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THE NONLOCAL PROBLEM FOR THE DIFFERENTIAL-OPERATOR EQUATION OF THE EVEN ORDER WITH THE INVOLUTION

In this paper, the problem with boundary non-self-adjoint conditions for differential-operator equations of the order $2n$ with involution is studied. Spectral properties of operator of the problem is investigated.

By analogy of separation of variables the nonlocal problem for the differential-operator equation of the even order is reduced to a sequence $\{L_k\}_{k=1}^{\infty}$ of operators of boundary value problems for ordinary differential equations of even order. It is established that each element L_k of this sequence is an isospectral perturbation of the self-adjoint operator $L_{0,k}$ of the boundary value problem for some linear differential equation of order $2n$.

We construct a commutative group of transformation operators whose elements reflect the system $V(L_{0,k})$ of the eigenfunctions of the operator $L_{0,k}$ in the system $V(L_k)$ of the eigenfunctions of the operators L_k. The eigenfunctions of the operator L of the boundary value problem for a differential equation with involution are obtained as the result of the action of some specially constructed operator on eigenfunctions of the sequence of operators $L_{0,k}$.

The conditions under which the system of eigenfunctions of the operator L of the studied problem is a Riesz basis is established.

Key words and phrases: operator of involution, differential-operator equation, eigenfunctions, Riesz basis.

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INTRODUCTION

The boundary value problems for linear differential-operator equations are used in the simulation of boundary value problems for differential equations with partial derivatives, in particular, in the study of nonlocal problems. Significant results concerning the theory of boundary value problems for differential-operator equations were obtained in the papers of Vishik M.I., Boehner M., Gorbachuk V.I., Gorbachuk M.L., Dezin O.O., Dubinsky Yu.V., Kochubei A.N., Lions J.L., Mamedov K.S., Romanko V.K., Shakhmurov Veli B., Triebel Kh., Yakubov S., Yurchuk N.Yu.

During recent years, the number of publications with the use of an involution operator in various sections of the theory of ordinary differential equations (see [4, 10, 12, 15, 16, 19]), of partial differential equations (see [3, 7, 9, 14, 16, 17, 20, 21]), of linear operators, T-invariant with respect to some group of homeomorphisms (see [8]), differential equations with operator coefficients (see [5–7]), PT-symmetric operators (see [1, 2]) increased significantly.

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1 STATEMENT OF PROBLEM

Let us make some notations. H is a separable Hilbert space; $A : D(A) \subset H \to H$ is the closed unbounded linear operator with the discrete spectrum $\sigma(A) \equiv \{z_k = \alpha k^\gamma, \alpha, \gamma > 0, k = 1,2,\dots\}$; $V(A) \equiv \{v_k \in H : k = 1,2,\dots\}$ is the system of the eigenfunctions which forms the Riesz basis in the space H; $H(A^s) \equiv \{h \in H : A^s h \in H\}$; $s \geq 0$; $W_1 \equiv L_2((0,1),H)$; $D_x : W_1 \to W_1$ is a strong derivative in the space W_1; $W_2 \equiv \{u \in W_1 : D_x^{2n} u \in W_1, A^{2n} u \in W_1\}$; $[H]$ is the algebra of the bounded linear operators $B : H \to H$; I is the operator of the involution in the space $L_2(0,1)$; $Iy(x) \equiv y(1-x)$; $p_j \equiv \frac{1}{2}(E + (-1)^j I)$ are the orthoprojectors of the space $L_2(0,1)$; $L_{2,j}(0,1) \equiv \{y \in L_2(0,1) : p_j y \equiv y\}$; $j = 0,1$; $W_2^{2n}(0,1) \equiv \left\{y \in L_2(0,1) : y^{(m)} \in C[0,1], m = 0,1,\dots,2n-1, y^{(2n)} \in L_2(0,1)\right\}$; $W^*(0,1)$ is the space of continuous linear functionals over the space $W_2^{2n}(0,1)$; $W_j^*(0,1) \equiv \{l \in W^*(0,1) : ly = 0, y \in L_{2,1-j}(0,1) \bigcap W_2^{2n}(0,1)\}$; $j = 0,1$.

We consider the following boundary problem

$$Lw \equiv (-1)^n D_x^{2n} w(x) + A^{2n} w(x) + \sum_{j=1}^{n} a_j \left(D_x^{2j-1} w(x) - D_x^{2j-1} w(1-x)\right) = f(x), \quad x \in (0,1), \tag{1}$$

$$\ell_j w \equiv D_x^{m_j} w(0) + (-1)^{m_j} D_x^{m_j} w(1) = \varphi_j, \quad j = 1,2,\dots,n, \tag{2}$$

$$\ell_{n+j} w \equiv D_x^{m_{n+j}} w(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}} w(1) + l_j^1 w = \varphi_{n+j}, \quad j = 1,2,\dots,n, \tag{3}$$

$$\ell_j^1 w \equiv \sum_{r=0}^{k_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)). \tag{4}$$

