T\}$;

$\rho(T) = \{\lambda \in \mathbb{C} |\, \ker(T - \lambda) = \{0\},\ R(T - \lambda) = H\}$ (the resolvent set of T);

1_X is the identity in X;

$+,\ \dot{+}$ are, respectively, the symbols of sum and direct sum in a linear space.

If X, Y are Hilbert spaces then $(\cdot|\cdot)_X$ is the symbol of scalar product in X, $\mathcal{B}(X, Y)$ is the set of linear bounded operators $A : X \to Y$ such that $D(A) = X$; $\mathcal{B}(X) \stackrel{def}{=} \mathcal{B}(X, X)$.

If $A_i : X \to Y_i\, (i = 1, 2)$ are linear operators then the notation $A = A_1 \oplus A_2$ means that $Ax = \begin{pmatrix} A_1 x \\ A_2 x \end{pmatrix}$ for every $x \in X$.

Definition 1 ([18]). *Let G be an (auxiliary) Hilbert space and $\Gamma \in \mathcal{B}(L, G)$. The pair (G, Γ) is called a boundary pair for (L, L_0) if $R(\Gamma) = G,\ \ker \Gamma = L_0$.*

Theorem 1 ([18, 19]). *There exist Hilbert spaces G_1, G_2 and the operators*

$$\Gamma_1 \in \mathcal{B}(L, G_1),\ \Gamma_2 \in \mathcal{B}(L, G_2),\ \tilde{\Gamma_1} \in \mathcal{B}(M, G_2),\ \tilde{\Gamma_2} \in \mathcal{B}(M, G_1)$$

such that

i) $(G_1 \oplus G_2,\ \Gamma_1 \oplus \Gamma_2)$ is a boundary pair for (L, L_0);

ii) $\left(G_2 \oplus G_1,\ \tilde{\Gamma}_1 \oplus \tilde{\Gamma}_2\right)$ is a boundary pair for (M, M_0);

iii) for all $\hat{y} = (y, y') \in L$, for all $\hat{z} = (z, z') \in M$ $(y'|z) - (y|z') = (\Gamma_1 \hat{y} | \tilde{\Gamma}_2 \hat{z})_{G_1} - (\Gamma_2 \hat{y} | \tilde{\Gamma}_1 \hat{z})_{G_2}$.

We suppose below that the resolvent set $\rho\,(L_2)$ of the relation $L_2 \stackrel{def}{=} \ker \Gamma_2$ is not empty and $\lambda \in \rho\,(L_2)$. Then $\overline{\lambda} \in \rho\,(M_2)$, where $M_2 \stackrel{def}{=} \ker \tilde{\Gamma}_2\ (= L_2^*)$ and

$$L_\lambda \overset{def}{=} (L_2-\lambda)^{-1} \in \mathcal{B}(H), \quad M_{\overline{\lambda}} \overset{def}{=} \left(M_2-\overline{\lambda}\right)^{-1} (= L_\lambda^*) \in \mathcal{B}(H).$$

Put for all $y \in H$ $\hat{L}_\lambda y = \begin{pmatrix} L_\lambda y \\ y+\lambda L_\lambda y \end{pmatrix}$, for all $z \in H$ $\hat{M}_{\overline{\lambda}} z = \begin{pmatrix} M_{\overline{\lambda}} z \\ z+\overline{\lambda} M_{\overline{\lambda}} z \end{pmatrix}$,
for all $\hat{y} = (y,y') \in H^2$ $\tilde{L}_\lambda \hat{y} = L_\lambda y + \left(y' + \lambda L_\lambda y'\right)$,
for all $\hat{z} = \left(z,z'\right) \in H^2$ $\tilde{M}_{\overline{\lambda}} \hat{z} = M_{\overline{\lambda}} z + \left(z' + \overline{\lambda} M_{\overline{\lambda}} z'\right)$

