ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publl. 2016, **8** (2), 183–194
doi:10.15330/cmp.8.2.183-194
http://www.journals.pu.if.ua/index.php/cmp
Карпатськi матем. публ. 2016, Т.8, №2, С.183–194

BARDYLA S.O., GUTIK O.V.

ON A COMPLETE TOPOLOGICAL INVERSE POLYCYCLIC MONOID

We give sufficient conditions when a topological inverse λ-polycyclic monoid P_λ is absolutely H-closed in the class of topological inverse semigroups. For every infinite cardinal λ we construct the coarsest semigroup inverse topology τ_{mi} on P_λ and give an example of a topological inverse monoid S which contains the polycyclic monoid P_2 as a dense discrete subsemigroup.

Key words and phrases: inverse semigroup, bicyclic monoid, polycyclic monoid, free monoid, semigroup of matrix units, topological semigroup, topological inverse semigroup, minimal topology.

Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: sbardyla@yahoo.com (Bardyla S.O.), o_gutik@franko.lviv.ua, ovgutik@yahoo.com (Gutik O.V.)

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [10, 12, 16, 31]. If A is a subset of a topological space X, then we denote the closure of the set A in X by $\operatorname{cl}_X(A)$. By $\mathbb{N}$ we denote the set of all positive integers and by ω the first infinite cardinal.

A semigroup S is called an *inverse semigroup* if every a in S possesses a unique inverse, i.e. if there exists a unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

A map that associates to any element of an inverse semigroup its inverse is called the *inversion*.

A *band* is a semigroup of idempotents. If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication. The semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [35, Definition II.5.12] a chain L is called ω-chain if L is order isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqslant$. Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [17] or [12, Section 2.1]):

$$a\mathscr{R}b \text{ if and only if } aS^1 = bS^1; \quad a\mathscr{L}b \text{ if and only if } S^1a = S^1b;$$
$$\mathscr{D} = \mathscr{L}\circ\mathscr{R} = \mathscr{R}\circ\mathscr{L}; \quad \mathscr{H} = \mathscr{L}\cap\mathscr{R}.$$

The $\mathscr{R}$-class (resp., $\mathscr{L}$-, $\mathscr{H}$-, or $\mathscr{D}$–class) of the semigroup S which contains an element a of S will be denoted by R_a (resp., L_a, H_a, or D_a).

УДК 512.53
2010 *Mathematics Subject Classification:* 22A15, 22A26, 54A10, 54D25, 54D35, 54H11.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [12]. Also the well known Andersen Theorem states that *a simple semigroup S with an idempotent is completely simple if and only if S does not contains an isomorphic copy of the bicyclic semigroup* (see [2] and [12, Theorem 2.54]).

Let λ be a non-zero cardinal. On the set $B_\lambda = (\lambda \times \lambda) \cup \{0\}$, where $0 \notin \lambda \times \lambda$, we define the semigroup operation " $\cdot$ " as follows

$$(a,b) \cdot (c,d) = \begin{cases} (a,d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}$$

and $(a,b) \cdot 0 = 0 \cdot (a,b) = 0 \cdot 0 = 0$ for $a,b,c,d \in \lambda$. The semigroup B_λ is called the *semigroup of $\lambda \times \lambda$-matrix units* (see [12]).

In 1970 Nivat and Perrot proposed the following generalization of the bicyclic monoid (see [34] and [31, Section 9.3]). For a non-zero cardinal λ, the polycyclic monoid on λ generators P_λ is the semigroup with zero given by

$$P_\lambda = \left\langle \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \right\rangle.$$

If $\lambda = 1$ the semigroup P_1 is isomorphic to the bicyclic semigroup with adjoined zero. For every finite non-zero cardinal $\lambda = n$ the polycyclic monoid P_n is congruence free, combinatorial, 0-bisimple, 0-E-unitary inverse semigroup (see [31, Section 9.3]).

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S,τ) is a topological (inverse) semigroup, then we shall call τ an (*inverse*) *semigroup topology* on S. A *semitopological semigroup* is a Hausdorff topological space endowed with a separately continuous semigroup operation.

Let $\mathfrak{STSG}_0$ be a class of topological semigroups. A semigroup $S \in \mathfrak{STSG}_0$ is called *H-closed in* $\mathfrak{STSG}_0$, if S is a closed subsemigroup of any topological semigroup $T \in \mathfrak{STSG}_0$ which contains S both as a subsemigroup and as a topological space. The H-closed topological semigroups were introduced by Stepp in [39], and there they were called *maximal semigroups*. A topological semigroup $S \in \mathfrak{STSG}_0$ is called *absolutely H-closed in the class* $\mathfrak{STSG}_0$, if any continuous homomorphic image of S into $T \in \mathfrak{STSG}_0$ is H-closed in $\mathfrak{STSG}_0$. Absolutely H-closed topological semigroups were introduced by Stepp in [40], and there they were called *absolutely maximal*.

Recall [1], a topological group G is called *absolutely closed* if G is a closed subgroup of any topological group which contains G as a subgroup. In our terminology such topological groups are called H-closed in the class of topological groups. In [36] Raikov proved that a topological group G is absolutely closed if and only if it is Raikov complete, i.e., G is complete with respect to the two-sided uniformity. A topological group G is called *h-complete* if for every continuous homomorphism $h\colon G \to H$ the subgroup $f(G)$ of H is closed [13]. In our terminology such topological groups are called absolutely H-closed in the class of topological groups. The h-completeness is preserved under taking products and closed central subgroups [13]. H-closed paratopological and topological groups in the class of paratopological groups were studied in [37]. The paper [7] contains a sufficient condition for a quasitopological group to be H-closed, which allowed us to solve a problem by Arhangel'skii and Choban [3] and show that a topological group G is H-closed in the class of quasitopological groups if and only if G is Raikov-complete. In [18] it is proved that a topological group G is H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if G is compact.