By solution of the problem (1)–(4) we mean a function that satisfies equalities

$$\|Lw - f; W_1\| = 0, \ \|l_j w - \varphi_j; H\left(A^{\beta_j}\right)\| = 0,$$
$$\beta_j = 2n - m_j - \frac{1}{2}, \ \beta_{n+j} = 2n - \max(m_{n+j}, k_j) - \frac{1}{2},$$
$$a_j, b_{j,r,s} \in \mathbb{R}, \ r = 0,1,\dots,k_j, \ s = 0,1, \ j = 1,2,\dots,n,$$
$$m_n < m_{n-1} < \dots < m_1, \ m_{2n} < m_{2n-1} < \dots < m_{n+1}.$$

2 AUXILIARY BOUNDARY VALUE PROBLEM

Consider the partial case of the problem (1)–(4), when $a_j = 0$, $b_{j,r,s} = 0$, $r = 0,1,\dots,k_j$, $s = 0,1$, $j = 1,2,\dots,n$,

$$(-1)^n D_x^{2n} u(x) + A^{2n} u(x) = f(x), \quad x \in (0,1), \tag{5}$$

$$\ell_{0,j} u \equiv D_x^{m_j} u(0) + (-1)^{m_j} D_x^{m_j} u(1) = 0, \tag{6}$$

$$\ell_{0,n+j} u \equiv D_x^{m_{n+j}} u(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}} u(1) = 0, \quad j = 1,2,\dots,n. \tag{7}$$

Remark 2.1. *The boundary conditions* (6)*,* (7) *are numbered so that the following conditions are satisfied*

$$l_j \in W_0^*(0,1), \quad l_{n+j} \in W_1^*(0,1), \ j = 1,2,\dots,n. \tag{8}$$

Let L_0 be the operator of the problem (5)–(7), $L_0 u \equiv (-1)^n \; D_x^{2n} u + \; A^{2n} u$, $u \in D(L_0)$, $D(L_0) \equiv \{u \in W_2 : l_j u = 0, j = 1, 2, \dots, 2n\}$. Consider the spectral problem for the operator L_0

$$(-1)^n \; D_x^{2n} u(x) + \; A^{2n} u(x) = \lambda u(x), \;\; l_j u = 0, \; \lambda \in \mathbb{C}, \; j = 1, 2, \dots, 2n. \tag{9}$$

The solution of the spectral problem (9) is defined as the product $u(x) = y(x) v_k$, $v_k \in V(A)$, $k = 1, 2, \dots$. To determine the unknown function $y \in W_2^{2n}(0,1)$, we obtain the spectral problem

$$(-1)^n \; y^{(2n)}(x) + \; z_k^{2n} y(x) = \lambda y(x), \;\; \lambda \in \mathbb{C}, \tag{10}$$

$$\ell_{0,j} y \equiv \; y^{(m_j)}(0) + \; (-1)^{m_j} y^{(m_j)}(1) = 0, \quad j = 1, 2, \dots, n, \tag{11}$$

$$\ell_{0,n+j} y \equiv \; y^{(m_{n+j})} y(0) - (-1)^{m_{n+j}} y^{(m_{n+j})}(1) = 0, \;\; j = 1, 2, \dots, n. \tag{12}$$

Let $L_{0,k}$ be the operator of the problem (10)–(12), $L_{0,k} y \equiv (-1)^n \; y^{(2n)}(x) + \; z_k^{2n} y(x)$; $y \in D(L_{0,k})$; $D(L_{0,k}) \equiv \{y \in W_2^{2n}(0,1) : l_{0,j} y = 0, j = 1, 2, \dots, 2n\}$.

Assumption B_1**.** The conditions (11), (12) are self-adjoint.

Assumption B_2**.** The boundary conditions (10), (11) are strongly regular according to Birkhoff (see [18]).

In what follows we assume that the assumptions B_1–B_2 are satisfied. The roots ρ_j of the characteristic equation $(-1)^n \rho^{2n} = \lambda - z_k^{2n}$, which corresponds to the differential equation

$$(-1)^n \; y^{(2n)}(x) + \; z_k^{2n} y(x) = \lambda y(x), \tag{13}$$

are determined by the relations $\rho_j = \omega_j \rho$, $\omega_1 = i$, $\omega_j = i \exp i \frac{\pi(j-1)}{2n}$, $j = 2, 3, \dots, n$.