(it is clear that $\hat{L}_\lambda^* = \tilde{M}_{\overline{\lambda}}$, $\hat{M}_{\overline{\lambda}}^* = \tilde{L}_\lambda$),

$$Z_\lambda = \left(\tilde{\Gamma}_1 \hat{M}_{\overline{\lambda}}\right)^* (= \tilde{L}_\lambda \tilde{\Gamma}_1^*), \quad \tilde{Z}_{\overline{\lambda}} = \left(\Gamma_1 \hat{L}_\lambda\right)^* (= \tilde{M}_{\overline{\lambda}} \Gamma_1^*), \quad \hat{Z}_\lambda = \begin{pmatrix} Z_\lambda \\ \lambda Z_\lambda \end{pmatrix}.$$

Note that in some articles Z_λ is said to be aγ-field.

Lemma 1 ([19]). *i)* $R(\hat{L}_\lambda) = L_2$, $R\left(\hat{M}_{\overline{\lambda}}\right) = M_2$;
ii) $Z_\lambda \in \mathcal{B}(G_2, H)$ *and* $R(Z_\lambda) = \ker(L-\lambda)$,
$\tilde{Z}_{\overline{\lambda}} \in \mathcal{B}(G_1, H)$ *and* $R(\tilde{Z}_{\overline{\lambda}}) = \ker(M-\overline{\lambda})$;
iii) $R(\hat{Z}_\lambda) \subset L$ *and* $\Gamma_2 \hat{Z}_\lambda = 1_{G_2}$.

Proposition 1 ([1,3,6]). *Let S be closed linear nonnegative, in symbols $S \geq 0$ (that is $(z|y) \geq 0$ for all $(y,z) \in S$), selfadjoint relation in H and $\lambda < 0$. Then*
i)

$$\lambda \in \rho(S), \quad \left\|(S-\lambda)^{-1}\right\| \leq \frac{1}{|\lambda|}. \tag{1}$$

ii) Put $S(0) = \{y \in H : (0,y) \in S\}$, $S_s = S \ominus (\{0\} \oplus S(0))$. S_s is the graph of selfadjoint operator $S_{op} : S(0)^\perp \to S(0)^\perp (\equiv \overline{D(S)})$ (which is said to be an operator part of S) with $D(S_{op}) = D(S)$ and $R\left((S-\lambda)_{op}\right) = S(0)^\perp$.

It is clear that (1) implies

$$\text{for all } f \in H \quad \lim_{\lambda \to -\infty} (S-\lambda)^{-1} f = 0. \tag{2}$$

Moreover, if S is an operator, then

$$\text{for all } f \in H \quad \lim_{\lambda \to -\infty} \left[\lambda (S-\lambda)^{-1} f + f\right] = 0. \tag{3}$$

Indeed, for each $g \in D(S)$ we have $\lambda (S-\lambda)^{-1} g + g = (S-\lambda)^{-1} Sg \underset{\lambda \to -\infty}{\to} 0$ (see (2)). Further, in view of (1) for all $\lambda \in (-\infty, 0)$ $\left\|\lambda (S-\lambda)^{-1} + 1_H\right\| \leq 2$.

Since $\overline{D(S)} = H$, two latter relations guarantee that (3) is true. It follows from the well known criterion of the strong convergence for the operator sequences (see [12, p. 59]).

2 AUXILIARY STATEMENTS

Remark 1. *Applying Hilbert resolvent identity for relations (see [6]) it is easy to prove that*

$$\textit{for all } \lambda, \mu \in \rho(L_2) \quad \tilde{L}_\lambda - \tilde{L}_\mu = (\lambda-\mu) L_\lambda \tilde{L}_\mu (= (\lambda-\mu) L_\mu \tilde{L}_\lambda). \tag{4}$$

