ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2016, 8 (1), 158–162
doi:10.15330/cmp.8.1.158-162

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2016, Т.8, №1, С.158–162

ROMANIV O.M., SAGAN A.V.

ω-EUCLIDEAN DOMAIN AND LAURENT SERIES

It is proved that a commutative domain R is ω-Euclidean if and only if the ring of formal Laurent series over R is ω-Euclidean domain. It is also proved that every singular matrice over ring of formal Laurent series R_X are products of idempotent matrices if R is ω -Euclidean domain.

Key words and phrases: ω-Euclidean domain, formal Laurent series, idempotent matrices.

Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: oromaniv@franko.lviv.ua (Romaniv O.M.), andrijsagan@gmail.com (Sagan A.V.)

INTRODUCTION

Let R will always denote a commutative domain with nonzero unit element. Let $\varphi : R \to \mathbb{Z}$ be a norm satisfying $\varphi(0) = 0$, $\varphi(a) > 0$ for $a \neq 0$, and $\varphi(ab) \geq \varphi(a)$.

Definition 1. *Domain R is called Euclidean if for any $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that*

$$a = bq + r \quad \text{and} \quad \varphi(r) < \varphi(b).$$

Let $a, b \in R$, $b \neq 0$, and k be an arbitrary positive integer. We talk about *k-term divisibility chain* [7] if there exists a finite sequence of equalities

$$a = bq_1 + r_1, b = r_1 q_2 + r_2, \dots, r_{k-2} = r_{k-1} q_k + r_k. \tag{1}$$

Definition 2. *Domain R is called ω-*Euclidean ring [7] *relatively to norm $\mathbb{N}$, if for every pair of elements $a, b \in R$, $b \neq 0$ can be found $k \in \mathbb{N}$ and such divisibility chain* (1) *of length k that*

$$\varphi(r_k) < \varphi(b).$$

Clearly, 1-Euclidean domain is an Euclidean domain. Now let $R_X = R[[X]][X^{-1}]$ be the ring of formal Laurent series with coefficient in R. P. Samuel in [6] proved that if R_X is euclidean, R is so. Also F. Dress proved the converse in [3]. Also in [1] it is proved similar results are for 2-Euclidean domain.

УДК 512.552.13
2010 *Mathematics Subject Classification:* 15S50, 16U80.

MAIN RESULTS

Let R be an integral domain with a norm map $\varphi : R \to \mathbb{Z}$ and let $R_X = R[[X]][X^{-1}]$ be the ring of formal Laurent series with coefficient in R.

For any element

$$f = \sum_{i \geq h} a_i X^i \in R_X,\ a_i \in R,\ h \in \mathbb{Z},\ a_h \neq 0$$

we put a norm map $\psi : R_X \to R$ satisfying $\psi(f) = a_h$ and $\psi(0) = 0$, where a_h be a variable coefficient in the lowest degree.

Proposition 1. *For any $f, g \in R_X$ with $g \neq 0$ we have that $f = gu$ or, $f = gu + v$, where $\psi(g) \nmid \psi(v)$.*

Proof. Let h (resp. k) be the lowest degree of f (resp. g). Set $\psi(f) = \psi(g)q + r$, where $q, r \in R$. Then we can write

$$v = f - qX^{h-k}g = rX^h + \text{higher degree terms}.$$

If $\psi(g) \nmid r$, we get $\psi(g) \nmid r = \psi(v)$.

If $\psi(g) \mid r$, we similarly construct $v_1 = v - q_1 X^{h_1 - k} g$, ($h_1 =$ order of v) and so on. If the process stops after a finite number of steps, we obtain

$$f = gu + v, \qquad \psi(g) \nmid \psi(v).$$

Otherwise the infinite sum

$$u = qX^{h-k} + q_1 X^{h_1-k} + \cdots + q_n X^{h_n-k} + \cdots$$

is true sense, and we obtain $f = gu$. ☐

Let a map $\varphi_x : R \to \mathbb{Z}$ by $\varphi_x(f) = \varphi(\psi(f))$. Then we obtain the following.