The fundamental system of the solutions of the differential equation (13) is defined by the formulas

$$y_j(x, \rho) \equiv \frac{1}{2} (\exp \omega_j \rho x + \exp \omega_j \rho (1 - x)), \tag{14}$$

$$y_{n+j}(x, \rho) \equiv \frac{1}{2} (\exp \omega_j \rho x - \exp \omega_j \rho (1 - x)), \;\; j = 1, 2, \dots, n. \tag{15}$$

Substituting the general solution of the differential equation (13) $y(x, \rho) = \sum\limits_{s=1}^{2n} C_s y_s(x, \rho)$ into the boundary conditions (11), (12) we obtain an equation for determining the eigenvalues of the operator $L_{0,k}$

$$\Delta(\rho) = \det(l_r y_j)_{r,j=1}^{2n} = 0. \tag{16}$$

From the conditions (8) and from the properties of the functions (14), (15), we obtain

$$l_{0,r} y_{n+j} = 0, \;\; l_{0,n+r} y_j = 0, \;\; j, r = 1, 2, \dots, n, \tag{17}$$

therefore,

$$\Delta(\rho) = \Delta_0(\rho) \Delta_1(\rho) = 0, \tag{18}$$

where $\Delta_s(\rho) = \det(l_{sn+r} y_{sn+j})_{r,j=1}^{n}$, $s = 0, 1$.

The operator $L_{0,k}$ is self-adjoint, therefore the roots of the equation (18) lie on the semiaxis $Imz = 0$, $Rez \geq 0$. For any $s \in 0, 1$, we number the roots $\rho_{s,q}$ of the equation in ascending order $\rho_{s,1} < \rho_{s,2} < \dots$.

Thus, the operator $L_{0,k}$ has the eigenvalues

$$\lambda_{s,q,k} = (\rho_{s,q})^{2n} + z_k^{2n}, \;\; s \in 0, 1, \;\; q = 1, 2, \dots. \tag{19}$$

Let $\beta_0 = m_1 + m_2 + \cdots + m_n,\ \ \beta_1 = m_{n+1} + m_{n+2} + \cdots + m_{2n}$. We define the eigenfunctions of the operator $L_{0,k}$, which are normalized in the space. Let $B(s, x, \rho)$ be a square matrix of the order n, the first row of which is determined by the functions $y_{sn+j}(x, \rho)$, and the r-th row is determined by the numbers $l_{sn+r}y_{sn+j},\ \ r = 2, 3, \ldots, n,\ \ s = 0, 1, j = 1, 2, \ldots, n$. Let

$$v_{s,q}(x, L_{0,k}) = (\rho_{s,q})^{-\beta_s}\theta_{s,q} \det B(s, x, \rho_{s,q}). \tag{20}$$

Then $\|v_{s,q}(x, L_{0,k}); L_2(0, 1)\| = 1,\ \ s = 0, 1,\ \ q = 1, 2, \ldots$.

Lemma 2.1. *Suppose that the assumptions B_1–B_2 hold. Then for each number $k \in \mathbb{N}$ the operator $L_{0,k}$ has the eigenvalues* (19)*, and it also has the system of the eigenfunctions* (20)*, which forms the orthogonal basis in the space $L_2(0, 1)$.*

Therefore, the operator L_0 has a system

$$V(L_0) \equiv \left\{v_{s,q,k}(x, L_0) \in W_1 : v_{s,q,k}(x, L_0) = v_{s,q}(x, L_{0,k})v_k,\ \ s = 0, 1,\ \ k, q = 1, 2, \ldots\right\}$$

of the eigenfunctions in the space W_1. The product of a system $V(A)$ and an orthonormal system $V(L_{0,k})$ is the Riesz basis in the space W_1. Thus, the following theorem is true.

Theorem 1. *Let the assumptions B_1–B_2 hold. The operator L_0 has the discrete spectrum*

$$\sigma(L_0) = \left\{\lambda_{s,q,k} = (\rho_{s,q})^{2n} + z_k^{2n},\ \ s = 0, 1,\ \ k, q = 1, 2, \ldots\right\}.$$

It also has the system of the eigenfunctions $V(L_0)$ which forms the Riesz basis in the space W_1.

We choose an arbitrary eigenvalue $\lambda_{0,q,k} \in \sigma(L_{0,k}),\ q \in \mathbb{N}$. Let

$$y_{n+j}(x, \rho_{0,q}) \equiv \frac{1}{2}(\exp \omega_j\rho_{0,q}x - \exp \omega_j\rho_{0,q}(1 - x)), j = 1, 2, \ldots, n,$$

$B_p(x, \rho_{0,q})$ is square matrix of the order n, the p-th row of which is defined by the functions $y_{n+j}(x, \rho_{0,q})$, and the r-th row is defined by the numbers $(\omega_j)^{m_{n+r+1}}(1 + (-1)^{m_{n+r+1}} \exp \omega_j\rho_{0,q})$, $r \neq p,\ \ j, r = 1, 2, \ldots, n,$

$$\begin{aligned} &y_{0,n+p}(x, \rho_{0,q}) \equiv \det B_p(x, \rho_{0,q}),\\ &\Delta_1(\rho_{0,q}) \equiv \det((\omega_j)^{m_{n+r+1}}(1 + (-1)^{m_{n+r+1}} \exp \omega_j\rho_{0,q}))_{r,j=1}^{n},\\ &y_{1,n+p}(x, \rho_{0,q}) \equiv (\Delta_1(\rho_{0,q}))^{-1}y_{0,n+p}(x, \rho_{0,q}). \end{aligned} \tag{21}$$