Indeed,

$$\begin{aligned}\tilde{L}_\lambda - \tilde{L}_\mu &= \left(L_\lambda - L_\mu,\ \lambda L_\lambda - \mu L_\mu\right) = \left(L_\lambda - L_\mu,\ (\lambda-\mu)\,L_\lambda + \mu\left(L_\lambda - L_\mu\right)\right)\\ &= \left((\lambda-\mu)\,L_\lambda L_\mu,\ (\lambda-\mu)\,L_\lambda + \mu\,(\lambda-\mu)\,L_\lambda L_\mu\right)\\ &= (\lambda-\mu)\,L_\lambda\left(L_\mu,\, 1_H + \mu L_\mu\right) = (\lambda-\mu)\,L_\lambda \tilde{L}_{\tilde{\mu}}.\end{aligned}$$

Similar arguments show that

$$\text{for all } \lambda,\ \mu \in \rho(L_2) \quad \hat{L}_\lambda - \hat{L}_\mu = (\lambda-\mu)\,\hat{L}_\lambda L_\mu (= (\lambda-\mu)\,\hat{L}_\mu L_\lambda).$$

Lemma 2. *Let $\lambda,\ \mu \in \rho(L_2)$. Then*

$$Z_\lambda - Z_\mu = (\lambda-\mu)\,L_\lambda Z_\mu (= (\lambda-\mu)\,L_\mu Z_\lambda), \tag{5}$$

$$\tilde{Z}^*_{\bar\lambda} - \tilde{Z}^*_{\bar\mu} = (\lambda-\mu)\,\tilde{Z}^*_{\bar\mu} L_\lambda (= (\lambda-\mu)\,\tilde{Z}^*_{\bar\lambda} L_\mu), \tag{6}$$

$$\hat{Z}_\lambda - \hat{Z}_\mu = (\lambda-\mu)\,\hat{L}_\lambda Z_\mu (= (\lambda-\mu)\,\hat{L}_\mu Z_\lambda). \tag{7}$$

Proof. Taking into account (4) we obtain

$$Z_\lambda - Z_\mu = \left(\tilde{L}_\lambda - \tilde{L}_\mu\right)\tilde{\Gamma}^*_1 = (\lambda-\mu)\,L_\lambda \tilde{L}_\mu \tilde{\Gamma}^*_1 = (\lambda-\mu)\,L_\lambda Z_\mu.$$

The equality (5) is proved. The proof of (6) is analogous. Furthermore,

$$\begin{aligned}\lambda Z_\lambda - \mu Z_\mu &= \lambda\left(Z_\lambda - Z_\mu\right) + (\lambda-\mu)\,Z_\mu\\ &= \lambda\,(\lambda-\mu)\,L_\lambda Z_\mu + (\lambda-\mu)\,Z_\mu = (\lambda-\mu)\,(1_H + \lambda L_\lambda)\,Z_\mu.\end{aligned}$$

The latter identity together (5) implies (7). □

Corollary 1. *For arbitrary $\lambda \in \rho(L_2),\ n \in \mathbb{N}$ we have*

$$Z^{(n)}_\lambda = n! L^n_\lambda Z_\lambda, \tag{8}$$

$$(\tilde{Z}^*_{\bar\lambda})^{(n)} = n! \tilde{Z}^*_{\bar\lambda} L^n_\lambda, \tag{9}$$

$$\hat{Z}^{(n)}_\lambda = n! \hat{L}_\lambda L^{n-1}_\lambda Z_\lambda. \tag{10}$$

Proof. First of all, note that

$$L^{(n)}_\lambda = n! L^{n+1}_\lambda, \quad \tilde{L}^{(n)}_\lambda = n! L^n_\lambda \tilde{L}_\lambda, \quad \hat{L}^{(n)}_\lambda = n! \hat{L}_\lambda L^n_\lambda. \tag{11}$$

In the case $n = 1$ these equalities follow immediately from the Hilbert resolvent identity. In the general case induction should be applied. The equalities (11) imply (8), (9). In order to prove (10) note that $(\lambda Z_\lambda)^{(n)} = n Z^{(n-1)}_\lambda + \lambda Z^{(n)}_\lambda$ (it can be shown by induction). The latter identity together with (9) imply (11). □

