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FG-COUPLED FIXED POINT THEOREMS IN CONE METRIC SPACES

The concept of FG -coupled fixed point introduced recently is a generalization of coupled fixed point introduced by Guo and Lakshmikantham. A point $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$, where X is a non empty set. In this paper, we introduce FG -coupled fixed point in cone metric spaces for the mappings $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ and establish some FG -coupled fixed point theorems for various mappings such as contraction type mappings, Kannan type mappings and Chatterjea type mappings. All the theorems assure the uniqueness of FG -coupled fixed point. Our results generalize several results in literature, mainly the coupled fixed point theorems established by Sabetghadam et al. for various contraction type mappings. An example is provided to substantiate the main theorem.

Key words and phrases: FG -coupled fixed point, cone metric space, contraction type mappings.

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1 INTRODUCTION

The classical Banach contraction theorem is proved to be one of the most fruitful and durable results in metric fixed point theory. Due to its enormous applications, several authors have studied and made very many generalizations of Banach contraction principle. In 2004 A.C.M. Ran and M.C.B. Reurings [1] proved an analogue of Banach contraction principle in partially ordered metric spaces and used the theorem to solve matrix equations. Following this, J.J. Nieto and R.R. Lopez [5, 6] established several fixed point theorems in partially ordered metric spaces and obtained applications to periodic boundary value problems. As an extension of fixed point, a new concept called coupled fixed point is introduced by D. Guo and V. Lakshmikantham [2]. They investigated some coupled fixed point theorems of mixed monotone operator, and applied their results to solve initial value problem of ordinary differential equations with discontinuous right hand sides. Using the notion of coupled fixed points they explored the existence and uniqueness of fixed point of non-monotone operator. Later T.G. Bhaskar and V. Lakshmikantham [13] established existence and uniqueness theorems of coupled fixed point for mixed monotone mappings defined on partially ordered complete metric spaces satisfying contraction type condition and applied their result to solve periodic boundary value problems. After the work of Gnana Bhaskar and Lakshmikantham, in 2009 V. Lakshmikantham and L. Ćirić [14] introduced a new mapping called mixed g -monotone mapping. Using this, they proved coupled coincidence and coupled common fixed point theorems

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which generalize the results of Gnana Bhaskar and Lakshmikantham. In 2007 L.G. Huang and X. Zhang [8] introduced a metric called cone metric by replacing the real line by a real Banach space equipped with a partial ordering with respect to the cone. They proved some fixed point theorems for contraction mappings defined on cone metric spaces. Following them several authors have proved various fixed point theorems in cone metric spaces [10–12]. Later in 2009 F. Sabetghadam et al. [4] introduced the concept of coupled fixed point in cone metric spaces, and proved several coupled fixed point theorems for different contraction type mappings. In 2011 M.O. Olatinwo [9] proved coupled fixed point theorems by considering two different cone metrics on the same ambient space. Recently E. Prajisha and P. Shaini [3] introduced a concept called FG -coupled fixed point in partially ordered metric spaces which is a generalization of coupled fixed point. They established some FG -coupled fixed point theorems, in which F and G satisfy different contraction type conditions. Subsequently, K. Deepa and P. Shaini [7] proved several FG -coupled fixed point theorems for various contractive and generalized quasi-contractive mappings.

In this paper we define FG -coupled fixed point in cone metric spaces and prove FG -coupled fixed point theorems for different contraction type mappings on complete cone metric spaces. Let us give some useful definitions.

Definition 1. A cone P is a subset of real Banach space E such that:

- (i) P is closed, nonempty and $P \neq \{0\}$;
- (ii) if a, b are non-negative real numbers and $x, y \in P$, then $ax + by \in P$;
- (iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, the partial ordering \leq with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. The notation $x \ll y$ stands for $y - x \in \text{int}P$ where $\text{int}P$ denotes the interior of P . Also we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies that $\|x\| \leq M \|y\|$. The least positive number satisfying the above is called the normal constant of P . The cone P is called regular if every increasing (decreasing) sequence which is bounded above (below) is convergent. It is known that every regular cone is normal.

Definition 2 ([8]). Let X be a non empty set and let E be a real Banach space equipped with the partial ordering \leq with respect to the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies the following conditions:

- (i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and the pair (X, d) is called a cone metric space.