In [40] Stepp studied H-closed topological semilattices in the class of topological semigroups. He proved that an algebraic semilattice E is algebraically h-complete in the class of topological semilattices if and only if every chain in E is finite. In [27] Gutik and Repovš studied the closure of a linearly ordered topological semilattice in a topological semilattice. They obtained a characterization of H-closed linearly ordered topological semilattices in the class of topological semilattices and showed that every H-closed linear topological semilattice is absolutely H-closed in the class of topological semilattices. Such semilattices were studied also in [11,20]. In [5] the closures of the discrete semilattices $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$ were described. In that paper the authors constructed an example of an H-closed topological semilattice in the class of topological semilattices, which is not absolutely H-closed in the class of topological semilattices. The constructed example gives a negative answer to Question 17 from [40]. H-closed and absolutely H-closed (semi)topological semigroups and their extensions in different classes of topological and semitopological semigroups were studied in [8,18,19,21–26]

In [6] we showed that the λ-polycyclic monoid for an infinite cardinal $\lambda \geqslant 2$ has similar algebraic properties to that of the polycyclic monoid P_n with finitely many $n \geqslant 2$ generators. In particular we proved that for every infinite cardinal λ the polycyclic monoid P_λ is congruence-free, combinatorial, 0-bisimple, 0-E-unitary, inverse semigroup. Also we showed that every non-zero element $x \in P_\lambda$ is an isolated point in (P_λ, τ) for every Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup; moreover, every locally compact Hausdorff semigroup topology on P_λ is discrete. The last statement extends results of the paper [32] treating topological inverse graph semigroups. We described all feebly compact topologies τ on P_λ such that (P_λ, τ) is a semitopological semigroup. Also in [6] we proved that for every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating and there exists no Hausdorff feebly compact topological semigroup containing P_λ as a dense subsemigroup.

This paper is a continuation of [6]. In this paper we give sufficient conditions on a topological inverse λ-polycyclic monoid P_λ to be absolutely H-closed in the class of topological inverse semigroups. For every infinite cardinal λ we construct the coarsest semigroup inverse topology τ_{mi} on P_λ and give an example of a topological inverse monoid S which contains the polycyclic monoid P_2 as a dense discrete subsemigroup.

It is well known that for an arbitrary topological inverse semigroup S and every element $x \in S$ the continuity of the semigroup operation and the inversion in S implies that any $\mathscr{L}$-class L_x and any $\mathscr{R}$-class R_x which contain the element x are closed subsets in S. Indeed, the Wagner–Preston Theorem (see Theorem 1.17 from [12]) implies that $L_x = L_{x^{-1}x}$ and $R_x = R_{xx^{-1}}$ for arbitrary $x \in S$ and since the maps $\varphi\colon S \to E(S)$ and $\psi\colon S \to E(S)$ defined by the formulae

$$(x)\varphi = xx^{-1} \qquad \text{and} \qquad (x)\psi = x^{-1}x$$

are continuous, we get that $L_x = (x^{-1}x)\psi^{-1}$ and $R_x = (xx^{-1})\varphi^{-1}$ are closed subsets of the topological semigroup S. This implies that for any idempotents e and f of a topological inverse semigroup S the following $\mathscr{H}$-classes of S:

$$H_e = R_e \cap L_e \qquad \text{and} \qquad H_{e,f} = R_e \cap L_f$$

are closed subsets of the topological inverse semigroup S too. Moreover, the relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ are closed subsets in $S \times S$, but $\mathscr{D}$ and $\mathscr{J}$ are not necessary closed subsets in $S \times S$ for an arbitrary topological inverse semigroup S (see [15, Section II]).

The following proposition describes $\mathscr{D}$-equivalent $\mathscr{H}$-classes in an arbitrary topological inverse semigroup.

Proposition 1. *Let S be a Hausdorff topological inverse semigroup and a, c be $\mathscr{D}$-equivalent elements of S. Then there exists $b \in S$ such that $a\mathscr{R}b$ and $b\mathscr{L}c$ in S, and hence $as = b$, $bs' = a$, $tb = c$, $t'c = b$, for some $s, s', t, t' \in S$. The mappings $\mathfrak{f}_{a,c} \colon H_a \to H_c \colon x \mapsto txs$ and $\mathfrak{f}_{c,a} \colon H_c \to H_a \colon x \mapsto t'xs'$ are continuous and mutually inverse, and hence are homeomorphisms of closed subspaces H_a and H_c of the topological space S. Moreover, if H_a and H_c are subgroups of S then H_a and H_c are topologically isomorphic closed topological subgroups in the topological inverse semigroup S.*

Proof. The above arguments imply that H_a and H_c are closed subspaces of S. Also, the algebraic part of the statement of our theorem follows from Theorem 2.3 of [12] and Theorem 1.2.7 from [28]. The continuity of the semigroup operation in S implies that the maps $\mathfrak{f}_{a,c} \colon H_a \to H_c$ and $\mathfrak{f}_{c,a} \colon H_c \to H_a$ are continuous and hence are homeomorphisms. Now, the proof of Theorem 1.2.7 from [28] implies that in the case when H_a and H_c are subgroups of S, then there exist $u, u' \in S$ such that the maps $\mathfrak{f}_{a,c} \colon H_a \to H_c \colon x \mapsto uxu'$ and $\mathfrak{f}_{c,a} \colon H_c \to H_a \colon x \mapsto u'xu$ are mutually inverse isomorphisms and the continuity of the semigroup operation in S implies that so defined maps are topological isomorphisms. □

Remark 1. *The proof of Proposition 1 implies that any two $\mathscr{D}$-equivalent $\mathscr{H}$-classes of a Hausdorff semitopological semigroup S are homeomorphic subspaces in S, but they are not necessary closed subspaces in S, and a similar statement holds for maximal subgroups in S (see [18]).*

Lemma 1. *Let T and S be a Hausdorff topological inverse semigroup such that S is an inverse subsemigroup of T. Let G be an $\mathscr{H}$-class in S which is a closed subset of the topological inverse semigroup T and D_G be a $\mathscr{D}$-class of the semigroup S which contains the set G. Then every $\mathscr{H}$-class $H \subseteq D_G$ of the semigroup S is a closed subset of the topological space T.*

Proof. First we consider the case when G has an idempotent, i.e., G is a maximal subgroup of the semigroup S (see Theorem 2.16 of [12]).