Lemma 3. *Suppose that $\lambda, \mu \in \rho(L_2)$. Then*

$$\left(\tilde{Z}^*_{\bar\mu} Z_\lambda\right)^{-1} \in \mathcal{B}\left(G_1, G_2\right) \Leftrightarrow R\left(L_0 - \mu\right) \dot{+} \ker\left(L - \lambda\right) = H. \tag{12}$$

Proof. It is sufficient to verify the next implications:

$$i)\ R(L_0-\mu)\cap\ker(L-\lambda)=\{0\}\Rightarrow\ker\left(\tilde Z^*_{\overline{\mu}}Z_\lambda\right)=\{0\},$$
$$ii)\ R(L_0-\mu)+\ker(L-\lambda)=H\Rightarrow R\left(\tilde Z^*_{\overline{\mu}}Z_\lambda\right)=G_1,$$
$$iii)\ \ker\left(\tilde Z^*_{\overline{\mu}}Z_\lambda\right)=\{0\}\Rightarrow R(L_0-\mu)\cap\ker(L-\lambda)=\{0\},$$
$$iv)\ R\left(\tilde Z^*_{\overline{\mu}}Z_\lambda\right)=G_1\Rightarrow R(L_0-\mu)+\ker(L-\lambda)=H.$$

Let us consider each of them.

i) Assume that for some $a\in G_2$ the equality $\tilde Z^*_{\overline{\mu}}Z_\lambda a\equiv\Gamma_1\hat L_\mu Z_\lambda a=0$ holds. Then $\hat L_\mu Z_\lambda a\in\ker\Gamma_1$. But $\hat L_\mu Z_\lambda a\in L_2=\ker\Gamma_2$, hence $\hat L_\mu Z_\lambda a\in L_0$. In other words, $\begin{pmatrix}L_\mu Z_\lambda a\\ Z_\lambda a+\mu L_\mu Z_\lambda a\end{pmatrix}\in L_0$. Consequently $\begin{pmatrix}L_\mu Z_\lambda a\\ Z_\lambda a\end{pmatrix}\in L_0-\mu$, in particular $Z_\lambda a\in R(L_0-\mu)$. But (see Lemma 1) $Z_\lambda a\in\ker(L-\lambda)$, therefore $Z_\lambda a=0,\ \ \hat Z_\lambda a=0$. Thus $a=\Gamma_2\hat Z_\lambda a=0$.

ii) For arbitrary $h\in G_1$ there exists $\hat y=\begin{pmatrix}y\\ y'\end{pmatrix}\in L_2=\ker\Gamma_2$ satisfying the equality $\Gamma_1\hat y=h$. We have: $\begin{pmatrix}y\\ y'-\mu y\end{pmatrix}\in L_2-\mu$, in particular, $y=L_\mu\left(y'-\mu y\right)$. Further, there exist $u\in R\left(L_0-\mu\right),\ a\in G_2$ such that $y'-\mu y=u+Z_\lambda a$. It means that for some $\hat y_0=\begin{pmatrix}y_0\\ y_0'\end{pmatrix}\in H^2$ the equalities $u=y_0'-\mu y_0,\ y_0=L_\mu\left(y_0'-\mu y_0\right)$ are fulfilled. Whence using these equalities we obtain