Definition 3 ([8]). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X . Then

- (i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$;

- (ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence is convergent.

Definition 4 ([4]). Let (X, d) be a cone metric space and $F : X \times X \rightarrow X$ be a mapping. An element $(x, y) \in X \times X$ is said to be coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 5 ([3]). Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two mappings, then for $n \geq 1$, $F^n(x, y) = F(F^{n-1}(x, y), G^{n-1}(y, x))$ and $G^n(y, x) = G(G^{n-1}(y, x), F^{n-1}(x, y))$ where $F^0(x, y) = x$ and $G^0(y, x) = y$ for all $x \in X$ and $y \in Y$.

In the next section we define FG -coupled fixed point on cone metric spaces and prove existence and uniqueness theorems of FG -coupled fixed point for different contraction type mappings. We consider $d_X : X \times X \rightarrow E$ and $d_Y : Y \times Y \rightarrow E$, where E is a real Banach space equipped with the partial ordering \leq with respect to the cone $P \subseteq E$ with $intP \neq \emptyset$.

2 MAIN RESULTS

Three main theorems on FG -coupled fixed point are investigated in this section. We define FG -coupled fixed point in cone metric spaces as follows:

Definition 6. Let (X, d_X) and (Y, d_Y) are cone metric spaces and $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ are two mappings. An element $(x, y) \in X \times Y$ is said to be an FG -coupled fixed point if $F(x, y) = x$ and $G(y, x) = y$.

Theorem 1. Let (X, d_X) and (Y, d_Y) be two complete cone metric spaces. Suppose that the mappings $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ satisfy the following conditions for all $x, u \in X, y, v \in Y$:

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, u) + l d_Y(y, v), \tag{1}$$

$$d_Y(G(y, x), G(v, u)) \leq k d_Y(y, v) + l d_X(x, u), \tag{2}$$

where k, l are non negative constants with $k + l < 1$. Then there exist a unique FG -coupled fixed point.

Proof. Take $x_0 \in X$ and $y_0 \in Y$. Construct sequences $\{x_n\}$ and $\{y_n\}$ by defining $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$ and $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \geq 0$.

We have,

$$d_X(x_{n+1}, x_n) = d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq k d_X(x_n, x_{n-1}) + l d_Y(y_n, y_{n-1}),$$

and

$$\begin{aligned} d_Y(y_{n+1}, y_n) &= d_Y(G(y_n, x_n), G(y_{n-1}, x_{n-1})) \\ &\leq k d_Y(y_n, y_{n-1}) + l d_X(x_n, x_{n-1}). \end{aligned}$$

By adding the above inequalities we get $d_n \leq (k + l) d_{n-1}$, where

$$d_n = d_X(x_{n+1}, x_n) + d_Y(y_{n+1}, y_n).$$

Continuing this process we get $d_n \leq \theta d_{n-1} \leq \theta^2 d_{n-2} \cdots \leq \theta^n d_0$, where $\theta = k + l < 1$. If $d_0 = 0$ then (x_0, y_0) is an FG-coupled fixed point. If $d_0 \neq 0$, then we have $d_0 > 0$.

We have for $m > n$, $d_X(x_n, x_m) \leq d_X(x_n, x_{n+1}) + d_X(x_{n+1}, x_{n+2}) + \cdots + d_X(x_{m-1}, x_m)$ and $d_Y(y_n, y_m) \leq d_Y(y_n, y_{n+1}) + d_Y(y_{n+1}, y_{n+2}) + \cdots + d_Y(y_{m-1}, y_m)$

$$\begin{aligned} \text{ie, } d_X(x_n, x_m) + d_Y(y_n, y_m) &\leq d_n + d_{n+1} + \cdots + d_{m-1} \\ &\leq \theta^n d_0 + \theta^{n+1} d_0 + \cdots + \theta^{m-1} d_0 \leq \frac{\theta^n}{1-\theta} d_0 \end{aligned}$$

Now, for $0 \ll c$ there exist $r > 0$ such that $y \ll c$ for $\|y\| < r$. Choose a positive integer N_c such that for all $n \geq N_c$, $\|\frac{\theta^n}{1-\theta} d_0\| < r$, which implies $\frac{\theta^n}{1-\theta} d_0 \ll c$, for $n \geq N_c$.