In the case when the $\mathscr{H}$-class H contains an idempotent, Theorem 2.16 in [12] implies that H is a maximal subgroup of S and hence H is a subgroup of topological inverse semigroup T. We put e and f are unit elements of the groups G and H, respectively. Since the idempotents e and f are $\mathscr{D}$-equivalent in S, Proposition 3.2.5 of [31] implies that there exists $a \in S$ such that $aa^{-1} = e$ and $a^{-1}a = f$. Now by Proposition 3.2.11(5) of [31] the idempotents e and f are $\mathscr{D}$-equivalent in the semigroup T. Put H_e^T and H_f^T be the $\mathscr{H}$-classes of idempotents e and f in the semigroup T, respectively. We define the maps $\mathfrak{f}_{e,f} \colon T \to T$ and $\mathfrak{f}_{f,e} \colon T \to T$ by the formulae

$(x)\mathfrak{f}_{e,f} = a^{-1}xa$ and $(x)\mathfrak{f}_{f,e} = axa^{-1}$, respectively. Then for any $s \in H_e^T$ and $t \in H_f^T$ we get the equalities

$$\begin{aligned}
(s)\mathfrak{f}_{e,f}\big((s)\mathfrak{f}_{e,f}\big)^{-1} &= a^{-1}sa(a^{-1}sa)^{-1} = a^{-1}saa^{-1}s^{-1}a = a^{-1}ses^{-1}a = a^{-1}ss^{-1}a = a^{-1}ea\\
&= a^{-1}a = f,\\
\big((s)\mathfrak{f}_{e,f}\big)^{-1}(s)\mathfrak{f}_{e,f} &= (a^{-1}sa)^{-1}a^{-1}sa = a^{-1}s^{-1}aa^{-1}sa = a^{-1}s^{-1}esa = a^{-1}s^{-1}sa = a^{-1}ea\\
&= a^{-1}a = f,\\
(t)\mathfrak{f}_{f,e}\big((t)\mathfrak{f}_{f,e}\big)^{-1} &= ata^{-1}(ata^{-1})^{-1} = ata^{-1}at^{-1}a^{-1} = atft^{-1}a^{-1} = att^{-1}a^{-1} = afa^{-1}\\
&= aa^{-1} = e,\\
\big((t)\mathfrak{f}_{f,e}\big)^{-1}(t)\mathfrak{f}_{f,e} &= (ata^{-1})^{-1}ata^{-1} = at^{-1}a^{-1}ata^{-1} = at^{-1}fta^{-1} = at^{-1}ta^{-1} = afa^{-1}\\
&= aa^{-1} = e,\\
\big((s)\mathfrak{f}_{e,f}\big)\mathfrak{f}_{f,e} &= aa^{-1}saa^{-1} = ese = s,\\
\big((t)\mathfrak{f}_{f,e}\big)\mathfrak{f}_{e,f} &= a^{-1}ata^{-1}a = ftf = t,
\end{aligned}$$

because $aa^{-1} = ss^{-1} = s^{-1}s = e$, $ea = a$, $af = a$ and $a^{-1}a = tt^{-1} = t^{-1} = f$. Similarly, for arbitrary $s, v \in H_e^T$ and $t, u \in H_f^T$ we have that

$$(s)\mathfrak{f}_{e,f}(v)\mathfrak{f}_{e,f} = a^{-1}saa^{-1}va = a^{-1}seva = a^{-1}sva = (sv)\mathfrak{f}_{e,f}$$

and

$$(t)\mathfrak{f}_{f,e}(u)\mathfrak{f}_{f,e} = ata^{-1}aua^{-1} = atfua^{-1} = atua^{-1} = (tu)\mathfrak{f}_{f,e}.$$

Hence the restrictions $\mathfrak{f}_{e,f}|_{H_e^T} \colon H_e^T \to H_f^T$ and $\mathfrak{f}_{f,e}|_{H_f^T} \colon H_f^T \to H_e^T$ are mutually invertible group isomorphisms. Also, since $a \in S$ we get that the restrictions $\mathfrak{f}_{e,f}|_G \colon G \to H$ and $\mathfrak{f}_{f,e}|_H \colon H \to G$ are mutually invertible group isomorphisms too. This and the continuity of left and right translations in T imply that H is a closed subgroup of the topological inverse semigroup T.

Next we consider the case when the $\mathscr{H}$-class H contains no idempotents. Then there exists distinct idempotents $e, f \in S$ such that $ss^{-1} = e$ and $s^{-1}s = f$ for all $s \in H$. Suppose to the contrary that H is not a closed subset of the topological inverse semigroup T. Then there exists an accumulation point $x \in T \setminus H$ of the set H in the topological space T. Since every $\mathscr{H}$-class of a topological inverse semigroup T is a closed subset of T we get that H and x are contained in a same $\mathscr{H}$-class H_x of the semigroup T. Then $xx^{-1} = e$ and $x^{-1}x = f$. Now the $\mathscr{H}$-class H_e^T in T which contains the idempotent $e \in S$ is a topological subgroup of the topological inverse semigroup T and by Proposition 1 the subspace H_e^T of the topological space T is homeomorphic to the subspace H_x of T. Moreover, Theorem 1.2.7 from [28] implies that there exists a homeomorphism $\mathfrak{f} \colon H_x\ rightarrow H_e^T$ such that the image $(H)\mathfrak{f}$ is a topological subgroup of the topological inverse semigroup T and $(H)\mathfrak{f}$ is topologically isomorphic to the topological group G. Then $(H)\mathfrak{f}$ is not a closed subgroup of T which contradicts our above part of the proof.

Assume that G has no idempotents. By the previous part of the proof it suffices to show that there exists a maximal subgroup H_e with an idempotent e in the $\mathscr{D}$-class D_G such that H_e is a closed subgroup of topological semigroup T. Suppose to the contrary that every maximal subgroup in the $\mathscr{D}$-class D_G is not a closed in T. Fix and arbitrary subgroup H_e with an idempotent e in the $\mathscr{D}$-class D_G such that $xx^{-1} = e$ for all $x \in G$. Then Proposition 3.2.11(3) of [31] implies that there exist $\mathscr{H}$-classes H^T_G and H^T_e in the semigroup T which contain the set G and group H_e. Since in the topological semigroup T every $\mathscr{H}$-class is a closed subset in T, we have that G is a closed subset of the space H^T_G and H_e is not a closed subgroup of the topological group H^T_e. Then Proposition 3.2.11 of [31] and Proposition 1 imply that there exist $s, s', t, t^{\,prime} \in S$ such that the maps $\mathfrak{f}_e\colon H^T_e \to H^T_G\colon x \mapsto txs$ and $\mathfrak{f}_G\colon H^T_G \to H^T_e\colon x \mapsto t'xs'$ are mutually invertible homeomorphisms of the topological spaces H^T_e and H^T_G such that the restrictions $\mathfrak{f}_e|_{H_e}\colon H^T_e \to G$ and $\mathfrak{f}_G|_G\colon G \to H_e$ are mutually invertible homeomorphisms. This is a contradiction, because H_e is not a closed subset of H^T_e. This completes proof of the lemma. □

Lemma 1 implies the following corollary.

