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# APPROXIMATION RELATIONS ON THE POSETS OF PSEUDOMETRICS AND OF PSEUDOULTRAMETRICS

We show that non-trivial "way below" and "way above" relations on the posets of all pseudometrics and of all pseudoultrametrics on a fixed set $X$ are possible if and only if the set $X$ is finite.

*Key words and phrases:* pseudometric, pseudoultrametric, way below, way above.

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## INTRODUCTION

It turned out (see [1]) that partial orders are closely related to topologies, in particular, a "decent" ordering of a set determines quite natural and useful topologies, e.g., Scott topology, upper/lower topology, Lawson topology etc. For these topologies to have nice properties, the original order has to satisfy certain requirements, mostly related to approximation relations.

Recall that a poset $(D, \leq)$ is directed (resp. filtered) if for all $d_1, d_2 \in D$ there is $d \in D$ such that $d_1, d_2 \leq d$ (resp. $d_1, d_2 \geq d$).

**Definition 1.** *An element $x_0$ is called to be way below an element $x_1$ (or approximates $x_1$ from below) in a poset $(X, \leq)$ (denoted $x_0 \ll x_1$) if for every non-empty directed subset $D \subset X$ such that $x_1 \leq \sup D$ there is an element $d \in D$ such that $x_0 \leq d$.*

**Definition 2.** *An element $x_0$ is called to be way above an element $x_1$ (or approximates $x_1$ from above) in a poset $(X, \leq)$ (denoted $x_0 \gg x_1$) if for every non-empty filtered subset $D \subset X$ such that $x_1 \geq \inf D$ there is an element $d \in D$ such that $x_0 \geq d$.*

Obviously $x_0 \ll x_1$ or $x_0 \gg x_1$ imply respectively $x_0 \leq x_1$ or $x_0 \geq x_1$ (see more in [1]).

A poset is called continuous (dually continuous) if each element is the least upper bound of all elements approximating it from below (resp. the greatest lower bound of all elements approximating it from above).

We are going to apply the above apparatus to the set of all pseudometrics on a fixed set, and to its subset that consists of all pseudoultrametrics. Ultrametrics (or non-Archimedean metrics [2]) are studied since the beginning of XX century, cf. a review in [3]. They found numerous applications, e.g., in computer science.

Monotone families of (pseudo-)ultrametrics were studied in [4], but approximation relations were out of the scope of the latter paper.

The following notion is a natural mixture of ones of ultrametric and pseudometric.

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**Definition 3.** *A mapping $d : X \times X \to \mathbb{R}$, that satisfies the conditions:*

- $d(x,y) \geq 0$ *for all $x, y \in X$ (nonnegativeness);*
- $d(x,x) = 0$ *for all $x \in X$ (identity);*
- $d(x,y) = d(y,x)$ *for all $x, y \in X$ (symmetry);*
- $d(x,y) \leq \max\{d(y,z), d(z,x)\}$ *for all $x, y, z \in X$ (strong triangle inequality);*

*is called a pseudoultrametric on the set $X$.*

It is just a pseudometric such that the usual triangle inequality $d(x,y) \leq d(y,z) + d(z,x)$ holds in a stronger form.

The main results of this paper are somewhat disappointing, but they show that, to obtain meaningful theory of approximation, narrower classes of pseudometrics should be considered.

## 1 POSET OF PSEUDOMETRICS

We denote by $Ps(X)$ the set of all pseudometrics on a set $X$. The partial order on $Ps(X)$ is defined pointwise: a pseudometric $d_1$ precedes a pseudometric $d_2$ (written $d_1 \leq d_2$) if $d_1(x,y) \leq d_2(x,y)$ holds for all points $x, y \in X$.

Obviously the trivial pseudometric $d \equiv 0$ is the least element of $Ps(X)$, hence $Ps(X)$ is bounded from below. The greatest lower bound for two pseudometrics is described with the following statement.

**Lemma 1.** *For $d_1, d_2 \in Ps(X)$ the function*

$$d_*(x,y) = \inf\left\{ \sum_{k=0}^{n-1} \{\min\{d_1(t_k, t_{k+1}), d_2(t_k, t_{k+1})\}\} | n \in \mathbb{N}, x = t_0, \{t_1, ..., t_{n-1}\} \subset X, t_n = y \right\}$$

*is the infimum of $d_1$, $d_2$ in the set $Ps(X)$.*

*Proof.* Properties of symmetry and identity clearly hold for $d_*$. To verify the triangle inequality

$$d_*(x,y) \leq d_*(x,z) + d_*(z,y),$$

recall that (after renumbering points in the second sum)

$$\begin{aligned}
d_*(x,z) + d_*(z,y) = \inf\{ &\sum_{k=1}^{m} \{\min\{d_1(t_{k-1}, t_k), d_2(t_{k-1}, t_k)\} | \\
& m \in \mathbb{N}, t_0, t_1, \ldots, t_m \in X, x = t_0, t_m = z\} \\
+ \inf\{ &\sum_{k=m+1}^{n} \min\{d_1(t_{k-1}, t_k), d_2(t_{k-1}, t_k)\} | \\
& m, n \in \mathbb{N}, 1 \leqslant m \leqslant n-1, t_m, \ldots, t_{n-1}, t_n \in X, t_m = z, t_n = y\} \\
\geq \inf\{ &\sum_{k=1}^{n} \{\min\{d_1(t_{k-1}, t_k), d_2(t_{k-1}, t_k)\} | \\
& m, n \in \mathbb{N}, 1 \leqslant m \leqslant n-1, t_0, \ldots, t_{n-1}, t_n \in X, t_0 = x, t_m = z, t_n = y\} \\
\geq \inf\{ &\sum_{k=1}^{n} \{\min\{d_1(t_{k-1}, t_k), d_2(t_{k-1}, t_k)\} | \\
& n \in \mathbb{N}, t_0, \ldots, t_{n-1}, t_n \in X, t_0 = x, t_n = y\} = d_*(x,y).
\end{aligned}$$

Hence $d_* \in Ps(X)$.