$$\begin{aligned}\hat L_\mu Z_\lambda a&=\hat L_\mu\left(y'-\mu y-u\right)=\hat L_\mu\left((y'-\mu y)-(y_0'-\mu y_0)\right)=\hat L_\mu\left(y'-\mu y\right)-\hat L_\mu\left(y_0'-\mu y_0\right)\\&=\begin{pmatrix}L_\mu(y'-\mu y)\\ y'-\mu y+\mu L_\mu(y'-\mu y)\end{pmatrix}-\begin{pmatrix}L_\mu(y_0'-\mu y_0)\\ y_0'-\mu y_0+\mu L_\mu(y_0'-\mu y_0)\end{pmatrix}\\&=\begin{pmatrix}y\\ y'-\mu y+\mu y\end{pmatrix}-\begin{pmatrix}y_0\\ y_0'-\mu y_0+\mu y_0\end{pmatrix}=\begin{pmatrix}y\\ y'\end{pmatrix}-\begin{pmatrix}y_0\\ y_0'\end{pmatrix}=\hat y-\hat y_0,\end{aligned}$$

consequently

$$\tilde Z^*_{\overline{\mu}}Z_\lambda a=\Gamma_1\hat L_\mu Z_\lambda a=\Gamma_1\left(\hat y-\hat y_0\right)=\Gamma_1\hat y=h.$$

iii) Assume that $y\in R\left(L_0-\mu\right)\cap\ker\left(L-\lambda\right)$. Then $y=Z_\lambda a$ for some $a\in G_2$ and

$$y\in R\left(L_0-\mu\right). \tag{13}$$

The inclusion (13) implies $y\in R\left(L_2-\mu\right)$. It is easy to see that

$$\left(L_\mu y,y\right)\in L_2-\mu. \tag{14}$$

Taking into account (13), (14) and the equality $\ker\left(L_2-\lambda\right)=\{0\}$, we obtain $\left(L_\mu y,y\right)\in L_0-\mu$. The latter inclusion yields $\hat L_\mu y=\begin{pmatrix}L_\mu y\\ y+\mu L_\mu y\end{pmatrix}\in L_0$, therefore $\Gamma_1\hat L_\mu y=\Gamma_1\hat L_\mu Z_\lambda a=0$. Now it is clear that $a=0,\ \ y=0$.

iv) For any $h\in H$ we have $\hat L_\mu h\in L_2$ (see Lemma 1). Put $\Gamma_1\hat L_\mu h=g$. There exists $u\in\ker\left(L-\lambda\right)=R\left(Z_\lambda\right)$ such that $\Gamma_1\hat L_\mu u=g$, consequently $\Gamma_1\hat L_\mu\left(h-u\right)=0$. Moreover, $\hat L_\mu\left(h-u\right)\in L_0$, i.e. $\begin{pmatrix}L_\mu(h-u)\\ h-u\end{pmatrix}\in L_0-\mu$. Thus $h=u+(h-u)\in\ker\left(L-\lambda\right)+R\left(L_0-\mu\right)$. □

Remark 2. *Assume that* $\lambda_0 \in \rho(L_2)$, $\left(\tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0}\right)^{-1} \in \mathcal{B}(G_1, G_2)$, *and*

$$0 < |\lambda - \lambda_0| < \min\left\{ \frac{1}{2\left\|L_{\lambda_0}\right\|}, \frac{1}{2\left\|\tilde{Z}^*_{\overline{\lambda_0}}\right\| \cdot \left\|Z_{\lambda_0}\right\| \cdot \left\|L_{\overline{\lambda_0}}\right\| \cdot \left\|\tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0}\right\|} \right\}. \tag{15}$$

Then $\left(\tilde{Z}^*_{\overline{\lambda}} Z_{\lambda_0}\right)^{-1} \in \mathcal{B}(G_1, G_2)$ *(here and below* $\|T\|$ *is the norm of operator* T*).*

Indeed, let $|\lambda - \lambda_0| < \frac{1}{2\left\|L_{\lambda_0}\right\|}$. Applying the theorem on perturbation of invertible in $\mathcal{B}(H)$ operator (see [11, pp. 228–229]) we see that $\left(1_H + (\lambda - \lambda_0) L_{\lambda_0}\right)^{-1} \in \mathcal{B}(H)$ and

$$\left\|\left(1_H + (\lambda - \lambda_0) L_{\lambda_0}\right)^{-1} - 1_H\right\| < 2|\lambda - \lambda_0| \left\|L_{\lambda_0}\right\|. \tag{16}$$