Corollary 1. *Let T and S be a Hausdorff topological inverse semigroup such that S is an inverse subsemigroup of T. Let G be a maximal subgroup in S which is H-closed in the class of topological inverse semigroups and D_G be a $\mathscr{D}$-class of the semigroup S which contains the group G. Then every $\mathscr{H}$-class $H \subseteq D_G$ of the semigroup S is a closed subset of the topological space T.*

Lemma 2. *Let S be a Hausdorff topological inverse semigroup such following conditions hold:*

(i) every maximal subgroup of the semigroup S is H-closed in the class topological groups;

(ii) all non-minimal elements of the semilattice $E(S)$ are isolated points in $E(S)$.

If there exists a topological inverse semigroup T such that S is a dense subsemigroup of T and $T \setminus S \neq \varnothing$ then for every $x \in T \setminus S$ at least one of the points $x \cdot x^{-1}$ or $x^{-1} \cdot x$ belongs to $T \setminus S$.

Proof. First we consider the case when the topological semilattice $E(S)$ does not have the smallest element. Then the space $E(S)$ is discrete and Theorem 3.3.9 of [16] implies that $E(S)$ is an open subset of the topological space $E(T)$ and hence every point of the set $E(S)$ is isolated in $E(T)$. Also by Proposition II.3 [15] we have that $\operatorname{cl}_T(E(S)) = \operatorname{cl}_{E(T)}(E(S))$ and hence the points of the set $E(T) \setminus E(S)$ are not isolated in the space $E(T)$.

Fix an arbitrary point $x \in T \setminus S$. By Corollary 1 every $\mathscr{H}$-class is a closed subset of the topological inverse semigroup T. Since x is an accumulation point of the set S in the topological space T we have that every open neighbourhood $U(x)$ of the point x in T intersects infinitely many $\mathscr{H}$-classes of the semigroup S. By Proposition II.1 of [15] the inversion on T is a homeomorphism of the topological space T and hence $(U(x))^{-1}$ is an open neighbourhood of the point x^{-1} in T which intersects infinitely many $\mathscr{H}$-classes of the semigroup S. Then the continuity of the semigroup operations and the inversion in T implies that at least one of the sets $\left(U(x)\,(U(x))^{-1}\right) \cap E(T)$ or $\left((U(x))^{-1}\,U(x)\right) \cap E(T)$ is infinite for every open neighbourhood $U(x)$ of the point x in the topological semigroup T. This implies that at least one of $x\ cdotx^{-1}$ or $x^{-1} \cdot x$ is a non-isolated point in the topological space $E(T)$.

In the case when the semilattice $E(S)$ has a minimal idempotent the presented above arguments imply that for arbitrary point $x \in T \setminus S$ and every open neighbourhood $U(x)$ of the point x in T one of the sets $\left(U(x)\,(U(x))^{-1}\right) \cap E(T)$ or $\left((U(x))^{-1}\,U(x)\right) \cap E(T)$ is infinite for every open neighbourhood $U(x)$ of the point x in the topological semigroup T. Since H_e is a minimal ideal of S and it is a Raĭkov complete topological group. Then there exists an open neighborhood $U(x)$ of x in T, such that $U(x) \cap H_e = \varnothing$. If $xx^{-1} = e$ or $x^{-1}x = e$ then $x = xx^{-1}x \in H_e$, which contradicts that $x \in T \setminus S$. Hence $xx^{-1} \in T \setminus S$ or $x^{-1}x \in T \setminus S$. □

Lemma 2 implies the following two corollaries.

Corollary 2. *Let S be a Hausdorff topological inverse semigroup satisfying the following conditions:*

(i) *every maximal subgroup of the semigroup S and the semilattice $E(S)$ are H-closed in the class of topological inverse semigroups;*

(ii) *all non-minimal elements of the semilattice $E(S)$ are isolated points in $E(S)$.*

Then S is H-closed in the class of topological inverse semigroups.

Corollary 3. *Let $\lambda \geqslant 2$ and let P_λ be a proper dense subsemigroup of a topological inverse semigroup S. Then either $xx^{-1} \in S \setminus P_\lambda$ or $x^{-1}x \in S \setminus P_\lambda$ for every $x \in S \setminus P_\lambda$.*

The following theorem gives sufficient condition when a topological inverse λ-polycyclic monoid P_λ is absolutely H-closed in the class of topological inverse semigroups.

Theorem 1. *Let λ be a cardinal $\geqslant 2$ and τ be a Hausdorff inverse semigroup topology on P_λ such that $U(0) \cap L$ is an infinite set for every open neighborhood $U(0)$ of zero 0 in (P_λ, τ) and every maximal chain L of the semilattice $E(P_\lambda)$. Then (P_λ, τ) is absolutely H-closed in the class of topological inverse semigroups.*

Proof. First we observe that the definition of the λ-polycyclic monoid P_λ implies that for every maximal chain L in $E(P_\lambda)$ the set $L \setminus \{0\}$ is an ω-chain. Then Theorem 2 of [5] implies that every maximal chain L in $E(P_\lambda)$ with the induced topology from (P_λ, τ) is an absolutely H-closed topological semilattice. Suppose that $E(P_\lambda)$ with the induced topology from (P_λ, τ) is not an H-closed topological semilattice. Then there exists a topological semilattice S which contains $E(P_\lambda)$ as a dense proper subsemilattice. Also the continuity of the semilattice operation in S implies that zero 0 of $E(P_\lambda)$ is zero in S. Fix an arbitrary element $x \in S \setminus E(P_\lambda)$. Then for an arbitrary open neighbourhood $U(x)$ of the point x in S such that $0 \notin U(x)$ the continuity of the semilattice operation in S implies that there exists an open neighbourhood $V(x)\ subseteq U(x)$ of x in S such that $V(x) \cdot V(x) \subseteq U(x)$. Now, the neighbourhood $V(x)$ intersects infinitely many maximal chains of the semilattice $E(P_\lambda)$, because all maximal chains in $E(P_\lambda)$ with the induced topology from (P_λ, τ) are absolutely H-closed topological semilattices. Then the semigroup operation of P_λ implies that $0 \in V(x) \cdot V(x) \subseteq U(x)$, which contradicts the choice of the neighbourhood $U(0)$. Therefore, $E(P_\lambda)$ with the induced topology from (P_λ, τ) is an H-closed topological semilattice.