Thus $d_X(x_n, x_m) + d_Y(y_n, y_m) \ll c$, for $m > n \geq N_c$. Since $d_X(x_n, x_m) \leq d_X(x_n, x_m) + d_Y(y_n, y_m)$ and $d_Y(y_n, y_m) \leq d_X(x_n, x_m) + d_Y(y_n, y_m)$, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. By the completeness of X and Y there exist $(x, y) \in X \times Y$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. ie, for all $0 \ll c$ there exist N' such that $d_X(x_n, x) \ll \frac{c}{2}$ for all $n \geq N'$ and there exist N'' such that $d_Y(y_n, y) \ll \frac{c}{2}$ for all $n \geq N''$. Take $N = \max\{N', N''\}$. We have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), x_{N+1}) + d_X(x_{N+1}, x) = d_X(F(x, y), F(x_N, y_N)) + d_X(x_{N+1}, x) \\ &\leq k d_X(x, x_N) + l d_Y(y, y_N) + d_X(x_{N+1}, x) \ll k \frac{c}{2} + l \frac{c}{2} + \frac{c}{2} < c \end{aligned}$$

Thus $F(x, y) = x$. Similarly we get $G(y, x) = y$.

Now we prove the uniqueness of FG-coupled fixed point. Let $(x, y) \neq (x', y') \in X \times Y$ such that $F(x', y') = x'$ and $G(y', x') = y'$. Then we have,

$$\begin{aligned} d_X(x, x') &= d_X(F(x, y), F(x', y')) \leq k d_X(x, x') + l d_Y(y, y') \text{ and} \\ d_Y(y, y') &= d_Y(G(y, x), G(y', x')) \leq k d_Y(y, y') + l d_X(x, x') \\ \text{ie, } d_X(x, x') + d_Y(y, y') &\leq (k + l) [d_X(x, x') + d_Y(y, y')] < d_X(x, x') + d_Y(y, y'). \end{aligned}$$

This is not possible. So $x = x'$ and $y = y'$. Hence the proof. □

Example 1. Let $X = [0, \infty)$ and $Y = (-\infty, 0]$. Let $E = C_{\mathbb{R}}^1$ with $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$ and $P = \{x \in E : x(t) \geq 0, t \in [0, 1]\}$. Define cone metric $d : X \times X \rightarrow E$ by $d(x, y) = |x - y|\varphi$ where $\varphi : [0, 1] \rightarrow \mathbb{R}$ such that $\varphi(t) = e^t$: see [15] Consider the mappings $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ defined as $F(x, y) = \frac{x - 4y}{6}$ and $G(y, x) = \frac{y - 4x}{6}$. Clearly F and G satisfy all the conditions given in Theorem 1, and it is easy to see that $(0, 0)$ is a unique FG-coupled fixed point.

Corollary 1 ([4, Theorem 2.2]). Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:

$$d(F(x, y), F(u, v)) \leq k d(x, u) + l d(y, v), \tag{3}$$

where k and l are non negative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Corollary 2 ([4, Corollary 2.3]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \tag{4}$$

where $k \in [0, 1)$. Then F has a unique coupled fixed point.

Theorem 2. *Let (X, d_X) and (Y, d_Y) be two complete cone metric spaces. Suppose that the mappings $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ satisfy the following condition for all $x, u \in X, y, v \in Y$:*

$$d_X(F(x, y), F(u, v)) \leq k d_X(F(x, y), x) + l d_X(F(u, v), u), \tag{5}$$

$$d_Y(G(y, x), G(v, u)) \leq k d_Y(G(y, x), y) + l d_Y(G(v, u), v), \tag{6}$$

where k, l are non negative constants with $k + l < 1$. Then there exist a unique FG-coupled fixed point.

Proof. As in the proof of previous theorem construct sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$, $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \geq 0$. Then we have

$$\begin{aligned} d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq k d_X(F(x_n, y_n), x_n) + l d_X(F(x_{n-1}, y_{n-1}), x_{n-1}) \\ &= k d_X(x_{n+1}, x_n) + l d_X(x_n, x_{n-1}). \end{aligned}$$

Therefore $d_X(x_{n+1}, x_n) \leq \frac{l}{1-k} d_X(x_n, x_{n-1})$. Similarly $d_Y(y_{n+1}, y_n) \leq \frac{l}{1-k} d_Y(y_n, y_{n-1})$. Repeating this process we get, $d_X(x_{n+1}, x_n) \leq \delta^n d_X(x_1, x_0)$ and $d_Y(y_{n+1}, y_n) \leq \delta^n d_Y(y_1, y_0)$, where $\delta = \frac{l}{1-k}$.