Further, taking into account (6) in which μ is replaced by λ_0 we conclude that

$$\tilde{Z}^*_{\overline{\lambda}} Z_{\lambda_0} - \tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0} = \tilde{Z}^*_{\overline{\lambda_0}} \left[\left(1_H + (\lambda - \lambda_0) L_{\lambda_0}\right)^{-1} - 1_H\right] Z_{\lambda_0}.$$

Whence using above-mentioned theorem and (16) we obtain the following: (12) implies $\left(\tilde{Z}^*_{\overline{\lambda}} Z_{\lambda_0}\right)^{-1} \in \mathcal{B}(G_1, G_2)$.

Proposition 2. *Suppose that* $M_0 = L_0 \geq 0$, $M = L$, $G_1 = G_2 \stackrel{def}{=} \mathcal{H}$, $\tilde{\Gamma}_1 = \Gamma_1$, $\tilde{\Gamma}_2 = \Gamma_2$ *(in other words,* $(\mathcal{H}, \Gamma_1, \Gamma_2)$ *is a boundary triple (boundary value space) of* L *[5, 9, 10, 15]). Let* $L_2 \stackrel{def}{=} \ker \Gamma_2$ *be a (selfadjoint) nonnegative extension of* L_0*, and* L_λ, $\tilde{L}_\lambda$, Z_λ *be as above.*

Under these assumptions

$$s - \lim_{\lambda \to -\infty} Z^*_\lambda = \Gamma_1 (0 \oplus Q), \qquad w - \lim_{\lambda \to -\infty} Z_\lambda = Q \pi_2 \Gamma^*_1, \tag{17}$$

where $s - \lim$ *and* $w - \lim$ *are respectively the symbols of strong and weak limits for operator-functions, while* Q *and* π_2 *are the orthoprojections* $H \to L_2(0)$ *and* $H^2 \to \{0\} \oplus H$*.*

Proof. Let $f \in H$, P be the orthoprojection $H \to R\left((L_2 - \lambda)_{op}\right) (= L_2(0)^\perp)$, and Q be the orthoprojection $H \to (L_2 - \lambda)(0) (= L_2(0))$. Then $f = Pf + Qf$. We obtain

$$L_\lambda f = L_\lambda P f + L_\lambda Q f = \left((L_2 - \lambda)_{op}\right)^{-1} P f.$$

But $(L_2 - \lambda)_{op} = L_{2,op} - \lambda$ (indeed,

for all $f \in D\left((L_2 - \lambda)_{op}\right) = D\left(L_{2,op} - \lambda\right) (= D(L_2))$ $\left(L_{2,op} - \lambda\right) f - (L_2 - \lambda)_{op} f \in L_2(0)^\perp$;

on the other hand, the inclusions $\left(f, \left(L_{2,op} - \lambda\right) f\right)$, $\left(f, (L_2 - \lambda)_{op} f\right) \in L_2 - \lambda$ imply $(L_2 - \lambda)_{op} f - \left(L_{2,op} - \lambda\right) f \in (L_2 - \lambda)(0) = L_2(0)$), therefore

$$\lambda L_\lambda f + f = \lambda \left(L_{2,op} - \lambda\right)^{-1} P f + P f + Q f.$$

Taking into account (3) with $S = L_{2,op}$ we see that $\lim\limits_{\lambda \to -\infty} (\lambda L_\lambda f + f) = Qf$, whence using (2) with $S = L_2$ we obtain $\lim\limits_{\lambda \to -\infty} \hat{L}_\lambda f = \begin{pmatrix} 0 \\ Qf \end{pmatrix}$, therefore $\lim\limits_{\lambda \to -\infty} Z^*_\lambda f = \lim\limits_{\lambda \to -\infty} \Gamma_1 \hat{L}_\lambda f =$ $= \Gamma_1 (0 \oplus Q) f$. The first of the equalities (17) has been proved. The second equality is a immediate consequence from the first one. □

3 MAIN RESULT

Definition 2 ([16]). *An operator-function* $M(\lambda) \overset{def}{=} \Gamma_1 \hat{Z}_\lambda$ $(\lambda \in \rho(L_2))$ *is called the Weyl function of the couple* (L, L_0) *corresponding to its boundary pair* $(G_1 \oplus G_2, \Gamma_1 \oplus \Gamma_2)$.