Now, by Corollary 2 the topological inverse semigroup (P_λ, τ) is H-closed in the class of topological inverse semigroups. Since the λ-polycyclic monoid P_λ is congruence free, every continuous homomorphic image of (P_λ, τ) is H-closed in the class of topological inverse semigroups. Indeed, if $h\colon (P_\lambda, \tau) \to T$ is a continuous (algebraic) homomorphism from (P_λ, τ) into a topological inverse semigroup T, then the set $U(h(0)) \cap h(L)$ is infinite for every open neighbourhood $U(h(0))$ of the image zero $h(0)$ in T. Then the previous part of the proof implies that $h(P_\lambda)$ is a closed subsemigroup of T. $\square$

Remark 2. *By Remark 2.6 from [30] (also see [30, p. 453], [29, Section 2.1] and [31, Proposition 9.3.1]) for every positive integer $n \geqslant 2$ any non-zero element x of the polycyclic monoid*

P_n has the form $u^{-1}v$, where u and v are elements of the free monoid $\mathscr{M}_n$, and the semigroup operation on P_n in this representation is defined in the following way:

$$a^{-1}b \cdot c^{-1}d = \begin{cases} (c_1a)^{-1}d, & \text{if } c = c_1b \quad \text{for some } c_1 \in \mathscr{M}_n; \\ a^{-1}b_1d, & \text{if } b = b_1c \quad \text{for some } b_1 \in \mathscr{M}_n; \\ 0, & \text{otherwise} \end{cases}$$
$$\text{and} \quad a^{-1}b \cdot 0 = 0 \cdot a^{-1}b = 0 \cdot 0 = 0. \tag{1}$$

Then Lemma 2.4 of [6] implies that every any non-zero element x of the polycyclic monoid P_λ has the form $u^{-1}v$, where u and v are elements of the free monoid $\mathscr{M}_\lambda$, and the semigroup operation on P_λ in this representation is defined by formula (1).

Now we shall construct a topology τ_{mi} on the λ-polycyclic monoid P_λ such that $(P_\lambda, \tau_{\mathsf{mi}})$ is absolutely H-closed in the class of topological inverse semigroups.

Example 1. *We define a topology τ_{mi} on the polycyclic monoid P_λ in the following way. All non-zero elements of P_λ are isolated point in (P_λ, τ_{mi}). For an arbitrary finite subset A of $\mathscr{M}_\lambda$ put*

$$U_A(0) = \left\{a^{-1}b : a, b \in M_\lambda \setminus A\right\}.$$

We put $\mathscr{B}_{mi} = \{U_A(0) \colon A \text{ is a finite subset of } \mathscr{M}_\lambda\}$ to be a base of the topology τ_{mi} at zero $0 \in P_\lambda$.

We observe that τ_{mi} is a Hausdorff topology on P_λ because $U_{\{a,b\}}(0) \not\ni a^{-1}b$ for every non-zero element $a^{-1}b \in P_\lambda$. Also, since $(U_A(0))^{-1} = U_A(0)$ for any $U_A(0) \in \mathscr{B}_{mi}$, the inversion is continuous in (P_λ, τ_{mi}). Fix an arbitrary $a^{-1}b \in P_\lambda$ and any basic neighbourhood $U_A(0)$ of zero in (P_λ, τ_{mi}). Let S_b be a set of all suffixes of the word b. Put $B = P_b \cup \{kb \in \mathscr{M}_\lambda \colon ka \in A\}$. It is obvious that the set B is finite and hence formula (1) implies that $a^{-1}b \cdot U_B(0) \subseteq U_A(0)$. Let S_a be a set of all suffixes of the word a. Put $D = S_a \cup \{ta \in \mathscr{M}_\lambda \colon tb \in A\}$. It is obvious that the set D is finite and hence formula (1) implies that $U_D(0) \cdot a^{-1}b \subseteq U_A(0)$. Also $U_T(0) \cdot U_T(0) \subseteq U_A(0)$ for $T = A \cup \{b \in \mathscr{M}_\lambda \colon\ b \text{ is a suffix of some } a \in A\}$. Therefore (P_λ, τ_{mi}) is a topological inverse semigroup.

Theorem 1 and Example 1 implies the following corollary.

Corollary 4. *The topological inverse semigroup $(P_\lambda, \tau_{\mathsf{mi}})$ is absolutely H-closed in the class of topological inverse semigroups.*

Definition 1 ([23])**.** *A Hausdorff topological (inverse) semigroup (S, τ) is said to be* minimal *if no Hausdorff semigroup (inverse) topology on S is strictly contained in τ. If (S, τ) is minimal topological (inverse) semigroup, then τ is called a* minimal (inverse) semigroup topology.

Minimal topological groups were introduced independently in the early 1970's by Doïtchinov [14] and Stephenson [38]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time (cf. [9]). More than 20 years earlier L. Nachbin [33] had studied minimality in the context of division rings, and B. Banaschewski [4] investigated minimality in the more general setting of topological algebras. In [23] on the infinite semigroup of $\lambda \times \lambda$-matrix units B_λ the minimal semigroup and the minimal semigroup inverse topologies were constructed.