If $x_1 = x_0$ and $y_1 = y_0$, then the result follows. Otherwise for $m > n$ consider,

$$\begin{aligned} d_X(x_n, x_m) &\leq d_X(x_n, x_{n+1}) + d_X(x_{n+1}, x_{n+2}) + \dots + d_X(x_{m-1}, x_m) \\ &\leq \delta^n d_X(x_1, x_0) + \delta^{n+1} d_X(x_1, x_0) + \dots + \delta^{m-1} d_X(x_1, x_0) \\ &\leq \frac{\delta^n}{1-\delta} d_X(x_1, x_0). \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X . Similarly we can prove that $\{y_n\}$ is a Cauchy sequence in Y . Now by the completeness of the spaces X and Y , there exist $(x, y) \in X \times Y$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. Hence for all $0 \ll c$ there exist N' such that $d_X(x_n, x) \ll \frac{(1-k)c}{3}$ for all $n \geq N'$ and there exist N'' such that $d_Y(y_n, y) \ll \frac{(1-k)c}{3}$ for all $n \geq N''$.

Therefor we have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), x_{N'+1}) + d_X(x_{N'+1}, x) = d_X(F(x, y), F(x_{N'}, y_{N'})) + d_X(x_{N'+1}, x) \\ &\leq k d_X(F(x, y), x) + l d_X(F(x_{N'}, y_{N'}), x_{N'}) + d_X(x_{N'+1}, x) \\ &\leq k d_X(F(x, y), x) + l [d_X(x_{N'+1}, x) + d_X(x, x_{N'})] + d_X(x_{N'+1}, x). \end{aligned}$$

Thus

$$d_X(F(x, y), x) \leq \frac{l+1}{1-k} d_X(x_{N'+1}, x) + \frac{l}{1-k} d_X(x_{N'}, x) \ll \frac{(l+1)c}{3} + \frac{lc}{3} < c.$$

Hence we have $F(x, y) = x$. Similarly, $G(y, x) = y$. If (x', y') is another FG -coupled fixed point, then we have

$$\begin{aligned} d_X(x, x') &= d_X(F(x, y), F(x', y')) \leq k d_X(F(x, y), x) + l d_X(F(x', y'), x') \\ &= k d_X(x, x) + l d_X(x', x') = 0. \end{aligned}$$

Thus $x = x'$. Similarly $y = y'$. Hence the proof. \square

Corollary 3 ([4, Theorem 2.5]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq k d(F(x, y), x) + l d(F(u, v), u), \quad (7)$$

where k and l are non negative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Corollary 4 ([4, Corollary 2.7]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(F(x, y), x) + d(F(u, v), u)], \quad (8)$$

where $k \in [0, 1)$. Then F has a unique coupled fixed point.

Theorem 3. *Let (X, d_X) and (Y, d_Y) be two complete cone metric spaces. Suppose that the mappings $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ satisfy the following conditions for all $x, u \in X, y, v \in Y$:*

$$d_X(F(x, y), F(u, v)) \leq k d_X(F(x, y), u) + l d_X(F(u, v), x) \quad (9)$$

$$d_Y(G(y, x), G(v, u)) \leq k d_Y(G(y, x), v) + l d_Y(G(v, u), y), \quad (10)$$

where $k, l \in [0, \frac{1}{2})$. Then there exist a unique FG -coupled fixed point.

Proof. By defining $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = G(y_n, x_n)$ as in the above theorems, we construct sequences $\{x_n\}$ and $\{y_n\}$. Now we have,

$$\begin{aligned} d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq k d_X(F(x_n, y_n), x_{n-1}) + l d_X(F(x_{n-1}, y_{n-1}), x_n) \\ &= k d_X(x_{n+1}, x_{n-1}) + l d_X(x_n, x_n) \leq k [d_X(x_{n+1}, x_n) + d_X(x_n, x_{n-1})]. \end{aligned}$$