Lemma 4. *For any* λ, $\mu \in \rho(L_2)$, *the equality*

$$M(\lambda) - M(\mu) = (\lambda - \mu)\, \tilde{Z}^*_{\overline{\mu}} Z_\lambda \left(= (\lambda - \mu)\, \tilde{Z}^*_{\overline{\lambda}} Z_\mu \right)$$

is true.

Proof. In view of (10) we obtain

$$M(\lambda) - M(\mu) = \Gamma_1 \left(\hat{Z}_\lambda - \hat{Z}_\mu \right) = (\lambda - \mu)\, \Gamma_1 \hat{L}_\lambda Z_\mu = (\lambda - \mu)\, \tilde{Z}^*_{\overline{\mu}} Z_\lambda \left(= (\lambda - \mu)\, \tilde{Z}^*_{\overline{\lambda}} Z_\mu \right).$$

□

Consider some analytic properties of the operator-function $M(\lambda)$.

Lemma 5. $M(\lambda)$ *is analytic* $\mathcal{B}(G_1, G_2)$*-valued function on* $\rho(L_2)$. *Moreover, for any* $n \in \mathbb{N}$

$$M^{(n)}(\lambda) = n! \tilde{Z}^*_{\overline{\lambda}} L^{n-1}_\lambda Z_\lambda, \tag{18}$$

in particular $M'(\lambda) = \tilde{Z}^*_{\overline{\lambda}} Z_\lambda$.

Proof. Since L_λ is a $\mathcal{B}(H)$-valued analytic function on $\rho(L_2)$, we conclude that

$$\hat{Z}_\lambda = \begin{pmatrix} L_\lambda & 1_H + \lambda L_\lambda \\ \lambda L_\lambda & \lambda 1_H + \lambda^2 L_\lambda \end{pmatrix} \tilde{\Gamma}^*_1$$

is an analytic $\mathcal{B}\left(G_2, H^2\right)$-valued function. But by virtue of Lemma 1 $R(\hat{Z}_\lambda) \subset L$, sequently $\hat{Z}_\lambda$ is a $\mathcal{B}(G_2, L)$-valued analytic function. Moreover (see (10)) $\hat{Z}^{(n)}_\lambda = n! \hat{L}_\lambda L^{n-1}_\lambda Z_\lambda$, therefore

$$M^{(n)}(\lambda) = \Gamma_1 \hat{Z}^{(n)}_\lambda = n! \Gamma_1 \hat{L}_\lambda L^{n-1}_\lambda Z_\lambda.$$

□

Theorem 2. *Suppose that* $\lambda_0 \in \rho(L_2)$, $R(L_0 - \lambda_0) \dot{+} \ker(L - \lambda) = H$, *and* (15) *holds. Then*

i) $\left(\tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0} \right)^{-1} \in \mathcal{B}(G_1, G_2)$, $\quad (M(\lambda) - M(\lambda_0))^{-1} \in \mathcal{B}(G_1, G_2)$;

ii) λ_0 *is a first-order pole for the function* $(M(\lambda) - M(\lambda_0))^{-1}$ *and*

$$\operatorname{res}|_{\lambda=\lambda_0} (M(\lambda) - M(\lambda_0))^{-1} = \left(\tilde{Z}^*_{\lambda_0} Z_{\lambda_0} \right)^{-1}.$$