Theorem 1. *If R is ω-Euclidean domain with respect to φ, then R_X is ω-Euclidean domain with respect to $\varphi_x = \varphi \cdot \psi$.*

Proof. By Proposition 1 for any $f, g \in R_X$ with $g \neq 0$ we have the following:

(1) $f = gu$, or

(2) $f = gu + v$, $\psi(g) \nmid \psi(v)$.

It is obvious that the case (1), R_X is Euclidean domain and thus R is ω-Euclidean.

In the case of (2) review:

a) if $\varphi(\psi(v)) < \varphi(\psi(g))$, then we have $\varphi_x(v) < \varphi_x(g)$ by definition, R_X is Euclidean domain and thus R is ω-Euclidean;

b) if $\varphi(\psi(v)) \geq \varphi(\psi(g))$, then

$$\psi(v) = \psi(g)q_1 + r_1,\ \psi(g) = r_1 q_2 + r_2, \ldots, r_{k-2} = r_{k-1} q_k + r_k, \tag{2}$$

and $\varphi(r_k) < \varphi(\psi(g))$, because R is ω-Euclidean domain.

Now if we set

$$v - q_1 X^{h_1-k} g = v_1, \quad (h_1 \text{ - order of } v),$$

we have $f = (u + q_1X^{h_1-k})g + v_1$ and $\psi(v_1) = r_1$. If we set

$$g - q_2X^{k-h_2}v_1 = v_2, \quad (h_2 \text{ - order of } v_1),$$

we have $g = q_2X^{k-h_2}v_1 + v_2$ and $\psi(v_2) = r_2$. Continuing this process in the k step we get

$$v_{k-2} - q_kX^{h_{k-1}-h_k}v_{k-1} = v_k, \quad (h_k \text{ — order of } v_{k-1}),$$

then $v_{k-2} = q_kX^{h_{k-1}-h_k}v_{k-1} + v_k$ and $\psi(v_k) = r_k$. If $r_k \neq 0$, we obtain

$$f = (u + q_1X^{h_1-k})g + v_1,\ g = q_2X^{k-h_2}v_1 + v_2, \dots, v_{k-2} = q_kX^{h_{k-1}-h_k}v_{k-1} + v_k,$$

and

$$\varphi_x(g) = \varphi(\psi(g)) > \varphi(r_k) = \varphi_x(v_k).$$

If $r_k = 0$, we have $r_{k-2} = r_{k-1}q_k$. Then we have.

If $\varphi(\psi(g)) > \varphi(r_{k-1})$, we obtain $(k-1)$-term divisibility chain, because

$$\varphi(\psi(g)) = \varphi_x(g) > \varphi_x(v_{k-1}) = \varphi(r_{k-1}).$$

On the other hand, since $\varphi(r_{k-1}) \geq \varphi(\psi(g))$, then with (2) we get $\psi(g) = r_{k-1}m$, where $m \in R$. Then $\varphi(m) = 1$.

Hence,

$$r_{k-1} = \psi(g)m^{-1}$$

and

$$\psi(v) = \psi(g)x,$$

for some $x \in R$. This is contradictory to for $\psi(g) \nmid \psi(v)$. □

Theorem 2. *If R_X is ω-Euclidean domain with respect to φ_x, then R is ω-Euclidean domain with respect to φ.*

Proof. Let $a, b \in R$, where $b \neq 0$. Since R_X is ω-Euclidean domain, there exist such $q_1, \dots, q_n, r_1, \dots, r_n \in R_X$ that

$$a = bq_1 + r_1, b = r_1q_2 + r_2, \dots, r_{n-2} = r_{n-1}q_n + r_n, \tag{3}$$

where $\varphi_x(r_n) < \varphi_x(b)$.

Note that

$$q_i = q'_{k_i}X^{k_i} + \text{higher degree terms}, \quad r_i = r'_{s_i}X^{s_i} + \text{higher degree terms}$$

(1) Let $\varphi_x(r_1) < \varphi_x(b)$. If $k_1 < 0$, we have $k_1 = s_1$ and $bq'_{k_1} + r'_{s_1} = 0$, and hence $\varphi_x(r_1) = \varphi(r'_{s_1}) = \varphi(-bq'_{k_1}) \geq \varphi(b) = \varphi_x(b)$. This is a contradiction. Therefore we get $k_1 \geq 0$, then $a = bq'_{k_0} + r'_{s_0}$, $\varphi(r'_{s_0}) = \varphi_x(r_1) < \varphi_x(b) = \varphi(b)$.