Substituting the expression (21) into the boundary conditions (11), (12), we see that

$$l_jy_{1,n+p}(x, \rho_{0,q}) = 0,\ \ j \neq n + p, j = 1, 2, \ldots, 2n, \tag{22}$$

$$l_{n+p}y_{1,n+p}(x, \rho_{0,q}) = (\rho_{0,q})^{m_{n+p}}. \tag{23}$$

3 NON SELF-ADJOINT BOUNDARY VALUE PROBLEM

For the differential-operator equation (5) we consider the following boundary value problem for arbitrary fixed $p \in \{1, 2, \dots, n\}$, $b \in \mathbb{R}$,

$$\ell_{1,j} u \equiv D_x^{m_j} u(0) + (-1)^{m_j} D_x^{m_j} u(1) = 0, \quad j = 1, 2, \dots, n, \tag{24}$$

$$\ell_{1,n+j} u \equiv D_x^{m_{n+j}} u(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}} u(1) = 0, \quad j = 1, 2, \dots, n, \ j \neq p, \tag{25}$$

$$\ell_{1,n+p} u \equiv D_x^{m_{n+p}} u(0) - (-1)^{m_{n+p}} D_x^{m_{n+p}} u(1) + l_p^2 u = 0, \tag{26}$$

$$\ell_p^2 u \equiv b(D_x^{m_{n+p}} u(0) + (-1)^{m_{n+p}} D_x^{m_{n+p}} u(1)) = 0. \tag{27}$$

Let L_1 be the operator of the problem (5), (24)–(27), $L_1 u \equiv (-1)^n D_x^{2n} u(x) + A^{2n} u(x)$, $u \in D(L_1)$, $D(L_1) \equiv \{u \in W_2 : l_{1,j} u = 0, \ j = 1, 2, \dots, 2n\}$. The solution of the spectral problem (9), (24)–(27) is defined as the product $u(x) = y(x) v_k, \ v_k \in V(A), \ k = 1, 2, \dots$. To determine the unknown function $y \in W_2^{2n}(0, 1)$, we obtain the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{28}$$

$$\ell_{1,j} y \equiv y^{(m_j)}(0) + (-1)^{m_j} y^{(m_j)}(1) = 0, \quad j = 1, 2, \dots, n, \tag{29}$$

$$\ell_{1,n+j} y \equiv y^{(m_{n+j})}(0) - (-1)^{m_{n+j}} y^{(m_{n+j})}(1) = 0, \quad j = 1, 2, \dots, n, j \neq p, \tag{30}$$

$$\ell_{1,n+p} y \equiv y^{(m_{n+p})}(0) - (-1)^{m_{n+p}} y^{m_{n+p}}(1) + b(y^{(m_{n+p})}(0) + (-1)^{m_{n+p}} y^{(m_{n+p})}(1)) = 0. \tag{31}$$

Let $L_{1,k}$ be the operator of the problem (28)–(31), $L_{1,k} y \equiv (-1)^n y^{(2n)}(x) + z_k^{2n} y(x)$; $y \in D(L_{1,k})$, $D(L_{1,k}) \equiv \{y \in W_2^{2n}(0, 1) : l_{1,j} y = 0, j = 1, 2, \dots, 2n\}$.

Theorem 2. *Suppose that the assumptions B_1–B_2 hold. Then, for the any arbitrary fixed numbers $p \in \{1, 2, \dots, n\}$, $b \in \mathbb{R}$,*

1) the eigenvalues of the operators $L_{0,k}$ and $L_{1,k}$ coincide;

2) the system $V(L_{1,k})$ of the eigenfunctions of the operator $L_{1,k}$ is the Riesz basis of the space $L_2(0, 1)$.

Proof. We show that the eigenvalues of the operators $L_{0,k}$ and $L_{1,k}$ coincide. We substitute the fundamental system (14), (15) of the solutions of the differential equation (28) into the boundary conditions (29)–(31).

$$\det(l_{1,j} y_r(x, \rho))_{j,r=1}^n = \det(l_{1,j} y_r(x, \rho))_{j,r=1}^n \det(l_{1,n+j} y_{n+r}(x, \rho))_{j,r=1}^n.$$

If $l_{p,b} y_{n+j}(x, \rho) = 0$, we obtain the same equations for determining the spectrum. Define the elements of the system $V(L_{1,k})$. Direct substitution shows that the functions $v_{1,q}(x, L_{0,k})$, $q = 1, 2, \dots$, satisfy the conditions (29)–(31). Therefore, the eigenfunction of the operator $L_{1,k}$ that corresponds to the eigenvalue $\lambda_{1,q,k}$ is defined by

$$v_{1,q}(x, L_{1,k}) = v_{1,q}(x, L_{0,k}), \quad q = 1, 2, \dots, \tag{32}$$

$$v_{0,q}(x, L_{1,k}) = v_{0,q}(x, L_{0,k}) - l_p^2(v_{0,q}(x, L_{0,k}))(l_{1,n+p} y_{1,n+p}(x, \rho_{0,q}))^{-1} y_{1,n+p}(x, \rho_{0,q}), \ q = 1, 2, \dots.$$