Thus $d_X(x_{n+1}, x_n) \leq \frac{k}{1-k} d_X(x_n, x_{n-1})$. Similarly $d_Y(y_{n+1}, y_n) \leq \frac{k}{1-k} d_Y(y_n, y_{n-1})$. Repeating this way we get

$$d_X(x_{n+1}, x_n) \leq \delta^n d_X(x_1, x_0)$$

and

$$d_Y(y_{n+1}, y_n) \leq \delta^n d_Y(y_1, y_0), \text{ where } \delta = \frac{k}{1-k}.$$

In the similar lines of Theorem 2 see that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Since (X, d_X) and (Y, d_Y) are complete, there exist $(x, y) \in X \times Y$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. Hence for all $0 \ll c$ there exist N' such that $d_X(x_n, x) \ll \frac{(1-k)c}{2}$ for all $n \geq N'$ and there exist N'' such that $d_Y(y_n, y) \ll \frac{(1-k)c}{2}$ for all $n \geq N''$.

Thus we have

$$\begin{aligned}
 d_X(F(x, y), x) &\leq d_X(F(x, y), x_{N'+1}) + d_X(x_{N'+1}, x) \\
 &= d_X(F(x, y), F(x_{N'}, y_{N'})) + d_X(x_{N'+1}, x) \\
 &\leq k d_X(F(x, y), x_{N'}) + l d_X(F(x_{N'}, y_{N'}), x) + d_X(x_{N'+1}, x) \\
 &\leq k [d_X(F(x, y), x) + d_X(x, x_{N'})] + l d_X(x_{N'+1}, x) + d_X(x_{N'+1}, x) \\
 \text{ie, } d_X(F(x, y), x) &\leq \frac{k}{1-k} d_X(x_{N'}, x) + \frac{l+1}{1-k} d_X(x_{N'+1}, x) \ll \frac{k c}{2} + \frac{(l+1) c}{2} < c.
 \end{aligned}$$

Hence we get $F(x, y) = x$. Similarly $G(y, x) = y$. If $(x, y) \neq (x', y')$ is another FG-coupled fixed point, then we have

$$\begin{aligned}
 d_X(x, x') &= d_X(F(x, y), F(x', y')) \leq k d_X(F(x, y), x') + l d_X(F(x', y'), x) \\
 &= k d_X(x, x') + l d_X(x', x) = (k + l) d_X(x', x) < d_X(x', x).
 \end{aligned}$$

This is not possible. Thus $x = x'$. Similarly $y = y'$. Hence the proof. □

Corollary 5 ([4, Theorem 2.6]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq k d(F(x, y), u) + l d(F(u, v), x), \tag{11}$$

where k and l are non negative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Corollary 6 ([4, Corollary 2.8]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(F(x, y), u) + d(F(u, v), x)], \tag{12}$$

where $k \in [0, 1)$. Then F has a unique coupled fixed point.

Remark 1. *If $F = G$ and $X = Y$ in Theorems 1, 2 and 3, then we get Corollaries 1, 3 and 5 respectively. In addition to this, if k and l are equal in Theorems 1, 2 and 3 then we get the corollaries 2, 4 and 6 respectively.*

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Праджіша Е., Шайні П. *Теореми про FG-спарену фіксовану точку в конічних метричних просторах* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 163–170.

Концепція *FG*-спареної фіксованої точки, яка введена недавно, є узагальненням спареної фіксованої точки, що введена Гуо і Лакшмікантамом. Точка $(x, y) \in X \times X$ називається спареною фіксованою точкою відображення $F : X \times X \rightarrow X$ якщо $F(x, y) = x$ і $F(y, x) = y$, де X непорожня множина. У цій статті ми вводимо *FG*-спарену фіксовану точку у конічних метричних просторах для відображень $F : X \times Y \rightarrow X$ і $G : Y \times X \rightarrow Y$ та встановлюємо деякі теореми про *FG*-спарену фіксовану точку для різних відображень, як от відображення стискуючого типу, відображення типу Канана та Чатержі. Усі ці теореми стосуються єдиності *FG*-спареної фіксованої точки. Наші результати узагальнюють кілька результатів, в основному результати Сабетхадана та ін., про теореми про спарену фіксовану точку для різних типів стискуючих відображень. Також наведено приклад для того, щоб проілюструвати основну теорему.

Ключові слова і фрази: *FG*-спарена фіксована точка, конічний метричний простір, відображення стискуючого типу.