(2) Let $\varphi_x(r_1) \geq \varphi_x(b)$. If $s_1 + k_2 < 0$, we get $s_1 + k_2 = s_2$ and $r'_{s_1}q'_{k_2} + r'_{s_2} = 0$ and note that a chain 3 we get $r_n = r_1x^* + r_2y^*$ for some $x^*, y^* \in R_X$. Then $\varphi_x(r_n) = \varphi_x(r_1x^* + r_2y^*) = \varphi((x^* - q'_{k_2}y^*)r'_{s_1}) \geq \varphi(r'_{s_1}) \geq \varphi_x(b)$.

Hence $\varphi_x(r_n) < \varphi_x(b)$, this is contradiction and we get $s_1 + k_2 \geq 0$. Then we can consider possibility.

Case 1) $r'_{s_2} \neq 0$.

If $k_1 < 0$, we get $bq'_{k_1} + r'_{s_1} = 0$. On the other hand with chain 3 we have $r_n = bx + r_1 y$, for some $x, y \in R_X$,

$$\varphi_x(r_n) = \varphi_x(bx + r_1 y) = \varphi((x' - q'_{k_1} y')b) \geq \varphi(b) = \varphi_x(b).$$

This is contradiction, because $\varphi_x(r_n) < \varphi_x(b)$. Hence we have $k_1 \geq 0$. The we obtain

$$a = bq'_{k_1} + r'_{s_1}, b = r'_{s_1} q'_{k_2} + r'_{s_2}, \ldots, r'_{s_{n-2}} = r'_{s_{n-1}} q'_{k_n} + r'_{s_n},$$

where $\varphi_x(r_n) = \varphi(r'_{s_n}) < \varphi(b) = \varphi_x(b)$.

Case 2) $r'_{s_2} = 0$.

In this case, we distinguish now two subcases.

1') If $k_1 \geq 0$, it is obvious that

$$a = bq'_{k_1} + r'_{s_1}, b = r'_{s_1} q'_{k_2} + 0,$$

and $\varphi(0) < \varphi(b)$.

2') If $k_1 < 0$ we have $k_1 = s_1 < 0$ and $bq'_{k_1} + r'_{s_1} = 0$.

On the other hand, since $b = r'_{s_1} q'_{k_2}$ we have $r'_{s_1} q'_{k_1} q'_{k_2} + r'_{s_1} = 0$ i $q'_{k_1} q'_{k_2} + 1 = 0$ and hence q'_{k_1}, q'_{k_2} are units. Then we can obtain:

$$\begin{aligned} b &= (r'_{s_1} X^{s_1} + \cdots)(q'_{k_1} X^{k_1} + \cdots) + (r'_{s_2} X^{s_2} + \cdots) = r'_{s_1} q'_{k_2} \\ &+ (r'_{s_1} q'_{k_2+1} + r'_{s_1+1} q'_{k_2})X + (r'_{s_1} q'_{k_2+2} + r'_{s_1+1} q'_{k_2+1} + r'_{s_1+2} q'_{k_2})X^2 + \cdots + (r'_{s_2} X^{s_2} + \cdots). \end{aligned}$$

Therefore we get the following equations:

$$\begin{cases} r'_{s_1} q'_{k_2+1} + r'_{s_1+1} q'_{k_2} = 0, \\ r'_{s_1} q'_{k_2+2} + r'_{s_1+1} q'_{k_2+1} + r'_{s_1+2} q'_{k_2} = 0, \\ \cdots\cdots\cdots\cdots \\ r'_{s_1} q'_{k_2+s_2} + r'_{s_1+1} q'_{k_2+s_2-1} + \cdots + r'_{s_1+s_2} q'_{k_2} + r'_{s_2} = 0. \end{cases} \tag{4}$$

Since q'_{k_1} is a unit, we have

$$r'_{s_1+1} = (q'_{k_1})^{-1} r'_{s_1} q'_{k_1+1} = (q'_{k_1})^{-1} q'_{k_1+1} (q'_{k_2})^{-1} b.$$

Hence we get $b \mid r'_{s_1+1}$. Similarly, we have

$$b \mid r'_{s_1+2}, \cdots, r'_{s_1+s_2-1}.$$

Then if $s_1 + s_2 < 0$, we have $bq'_{s_1+s_2} + r'_{s_1+s_2} = 0$ and hence $b \mid r'_{s_1+1}$. By above equations (4), $b \mid r'_{s_2}$ and $\varphi(r'_{s_2}) \geq \varphi(b)$. This is a contradiction with $\varphi(r'_{s_2}) < \varphi(b)$. Therefore we get $s_1 + s_2 \geq 0$.