Theorem 2. *For any infinite cardinal λ, τ_{mi} is the coarsest inverse semigroup topology on P_λ, and hence $(P_\lambda, \tau_{\mathsf{mi}})$ is a minimal topological inverse semigroup.*

Proof. The definition of the topology τ_{mi} on P_λ implies that the subsemigroup of idempotents $E(P_\lambda)$ of the semigroup P_λ is a compact subset of the space $(P_\lambda, \tau_{\mathsf{mi}})$. By Proposition 3.1 of [6] every non zero-element of a semitopological monoid (P_λ, τ) is an isolated point in the space (P_λ, τ). This and above arguments imply that the topology τ_{mi} on P_λ induces the coarsest semigroup topology on the semilattice $E(P_\lambda)$. Also by Remark 2.6 from [30] (also see [30, p. 453], [29, Section 2.1] and [31, Proposition 9.3.1]) we have that every non-zero element of the semilattice $E(P_\lambda)$ can be represented in the form $a^{-1}a$ where a are elements of the free monoid $\mathscr{M}_n$, and the semigroup operation on $E(P_\lambda)$ in this representation is defined by formula (1).

Also, we observe that for any topological inverse semigroup S the following maps $\varphi\colon S \to E(S)$ and $\psi\colon S \to E(S)$ defines by the formulae $\varphi(x) = xx^{-1}$ and $\psi(x) = x^{-1}x$, respectively, are continuous. Since the inverse element of $u^{-1}v$ in P_λ is equal to $v^{-1}u$, we have that $U_A = P_\lambda \setminus \left(\varphi^{-1}(A) \cup \psi^{-1}(A)\right)$, for any finite subset A of the free monoid $\mathscr{M}_n$. This implies that $U_A(A) \in \tau$ for every inverse semigroup topology τ on P_λ, where A is an arbitrary finite subset of $\mathscr{M}_n$. Thus, τ_{mi} is the coarsest inverse semigroup topology on the λ-polycyclic monoid P_λ. □

In the next example we construct a topological inverse monoid S which contains the polycyclic monoid $P_2 = \left\langle p_1, p_2 \mid p_1p_1^{-1} = p_2p_2^{-1} = 1, p_1p_2^{-1} = p_2p_1^{-1} = 0\right\rangle$ as a dense discrete subsemigroup, i.e., the polycyclic monoid P_2 with the discrete topology is not H-closed in the class of topological inverse semigroups. Also, later we assume that the free monoid $\mathscr{M}_2$ is generated by two element p_1 and p_2.

Example 2. *Let $\mathscr{F}$ be the filter on the bicyclic semigroup $\mathscr{C}(p_1, p_1^{-1}) = \left\langle p_1, p_1^{-1} \mid p_1p_1^{-1} = 1\right\rangle$, generated by the base $\mathscr{B} = \{U_n \colon n \in \mathbb{N}\}$, where $U_n = \left\{p_1^{-k}p_1^m \colon k, m > n\right\}$. We denote*

$$A = \left\{a^{-1}b \in P_2 \colon a \neq p_1a_1 \text{ and } b \neq p_1b_1 \text{ for any } a_1, b_1 \in \mathscr{M}_2\right\}.$$

For any element $a^{-1}b$ of the set A let $\mathscr{F}_{a^{-1}b}$ be the filter on P_2, generated by the base $\mathscr{B}_{a^{-1}b} = \{V_n \colon n \in \mathbb{N}\}$, where $V_n = a^{-1}U_nb = \left\{(p_1^ka)^{-1}p_1^mb \colon k, m > n\right\}$. It is obvious that $\mathscr{F} = \mathscr{F}_{1^{-1}1}$, where 1 is the unit element of the free monoid $\mathscr{M}_2$.

We extend the binary operation from P_2 onto $S = P_2 \cup \left\{\mathscr{F}_{a^{-1}b} \colon a^{-1}b \in A\right\}$ by the following formulae:

$$\text{(I)}\;\; a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = \begin{cases} \mathscr{F}_{(ea)^{-1}d}, & \text{if } c = eb; \\ \mathscr{F}_{(e)^{-1}d}, & \text{if } b = p_1^nc \text{ for some } n \in \mathbb{N}, \text{ where } e \text{ is the longest suffix} \\ & \text{ of } a \text{ such that } e \neq p_1f \text{ for some } f \in M_2; \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{(II)}\;\; \mathscr{F}_{c^{-1}d} \cdot a^{-1}b = \begin{cases} \mathscr{F}_{c^{-1}eb}, & \text{if } d = ea; \\ \mathscr{F}_{c^{-1}e}, & \text{if } a = p_1^nd \text{ for some } n \in \mathbb{N}, \text{ where } e \text{ is the longest suffix} \\ & \text{ of } b \text{ such that } e \neq p_1f \text{ for some } f \in M_2; \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{(III)}\;\; \mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{c^{-1}d} = \begin{cases} \mathscr{F}_{a^{-1}d}, & \text{if } b = c; \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that the subset $T = S \setminus P_2 \cup \{0\}$ with the induced binary operation from S is isomorphic to the semigroup of $\omega \times \omega$-matrix units B_ω and moreover we have that $(\mathscr{F}_{a^{-1}b})^{-1} = \mathscr{F}_{b^{-1}a}$ in T.

We determine a topology τ on the set S in the following way: assume that the elements of the semigroup P_2 are isolated points in (S,τ) and the family

$$\mathscr{B}(\mathscr{F}_{a^{-1}b}) = \{U_n(\mathscr{F}_{a^{-1}b}) : U_n \in \mathscr{B}_{a^{-1}b}\}$$

of the set $U_n(\mathscr{F}_{a^{-1}b}) = U_n \cup \{\mathscr{F}_{a^{-1}b}\}$ is a neighborhood base of the topology τ at the point $\mathscr{F}_{a^{-1}b} \in S$.

Now we show that so defined binary operation on (S,τ) is continuous.

In case (I) we consider three cases.

If $a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = 0$ then we have that $a^{-1}b \cdot U_n(\mathscr{F}_{c^{-1}d}) = \{0\}$ for any positive integer n.