Taking into account the formulas (31), (21), and the inequalities

$$|l_p^2(v_{0,q}(x, L_{0,k}))| \leq K_1(\rho_{0,q})^{m_{n+p}},$$

we obtain the estimates

$$|l_p^2(v_{0,q}(x,L_{0,k}))(l_{1,n+p}y_{1,n+p}(x,\rho_{0,q}))^{-1}| \le K_2|b|.$$

For the problem (29)–(31), there exists an adjoint problem whose system of the eigenfunctions $W(L_{1,k})$ form the biorthogonal system to the $V(L_{1,k})$. The boundary conditions (29)–(31) are strongly regular according to Birkhoff. Therefore, according to the Kesselman-Mikhailov's Theorem [18], the system $V(L_{1,k})$ is the Riesz basis of the space $L_2(0,1)$. □

4 TRANSFORMATION OPERATORS

For any fixed $k \in \mathbb{N}$, $p \in \{1,2,\dots,n\}$, we consider the operator $B_p : L_2(0,1) \to L_2(0,1)$, the eigenvalues of which coincide with the eigenvalues of the operator $L_{0,k}$, and the eigenfunctions are defined by

$$v_{1,q}(x,B_p) \equiv v_{1,q}(x,L_{0,k}), v_{0,q}(x,B_p) \equiv v_{0,q}(x,L_{0,k}) + c_q(B_p)y_{1,n+p}(x,\rho_{0,q}), \tag{33}$$

$c_q(B_p) \in \mathbb{R}$, $q = 1,2,\dots$.

The operator that maps the system $V(L_{0,k})$ into the system $V(B_p)$ of the eigenfunctions of the operator B_p is denoted by $R(B_p) \equiv E + S(B_p)$, $S(B_p) : L_{2,0}(0,1) \to L_{2,1}(0,1)$, $S(B_p) : L_{2,1}(0,1) \to 0$.

We consider the set $G_p(L_{0,k})$ of the operators $R(B_p)$ such that the eigenfunctions of the operator B_p are defined by the equalities (33).

Lemma 4.1. *Suppose that, the assumptions B_1–B_2 hold, $R(B_p) \in G_p(L_{0,k})$. Then the system of the functions $V(B_p)$ is complete and minimal in the space $L_2(0,1)$.*

Taking into account the uniqueness of the operator $R(B_p)^{-1} \equiv E - S(B_p)$, we obtain the statement of the lemma. Suppose that U is the set of systems of functions $(u_m)_{m=1}^{\infty} \subset L_2(0,1)$, that are complete and minimal in space $L_2(0,1)$, $Q(I)$ is a set of operators $R = E + S$, such that $S : L_{2,0}(0,1) \to L_{2,1}(0,1)$, $S : L_{2,1}(0,1) \to 0$, $Q_c(I) \equiv [L_2(0,1)] \bigcap Q(I)$.

Taking into account equality $S^2(B_p) = 0$, $R(B_p) \in G_p(L_{0,k}) \subset Q(I)$ on the set $Q(I)$, we can define the operation of multiplication

$$R_1R_2 \equiv (E+S_1)(E+S_2) = E + S_1 + S_2, \quad R_1, R_2 \in Q(I).$$

In particular, $Q(I) = Q(I_0)$, $(E+S)(E-S) = E - S^2 = E, E + S = R \in Q(I)$. Therefore, for each operator $R = E + S \in Q(I)$ there exists a unique inverse operator $R^{-1} = E - S$.

According to the definition of the operator B_p and of the set $G_p(L_{0,k})$ we have the inclusions

$$G_p(L_{0,k}) \subset Q(I),\ G_{c,p}(L_{0,k}) \subset Q_c(I),\ p \in \{1,2,\dots,n\}.$$

Thus, the set $Q(I)$ is an Abelian group which contains the Abelian subgroups $Q_c(I)$, $G_p(L_{0,k})$, $G_{c,p}(L_{0,k})$, $p \in \{1,2,\dots,n\}$. Therefore, for any operators $R_j = E + S_j \in Q_0(I)$, $j = 1,2\dots,d$, $d \in \mathbb{N}$, the following equality is satisfied

$$\prod_{j=1}^{d} R_j \equiv \prod_{j=1}^{d} (E+S_j) = E + \sum_{j=1}^{d} S_j, \quad d \in \mathbb{N}.$$

Lemma 4.2. *Suppose that the assumptions B_1–B_2 hold, $R(B_p) \in G_p(L_{0,k})$. The system of the functions $V(B_p)$ is the Riesz basis of the space $L_2(0,1)$ if and only if the sequence $\{c_q(B_p)\}$ is bounded.*

The proof of the lemma is carried out analogously in [13]. Therefore, the operator L_1 has the system

$$V(L_1) \equiv \{v_{s,q,k}(x, L_1) \in W_1 : v_{s,q,k}(x, L_1) = v_{s,q}(x, L_{1,k})v_k,\ s = 0,1,\ q,k = 1,2,\dots\}$$

of the eigenfunctions in the space W_1. The product of the system $V(A)$ and the system $V(L_{1,k})$ is a Riesz basis in the space W_1. Thus, the following theorem is true.