Proof. i) This statement is a direct consequence of Lemma 3, Remark 2 and Lemma 4.

ii) Put

$$\Pi(\lambda) = \begin{cases} (\lambda - \lambda_0)^{-1} (M(\lambda) - M(\lambda_0)), & \lambda \neq \lambda_0 \\ M'(\lambda_0) = \tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0}, & \lambda = \lambda_0 \end{cases}.$$

It is clear that $\lim\limits_{\lambda \to \lambda_0} \Pi(\lambda) = M'(\lambda_0) = \tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0}$ (with respect to uniform operator convergence). Hence, $\lim\limits_{\lambda \to \lambda_0} \left[(\lambda - \lambda_0)(M(\lambda) - M(\lambda_0)^{-1} \right] = \lim\limits_{\lambda \to \lambda_0} \Pi(\lambda)^{-1} = (\tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0})^{-1}$. The theorem is proved. □

Remark 3. *Theorem 2 yields that in some neighbourhood of the point $\lambda_0 \in \rho(L_2)$ such that $R\,(L_0 - \lambda_0) \dot{+} \ker\,(L - \lambda) = H$, the following expansion takes place:*

$$(M(\lambda) - M(\lambda_0))^{-1} = \frac{1}{\lambda - \lambda_0}\left(\tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0}\right)^{-1} + \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R^{(n)}, \qquad (19)$$

where $R^{(n)} \in \mathcal{B}(G_1, G_2)$, $n = 0, 1, 2, \ldots$ On the other hand, in view of (18) *we obtain*

$$M(\lambda) - M(\lambda_0) = \sum_{n=1}^{\infty} (\lambda - \lambda_0)^n \cdot \tilde{Z}^*_{\overline{\lambda_0}} L^{n-1}_{\lambda_0} Z_{\lambda_0}.$$

Multiplying both sides of two latter equalities we obtain the recurrent relations for the coefficients $R^{(n)}$ in (19)*:*

$$\sum_{m=0}^{n} \tilde{Z}^*_{\overline{\lambda_0}} L^m_{\lambda_0} Z_{\lambda_0} \cdot R^{(n-m-1)} = 0 \quad (n \in \mathbb{N}), \qquad R^{(-1)} = (\tilde{Z}^*_{\overline{\lambda_0}} Z_{\lambda_0})^{-1}.$$

In particular, $R^{(0)} = -R^{(-1)} \cdot \tilde{Z}^*_{\overline{\lambda_0}} L_{\lambda_0} Z_{\lambda_0} \cdot R^{(-1)}$.

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Сторож О.Г. *Деякі аналітичні властивості функції Вейля замкненого лінійного відношення* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 197–205.

Нехай L та L_0, де $L_0 \subset L$, — замкнені лінійні відношення (багатозначні оператори) у комплексному гільбертовому просторі H. У термінах абстрактних граничних операторів (тобто у вигляді, який у випадку диференціальних операторів приводить безпосередньо до граничних умов) досліджуються деякі аналітичні властивості функції Вейля $M(\lambda)$, яка відповідає деякій граничній парі (L, L_0).

Зокрема, застосовуючи резольвентну тотожність Гільберта для відношень, встановлено критерій оборотності у алгебрі обмежених лінійних операторів, діючих у H, для відображення $M(\lambda) - M(\lambda_0)$ у деякому достатньо малому проколеному околі точки λ_0. Доведено, що в цьому випадку λ_0 є полюсом першого порядку для оператор-функції $(M(\lambda) - M(\lambda_0))^{-1}$. Знайдено відповідні лишок та розвинення у ряд Лорана.

При деяких додаткових припущеннях досліджується поведінка при $\lambda \to -\infty$ так званого γ-поля Z_λ, яке являє собою оператор-функцію, тісно пов'язаною з $M(\lambda)$.

Ключові слова і фрази: гільбертів простір, відношення, оператор, розширення, полюс.