If $a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = \mathscr{F}_{(ea)^{-1}d}$ then $c = eb$. We claim that $a^{-1}b \cdot U_n(\mathscr{F}_{c^{-1}d}) \subseteq U_n(\mathscr{F}_{(ea)^{-1}d})$ for any open basic neighbourhood $U_n(\mathscr{F}_{(ea)^{-1}d})$ of the point $\mathscr{F}_{(ea)^{-1}d}$ in (S,τ). Indeed, if $x \in U_n(\mathscr{F}_{c^{-1}d})$ then $x = (p_1^m c)^{-1} p_1^k d$ for some positive integers $m, k > n$, and hence we have that

$$a^{-1}b \cdot (p_1^m c)^{-1} p_1^k d = a^{-1}b \cdot (p_1^m eb)^{-1} p_1^k d = (p_1^m ea)^{-1} p_1^k d \in U_n(\mathscr{F}_{(ea)^{-1}d}).$$

If $a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = \mathscr{F}_{e^{-1}d}$, then e is the longest suffix of the word a in $\mathscr{M}_2$ which is not equal to the word $p_1 f$ for some $f \in \mathscr{M}_2$. This holds when $b = p_1^t c$ for some positive integer t. We claim that $a^{-1}b \cdot U_{n+t}(\mathscr{F}_{c^{-1}d}) \subseteq U_n(\mathscr{F}_{e^{-1}d})$ for any open basic neighbourhood $U_n(\mathscr{F}_{e^{-1}d})$ of the point $\mathscr{F}_{e^{-1}d}$ in (S,τ). Indeed, if $x \in U_{n+t}(\mathscr{F}_{c^{-1}d})$, then $x = (p_1^{m+t} c)^{-1} p_1^{k+t} d$ for some positive integers $m, k > n$, and hence we have that

$$a^{-1}b \cdot (p_1^{m+t} c)^{-1} p_1^{k+t} d = e^{-1} p_1^{-l} p_1^t c \cdot (p_1^{m+t} c)^{-1} p_1^{k+t} d = (p_1^{m+l} e)^{-1} p_1^{k+t} d \in U_n(\mathscr{F}_{e^{-1}d}).$$

In case (II) the proof of the continuity of binary operation in (S,τ) is similar to case (I).

Now we consider case (III).

If $\mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{c^{-1}d} = 0$ then $U_n(\mathscr{F}_{a^{-1}b}) \cdot U_n(\mathscr{F}_{c^{-1}d}) \subseteq \{0\}$, for any open basic neighbourhoods $U_n(\mathscr{F}_{a^{-1}b})$ and $U_n(\mathscr{F}_{c^{-1}d})$ of the points $\mathscr{F}_{a^{-1}b}$ and $\mathscr{F}_{c^{-1}d}$ in (S,τ), respectively.

If $\mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{c^{-1}d} = \mathscr{F}_{a^{-1}d}$ then $b = c$ and for every any open basic neighbourhood $U_n(\mathscr{F}_{a^{-1}d})$ of the point $\mathscr{F}_{a^{-1}d}$ in (S,τ) we have that $U_n(\mathscr{F}_{a^{-1}b}) \cdot U_n(\mathscr{F}_{b^{-1}d}) \subseteq U_n(\mathscr{F}_{a^{-1}d})$. Indeed if $(p_1^k a)^{-1} p_1^t b \in U_n(\mathscr{F}_{a^{-1}b})$ and $(p_1^l b)^{-1} p_1^m d \in U_n(\mathscr{F}_{b^{-1}d})$ then

$$(p_1^k a)^{-1} p_1^t b \cdot (p_1^l b)^{-1} p_1^m d = (p_1^k a)^{-1} p_1^t (b \cdot b^{-1}) p_1^{-l} p_1^m d = (p_1^s a)^{-1} p_1^z d,$$

for some positive integers $s, z > n$, and hence $(p_1^s a)^{-1} p_1^z d \in U_n(\mathscr{F}_{a^{-1}d})$.

Thus, we proved that the binary operation on (S,τ) is continuous. Taking into account that P_2 is a dense subsemigroup of (S,τ) we conclude that (S,τ) is a topological semigroup. Also, since $T = S \setminus P_2 \cup \{0\}$ with the induced binary operation from S is isomorphic to the semigroup of $\omega \times \omega$-matrix units B_ω we have that idempotents in S commute and moreover $\mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{b^{-1}a} \cdot \mathscr{F}_{a^{-1}b} = \mathscr{F}_{b^{-1}a}$. This implies that S is an inverse semigroup. Also, for every open basic neighbourhood $U_n(\mathscr{F}_{a^{-1}b})$ of the point $\mathscr{F}_{a^{-1}b}$ in (S,τ) we have that $(U_n(\mathscr{F}_{a^{-1}b}))^{-1} = U_n(\mathscr{F}_{b^{-1}a})$ for all $n \in \mathbb{N}$ and hence the inversion in (S,τ) is continuous.

ACKNOWLEDGEMENTS

Authors are very gratefull to Taras Banakh for his comments and suggestions.

REFERENCES

[1] Aleksandrov A.D. *On an extension of Hausdorff space to H-closed*. Dokl. Akad. Nauk SSSR 1942, **37**, 138–141. (in Russian)

[2] Andersen O. Ein Bericht über die Struktur abstrakter Halbgruppen. PhD Thesis, Hamburg, 1952.

[3] Arhangel'skii A., Choban M. *Semitopological groups and the theorems of Montgomery and Ellis*. C. R. Acad. Bulg. Sci. 2009, **62** (8), 917–922.

[4] Banaschewski B. *Minimal topological algebras*. Math. Ann. 1974, **211** (2), 107–114. doi:10.1007/BF01344165

[5] Bardyla S., Gutik O. *On $\mathscr{H}$-complete topological semilattices*. Mat. Stud. 2012, **38** (2), 118–123.

[6] Bardyla S., Gutik O. *On a semitopological polycyclic monoid*. Algebra Discr. Math. 2016, **21** (2), 163–183.

[7] Bardyla S., Gutik O., Ravsky A. *H-closed quasitopological groups*. Topology Appl. (arXiv: 1506.08320) (to appear)

[8] Berezovski T., Gutik O., Pavlyk K. *Brandt extensions and primitive topological inverse semigroups*. Int. J. Math. Math. Sci. 2010, **2010**, Article ID 671401, 12 pages. doi:10.1155/2010/671401

[9] Berri M.P., Porter J.R., Stephenson R.M., Jr. *A survey of minimal topological spaces*. In: Franklin S.P., Frolik Z., Koutnik V. (Eds.) Proc. of Topol. Conf. "General Topology and its Relations to Modern Analisys and Algebra". Kanpur, India, 1968. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1971, 93–114.