Theorem 3. *Suppose that the assumptions B_1–B_2 hold. Then for arbitrary fixed numbers $p \in \{1,2,\dots,n\}$, $b \in \mathbb{R}$, the system of the functions $V(L_1)$ is the Riesz basis of the space W_1.*

5 THE SPECTRAL PROBLEM FOR A DIFFERENTIAL-OPERATOR EQUATION

For the differential-operator equation (5) for arbitrary fixed $b_{p,r,s} \in \mathbb{R}$, $p \in \{1,2,\dots,n\}$, $r = 0,1,\dots, k_j, s = 0,1, j = 1,2,\dots,n$, we consider problem, generated by nonlocal conditions

$$\ell_{2,j}w \equiv D_x^{m_j}w(0) + (-1)^{m_j} D_x^{m_j}w(1) = 0, \quad j = 1,2,\dots,n, \tag{34}$$

$$\ell_{2,n+j}w \equiv D_x^{m_{n+j}}w(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}}w(1) = 0, j \neq p, \quad j = 1,2,\dots,n, \tag{35}$$

$$\ell_{2,n+p}w \equiv D_x^{m_{n+p}}w(0) - (-1)^{m_{n+p}} D_x^{m_{n+p}}w(1) + l_p^1 w = 0, \quad j = 1,2,\dots,n, \tag{36}$$

$$\ell_p^1 w \equiv \sum_{r=0}^{k_j}(b_{p,r,0}D_x^r w(0) + b_{p,r,1}D_x^r w(1)). \tag{37}$$

Assumption B_3**.** $b_{p,r,0} = (-1)^r b_{p,r,1}$, $r = 0,1,\dots,k_j$, $p = 1,2,\dots,n$.

Remark 5.1. *Assumption B_3 implies that $l_p^1 \in W_0^*$, $p = 1,2,\dots,n$.*

In what follows we assume that the assumptions B_1–B_3 are satisfied. Let L_2 be the operator of the problem (5), (34)–(37),

$$L_2 u \equiv (-1)^n D_x^{2n}u(x) + A^{2n}u(x), \quad u \in D(L_1),$$

$$D(L_2) \equiv \{u \in W_2 : l_{2,j}u = 0,\ j = 1,2,\dots,2n\}.$$

The solution of the spectral problem (5), (34)–(37) is defined as the product $u(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1,2,\dots$. To determine the unknown function $y \in W_2^{2n}(0,1)$, we obtain the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n}y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{38}$$

$$\ell_{2,j}y \equiv y^{(m_j)}(0) + (-1)^{m_j} y^{(m_j)}(1) = 0, \quad j = 1,2,\dots,n, \tag{39}$$

$$\ell_{2,n+j}y \equiv y^{(m_{n+j})}(0) - (-1)^{m_{n+j}} y^{(m_{n+j})}(1) = 0, \quad j = 1,2,\dots,n, \quad j \neq p, \tag{40}$$

$$\ell_{2,n+p}y \equiv y^{(m_{n+p})}(0) - (-1)^{m_{n+p}} y^{m_{n+p}}(1) + l_p^1 y = 0, \tag{41}$$

$$\ell_p^1 y \equiv \sum_{r=0}^{k_j}(b_{p,r,0}D_x^r y(0) + b_{p,r,1}D_x^r y(1)). \tag{42}$$

Let $L_{2,k}$ be the operator of the problem (38)–(42), $L_{2,k}y \equiv (-1)^n\ y^{(2n)}(x) + z_k^{2n}y(x)$; $y \in D(L_{2,k})$, $D(L_{2,k}) \equiv \{y \in W_2^{2n}(0,1) : l_{2,j}y = 0, j = 1,2,\dots,2n\}$.

Theorem 4. *Suppose that the assumptions B_1–B_3 hold. Then for arbitrary fixed numbers $b_{p,r,s} \in \mathbb{R}$, $x_s \in (0,1)$, $s = 0,1$, $r = 0,1,\dots,k_p$, $p \in \{1,2,\dots,n\}$,*

1) the eigenvalues of the operators $L_{0,k}$ and $L_{2,k}$ coincide;

2) the system $V(L_{2,k})$ of the eigenfunctions of the operator $L_{2,k}$ is complete and minimal in the space $L_2(0,1)$;

3) if $k_p \leq m_p$ then the system $V(L_{2,k})$ is the Riesz basis of the space $L_2(0,1)$.

Proof. The proof of part 1 of the theorem is carried out analogously in Theorem 2. Define the elements of the system $V(L_{2,k})$. Direct substitution shows that the functions $v_{2,q}(x, L_{0,k}), q = 1,2,\dots,$ satisfy the conditions (34)–(37).