Now, if $s_1 + s_2 > 0$, there exist an integer h such that $r'_{s_1} q'_{k_2+h} + r'_{s_1+h} q'_{k_2} = 0$ and $b \mid r'_{s_1+h} = r'_0$. Hence we obtain $a = bq'_0 + r'_0 = bq^*$.

If $s_1 + s_2 = 0$, the equation (4) we have

$$r'_{s_1} q'_{k_2+s_2} + \cdots + r'_{s_1+s_2} q'_{k_2} = r'_{s_1} q'_{k_2+s_2} + \cdots + (a - bq'_0) q'_{k_2} + r'_{s_2} = 0.$$

Then we obtain

$$a = bq'_0 + (q'_{k_2})^{-1}(-r'_{s_1} q'_{k_2+s_2} - \cdots - r'_{s_2}) = bq' + (q'_{k_2})^{-1}(-r'_{s_2})$$

and $\varphi((q'_{k_2})^{-1}(-r'_{s_2})) = \varphi(r'_{s_2}) < \varphi(b)$. □

As a consequent we obtain the following.

Theorem 3. *R is ω-Euclidean domain if and only if R_X is ω-Euclidean domain.*

A ring R has IP_n-property, if every square singular matrix of n order over R is a product of idempotent matrices. If this is true for any singular matrix over R, then the ring R has IP-property.

Theorem 4. *Let R is Bezout domain with IP_2-property, then R_X is a domain with IP-property.*

Proof. Let R be Bezout domain with IP_2-property, then R is GE_2-ring [4]. Since the condition GE_2-ring over Bezout domain implies the presence of the infinite divisibility chain for any two elements with R, hence R is ω-Euclidean domain. According to Theorem 1, R_X is ω-Euclidean domain, then from [2] for any two elements of R_X there exists the infinite divisibility chain. Then, according to Theorem 6.2 and Proposition 2.4 of [5] implies that R_X has IP-property. □

Given from theorem 2, consequently the following result is true.

Theorem 5. *Let R_X — ω-Euclidean domain, then R has IP-property.*

REFERENCES

[1] Amano K. *On 2-stage euclidean ring and Laurent series*. Bull. Fac. Gen. Ed. Gifu Univ. 1986, **22**, 83–86.

[2] Cooke G. E. *A weakening of the euclidean property for integral domains and applications to algebraic number theory. I*. J. Reine Angew. Math. 1976, **282**, 133–156.

[3] Dress F. *Stathmes euclidiens et series formelles*. Acta Arith. 1971, **19**, 261–265.

[4] Ruitenberg W. *Products of idempotent matrices over Hermite domains*. Semigroup Forum 1993, **46** (1), 371–378. doi:10.1007/BF02573579

[5] Salce L., Zanardo P. *Products of elementary and idempotent matrices over integral domains*. Linear Algebra Appl. 2014, **452**, 130–152. doi:10.1016/j.laa.2014.03.042

[6] Samuel P. *About Euclidean Rings*. J. Algebra 1971, **19**, 282–301.

[7] Zabavskii B.V., Romaniv O.M. *Noncommutative rings with elementary reduction of matrices*. Visn. Lviv Univ. 1998, **49**, 16–20.

Received 20.04.2016

Романів О.М., Саган А.В. *ω-евклідові області і Лоранові ряди* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 158–162.

Доведено, що комутативна область є ω-евклідовою тоді і тільки тоді, коли кільце формальних Лоранових рядів є ω-евклідовою областю. Також показано, що довільна особлива матриця над кільцем формальних Лоранових рядів R_X є добутком ідемпотентних матриць, якщо R є ω-евклідове кільце.

Ключові слова і фрази: ω-евклідова область, кільце формальних Лоранових рядів, ідемпотентні матриці.