[10] Carruth J.H., Hildebrant J.A., Koch R.J. The theory of topological semigroups. Marcel Dekker, Inc., New York – Basel, Vol. I 1983, Vol. II 1986.

[11] Chuchman I., Gutik O. *On H-closed topological semigroups and semilattices*. Algebra Discr. Math. 2007, **1**, 13–23.

[12] Clifford A.H., Preston G.B. The algebraic theory of semigroups. Amer. Math. Soc. Surveys 7, Providence, R.I., Vol. I 1961, Vol. II 1967.

[13] Dikranjan D.N., Uspenskij V.V. *Categorically compact topological groups*. J. Pure Appl. Algebra 1998, **126** (1–3), 149–168. doi:10.1016/S0022-4049(96)00139-9

[14] Doïtchinov D. *Produits de groupes topologiques minimaux*. Bull. Sci. Math. (2) 1972, **97**, 59–64.

[15] Eberhart C., Selden J. *On the closure of the bicyclic semigroup*. Trans. Amer. Math. Soc. 1969, **144**, 115–126. doi:10.2307/1995273

[16] Engelking R. General topology. 2nd ed., Heldermann, Berlin, 1989.

[17] Green J.A. *On the structure of semigroups*. Ann. Math. (2) 1951, **54** (1), 163–172. doi:10.2307/1969317

[18] Gutik O. *On closures in semitopological inverse semigroups with continuous inversion*. Algebra Discr. Math. 2014, **18** (1), 59–85.

[19] Gutik O., Lawson J., Repovš D. *Semigroup closures of finite rank symmetric inverse semigroups*. Semigroup Forum 2009, **78** (2), 326–336. doi:10.1007/s00233-008-9112-2

[20] Gutik O., Pagon D., Repovš D. *On chains in H-closed topological pospaces*. Order 2010, **27** (1), 69–81. doi:10.1007/s11083-010-9140-x

[21] Gutik O.V., Pavlyk K.P. *H-closed topological semigroups and topological Brandt λ-extensions*. Mat. Metody Fiz.-Mekh. Polya 2001, **44** (3), 20–28. (in Ukrainian)

[22] Gutik O., Pavlyk K. *Topological Brandt λ-extensions of absolutely H-closed topological inverse semigroups*. Visn. L'viv. Univ., Ser. Mekh.-Mat. 2003, **61**, 98–105.

[23] Gutik O.V., Pavlyk K.P. *On topological semigroups of matrix units*. Semigroup Forum 2005, **71** (3), 389–400. doi:10.1007/s00233-005-0530-0

[24] Gutik O., Pavlyk K., Reiter A. *Topological semigroups of matrix units and countably compact Brandt λ^0-extensions*. Mat. Stud. 2009, **32** (2), 115–131.

[25] Gutik O.V., Reiter A.R. *Symmetric inverse topological semigroups of finite rank* $\leqslant n$. J. Math. Sci. 2010, **171** (4), 425–432. doi:10.1007/s10958-010-0147-z (translation of Mat. Metody Fiz.-Mekh. Polya 2009, **52** (3), 7–14 (in Ukrainian))

[26] Gutik O., Reiter A. *On semitopological symmetric inverse semigroups of a bounded finite rank*. Visn. L'viv. Univ., Ser. Mekh.-Mat. 2010, **72**, 94–106. (in Ukrainian)

[27] Gutik O., Repovš D. *On linearly ordered H-closed topological semilattices*. Semigroup Forum 2008, **77** (3), 474–481. doi:10.1007/s00233-008-9102-4

[28] Higgins P.M. Techniques of semigroup theory. Oxford Univ. Press, New York, 1992.

[29] Jones D.G. Polycyclic monoids and their generalizations. PhD Thesis, Heriot-Watt University, 2011.

[30] Jones D.G., Lawson M.V. *Graph inverse semigroups: Their characterization and completion*. J. Algebra 2014, **409**, 444–473. doi:10.1016/j.jalgebra.2014.04.001

[31] Lawson M. Inverse semigroups. The theory of partial symmetries. World Scientific, Singapore, 1998.

[32] Mesyan Z., Mitchell J.D., Morayne M., Péresse Y.H. *Topological graph inverse semigroups*. Topology Appl. 2016, **208**, 106–126. doi:10.1016/j.topol.2016.05.012

[33] Nachbin L. *On strictly minimal topological division rings*. Bull. Amer. Math. Soc. 1949, **55**, 1128–1136. doi:10.1090/S0002-9904-1949-09339-4

[34] Nivat M., Perrot J.-F. *Une généralisation du monoide bicyclique*. C. R. Acad. Sci., Paris, Sér. A 1970, **271**, 824–827.

[35] Petrich M. Inverse semigroups. John Wiley & Sons, New York, 1984.

[36] Raikov D. A. *On a completion of topological groups*. Izv. Akad. Nauk SSSR 1946, **10**, 513–528. (in Russian)

[37] Ravsky O. *On H-closed paratopological groups*. Visn. L'viv. Univ., Ser. Mekh.-Mat. 2003, **61**, 172–179.

[38] Stephenson R.M., Jr. *Minimal topological groups*. Math. Ann. 1971, **192**, 193–195. doi:10.1007/BF02052870

[39] Stepp J.W. *A note on maximal locally compact semigroups*. Proc. Amer. Math. Soc. 1969, **20** (1), 251–253. doi:10.2307/2036002

[40] Stepp J.W. *Algebraic maximal semilattices*. Pacific J. Math. 1975, **58** (1), 243–248.

Received 30.06.2016

Бардила С.О., Гутік О.В. *Про повний топологічний інверсний поліциклічний моноїд* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 183–194.

Вказано достатні умови, за яких топологічний інверсний λ-поліциклічний моноїд P_λ є абсолютно H-замкненим в класі топологічних інверсних напівгруп. Для довільного нескінченного кардиналу λ побудовано найслабшу напівгрупову інверсну топологію τ_{mi} на P_λ та наведено приклад топологічного інверсного моноїда S, що містить поліциклічний моноїд P_2 як щільну дискретну піднапівгрупу.

Ключові слова і фрази: інверсна напівгрупа, біциклічний моноїд, поліциклічний, вільний моноїд, напівгрупа матричних одиниць, топологічна напівгрупа, топологічна інверсна напівгрупа, мінімальна топологія.