Therefore, the eigenfunction of the operator $L_{2,k}$, that corresponds to the eigenvalue $\lambda_{q,k}$ is defined by

$$v_{1,q}(x, L_{2,k}) = v_{1,q}(x, L_{0,k}), \quad q = 1,2,\dots, \tag{43}$$

$$v_{0,q}(x, L_{2,k}) = v_{0,q}(x, L_{0,k}) - l_p^1(v_{0,q}(x, L_{0,k}))(l_{2,n+p}y_{1,n+p}(x, \rho_{0,q}))^{-1}y_{1,n+p}(x, \rho_{0,q}).$$

Consequently $L_{2,k} \in Q(I)$. Taking into account Lemma 4.1, we obtain the second statement of the theorem. Taking into account the formulas (31), (21), and the inequalities $|l_p^1 v_{0,q}| \leq K_1(\rho_{0,q})^{m_{n+p}}$, we obtain the estimates

$$|l_p^1(v_{0,q}(x, L_{0,k}))(l_{2,n+p}y_{1,n+p}(x, \rho_{0,q}))^{-1}| \leq K_2|b|. \tag{44}$$

Taking into account Lemma 4.2, we obtain the third statement of the theorem. □

6 THE SPECTRAL BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL-OPERATOR EQUATION WITH INVOLUTION

Consider the spectral problem

$$Lu \equiv (-1)^n\, D_x^{2n}u(x) + A^{2n}u(x) + \sum_{j=1}^{n} a_j \left(D_x^{2j-1}u(x) - D_x^{2j-1}u(1-x)\right) = \lambda u(x), \quad \lambda \in \mathbb{C}, \tag{45}$$

$$\ell_j u \equiv D_x^{m_j}u(0) + (-1)^{m_j} D_x^{m_j}u(1) = 0, \tag{46}$$

$$\ell_{n+j}u \equiv D_x^{m_{n+j}}u(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}}u(1) + l_j^1 u = 0, \quad j = 1,2,\dots,n. \tag{47}$$

The solution of the spectral problem (45)–(47) is defined as the product $u(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1,2,\dots$. To determine the unknown function $y \in W_2^{2n}(0,1)$ we obtain the spectral problem

$$(-1)^n\, y^{(2n)}(x) + z_k^{2n}y(x) + \sum_{j=1}^{n} a_j \left(y^{(2j-1)}y(x) - y^{(2j-1)}(1-x)\right) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{48}$$

$$\ell_j y \equiv y^{(m_j)}(0) + (-1)^{m_j} y^{(m_j)}(1) = 0, \tag{49}$$

$$\ell_{n+j}y \equiv y^{(m_{n+j})}(0) - (-1)^{(m_{n+j})} y^{(m_{n+j})}(1) + l_j^1 y = 0, \quad j = 1,2,\dots,n. \tag{50}$$

Let $L_{3,k}$ be the operator of the problem (48)–(50);

$$L_{3,k}y \equiv (-1)^n\, y^{(2n)}y(x) + z_k^{2n}y(x) + \sum_{j=1}^{n} a_j \left(y^{(2j-1)}(x) + y^{(2j-1)}(1-x)\right), \quad y \in D(L_{3,k});$$

$$D(L_{3,k}) \equiv \{y \in W_2^{2n}(0,1) : l_j y = 0, j = 1,2,\dots,2n\};$$

$V(L_{3,k})$ is the system of the eigenfunctions of the operator $L_{3,k}$.

Theorem 5. *Suppose that $k_p \leq m_p$ and the assumptions B_1–B_3 hold. Then the system of the functions $V(L_{3,k})$ is the Riesz basis of the space $L_2(0,1)$.*

Proof. Define the elements of the system $V(L_{3,k})$. The functions $v_{1,q}(x, L_{0,k})$ satisfy the conditions (49)–(50), $q = 1, 2, \ldots$. Therefore, the eigenfunction of the operator $L_{3,k}$, that corresponds to the eigenvalue $\lambda_{1,q,k}$ is defined by

$$v_{1,q}(x, L_{3,k}) \equiv v_{1,q}(x, L_{0,k}), q = 1, 2, \ldots . \tag{51}$$

For convenience we consider the representation of an eigenfunction of the operator $L_{0,k}$ according to the formula

$$v_{0,q}(x, L_{0,k}) \equiv \theta_{0,q} \sum_{r=1}^{n} \Delta_0^{1,r}(\rho_{0,q}) y_r(x, \rho_{0,q}), \quad q = 1, 2, \ldots . \tag{52}$$

Let

$$y^1_{n+j}(x, \rho_{0,q}) \equiv (x - \frac{1}{2})(\exp \omega_j \rho_{0,q} x + \exp \omega_j \rho_{0,q}(1 - x)), \quad j = 1, 2, \ldots, n, \quad q = 1, 2, \ldots, \tag{53}$$

$$y^{1,1}(x, \rho_{0,q}) \equiv \sum_{r=1}^{n} h^{1,1}_{r,q} y^1_{n+r}(x, \rho_{0,q}) \in H_1, \quad q = 1, 2, \ldots, \tag{54}$$

be the linear combination of the functions (53) with the indeterminate coefficients $h^{1,1}_{r,q}$, and

$$y^{1,2}(x, \rho_{0,q}) \equiv \sum_{r=1}^{n} h^{1,2}_{r,q} y_{1,n+r}(x, \rho_{0,q}) \in H_1, \quad q = 1, 2, \ldots, \tag{55}$$

be the linear combination of the functions $y_{1,n+r}(x, \rho_{0,q})$ with the indeterminate coefficients $h^{1,2}_{r,q}$.

The eigenfunction $v_{0,q}(x, L_{3,k})$ of the operator $L_{3,k}$ is given by

$$v_{0,q}(x, L_{3,k}) \equiv v_{0,q}(x, L_{2,k}) + y^{1,1}(x, \rho_{0,q}) + y^{1,2}(x, \rho_{0,q}), \quad q = 1, 2, \ldots, \tag{56}$$

where

$$h^{1,1}_{r,q} = -\frac{1}{2n} \theta_{0,q} \sum_{j=1}^{n} a_j (\rho_{0,q})^{2j-2n} (\omega_r)^{2j-2} \Delta_0^{1,r}(\rho_{0,q}), \quad q = 1, 2, \ldots, \tag{57}$$

$$h^{1,2}_{r,q} = -(\rho_{0,q})^{-m_{n+r}} (\Delta_0(\rho_{0,q}))^{-1} \Delta_0^{1,r}(\rho_{0,q}) l_{n+r} y^{1,1}(x, \rho_{0,q}), \quad q = 1, 2, \ldots . \tag{58}$$

Let $\Delta_0^{1,r} = \lim\limits_{q \to \infty} \Delta_0^{1,r}(\rho_{0,k}), k = 1, 2, \ldots$; V_k be the system, whose elements are the functions

$$v_{1,q}(x) \equiv v_{1,q}(x, L_{0,k}), \, v_{0,q}(x) \equiv v_{0,q}(x, L_{0,k}) + \Delta_0^{1,r} y_{1,n+1}(x, \rho_{0,q}), \quad q = 1, 2, \ldots .$$

Using inequality $|\Delta_0^{1,r}| < K_3 < \infty$, and Lemma 4.2, we obtain the statement: V_k is Riesz basis of the space $L_2(0,1)$. Taking into account the quadratic proximity of the system V_k and complete the system $V(L_{2,k})$ in the space $L_2(0,1)$ and according to N.K.Bari's Theorem [11], we prove the Theorem. □

Therefore, the operator L has a system of the eigenfunctions

$$V(L) \equiv \{v_{s,q,k}(x,L) \in W_1 : v_{s,q,k}(x,L) \equiv v_{s,q}(x,L_{3,k})v_k, s = 0,1,k,q = 1,2,\dots\},$$

$\lambda_{s,q,k} = (\rho_{s,q})^{2n} + z_k^{2n}$ are the eigenvalues of the operator L, $s = 0,1, q = 1,2,\dots$.

Taking into account the formulas (56)–(58), we obtain the following statement: the sequence of the operators $\{R(L_{3,k}), k = 1,2,\dots\}$ is uniformly bounded by the norm $[L_2(0,1)]$. Thus, the following theorem is true.

Theorem 6. *Suppose that $k_p \leq m_p$ and the assumptions B_1–B_3 hold. Then the system of the functions $V(L)$ is the Riesz basis of the space W_1.*

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Баранецький Я.О., Каленюк П.І., Коляса Л.І., Копач М.І. *Нелокальна задача для диференціально-операторного рівняння парного порядку з інволюцією* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 109–119.

У роботі досліджується задача з крайовими несамоспряженими умовами диференціально-операторних рівнянь порядку $2n$ з інволюцією. Досліджено спектральні властивості оператора задачі.

Аналогічно методу відокремлення змінних, крайова задача для диференціально-операторного рівняння парного порядку, зведена до послідовності операторів $\{L_k\}_{k=1}^{\infty}$ крайових задач для звичайних диференціальних рівнянь парного порядку. Встановлено, що кожен елемент L_k цієї послідовності є ізоспектральним збуренням оператора $L_{0,k}$ самоспряженої крайової задачі для деякого лінійного звичайного диференціального рівняння порядку $2n$.

Побудовано комутативну групу операторів перетворення, елементи якої відображають систему $V(L_{0,k})$ власних функцій оператора $L_{0,k}$ у систему $V(L_k)$ власних функцій операторів L_k. Власні функції оператора крайової задачі для диференціально - операторного рівняння з інволюцією отримано, як результат дії деякого спеціально побудованого оператора на власні функції послідовності операторів $\{L_k\}_{k=1}^{\infty}$.

Встановлено достатні умови, при яких система власних функцій оператора задачі є базисом Рісса.

Ключові слова і фрази: оператор інволюції, диференціально-операторне рівняння, власні функції, базис Рісса.