The simplest sequence $t_0, t_1, \ldots, d_n$ that satisfies the above conditions is $t_0 = x$, $t_1 = y$ (for $n = 1$). It implies $d_*(x,y) \leq \min\{d_1(x,y), d_2(x,y)\}$, i.e., $d_*$ is a lower bound of the pseudometrics $d_1, d_2$.

Show that $d_*$ is the greatest lower bound. For all $x, y \in X$ and $d' \in Ps(X)$ such that $d' \leq d_1$, $d' \leq d_2$ we obtain

$$
\begin{aligned}
d'(x,y) &= \inf\{\sum_{k=1}^{n} d'(t_{k-1}, t_k) | n \in \mathbb{N}, t_0, \ldots, t_{n-1}, t_n \in X, t_0 = x, t_n = y\} \\
&\leqslant \inf\{\sum_{k=1}^{n} \{\min\{d_1(t_{k-1}, t_k), d_2(t_{k-1}, t_k)\} | n \in \mathbb{N}, t_0, \ldots, t_{n-1}, t_n \in X, t_0 = x, t_n = y\} \\
&= d_*(x,y).
\end{aligned}
$$

□

The least upper bound of pseudometrics $d_1, d_2$ is the pointwise minimum $d^*(x,y) = \max\{(d_1(x,y), d_2(x,y)\}$ for all $x, y \in X$, thus $Ps(X)$ is a lattice with the least element $d \equiv 0$, but obviously without a greatest element for $|X| > 1$. Being a lattice, $Ps(X)$ is both directed and filtered.

This lattice is not distributive.

**Example 1.** *Consider, e.g., the set $X = \{x_1, x_2, x_3\}$ and the pseudometrics*

$$
d_1(a,b) = \begin{cases} 0, & \{a,b\} = \{x_2, x_3\} \text{ or } a = b, \\ 1 & \text{otherwise,} \end{cases}
$$

$$
d_2(a,b) = \begin{cases} 0, & \{a,b\} = \{x_1, x_3\} \text{ or } a = b, \\ 1 & \text{otherwise,} \end{cases}
$$

$$
d_3(a,b) = \begin{cases} 0, & \{a,b\} = \{x_1, x_2\} \text{ or } a = b, \\ 1 & \text{otherwise,} \end{cases}
$$

*for all $a, b \in X$. Then*

$$
d_1 \vee d_2(a,b) = \begin{cases} 0, & a = b, \\ 1 & \text{otherwise,} \end{cases} \quad \text{hence } (d_1 \vee d_2) \wedge d_3 = d_3.
$$

*On the other hand*

$$
d_1 \wedge d_3 = d_2 \wedge d_3 \equiv 0, \text{ hence } (d_1 \wedge d_3) \vee (d_2 \wedge d_3) \equiv 0.
$$

*Therefore $(d_1 \vee d_2) \wedge d_3 \neq (d_1 \wedge d_3) \vee (d_2 \wedge d_3)$.*

Not having a greatest element, the lattice $Ps(X)$ cannot be complete. Nevertheless, it is straightforward to verify that $Ps(X)$ is a conditionally complete upper semilattice, i.e., each non-empty set $D$ of pseudometrics that is bounded from above by a pseudometric $d_0$ has a supremum which is calculated pointwise: $(\sup D)(x,y) = \sup\{d(x,y) \mid d \in D\}$ for all $x, y \in X$. The latter supremum exists because the set in the curly braces is bounded by $d_0(x,y)$. The infimum of a set $D$ (which is always bounded from below by $d_0 \equiv 0$) is similar to the one in Lemma 1:

$$(\inf D)(x,y) = \inf\Big\{ \sum_{k=1}^{n} \inf\{d(t_{k-1},t_k) \mid d \in D\} \mid n \in \mathbb{N}, x = t_0, \{t_1,...,t_{n-1}\} \subset X, t_n = y \Big\}.$$

Thus $Ps(X)$ is a complete lower semilattice.

Let us start with a simple but important observation.

**Lemma 2.** *Let pseudometrics $d_0, d_1$ in $X$ be such that $d_0(x,y) \geq d_1(x,y) > 0$ for some $x,y \in X$. Then neither $d_0 \ll d_1$ nor $d_1 \gg d_0$ is valid.*

*Proof.* Choose the set $D = \{(1-\frac{1}{n}) \cdot d_1 | n \in \mathbb{N}\}$ of pseudometrics. It is directed, its supremum is equal to $d_1$, but $(1-\frac{1}{n}) \cdot d_1(x,y) < d_1(x,y) \leqslant d_0(x,y)$, hence $(1-\frac{1}{n})d_1 \not\geqslant d_0$, thus $d_0 \not\ll d_1$. Similarly the set $D' = \{(1+\frac{1}{n}) \cdot d_0 | n \in \mathbb{N}$ is filtered with the greatest lower bound $d_0$, but neither of its element precedes $d_1$, hence $d_1 \not\gg d_0$. ☐

It is easy to see that pseudometrics on a finite set are in the "way below" relation if and only if the above double inequality does not hold for all pairs of points.

**Proposition 1.** *For pseudometrics $d_0$ and $d_1$ on a finite set $X$ the following statements are equivalent:*

*(1) $d_0 \ll d_1$ in $Ps(X)$;*

*(2) $d_1 \gg d_0$ in $Ps(X)$;*

*(3) for all $x,y \in X$ either $d_0(x,y) = d_1(x,y) = 0$ or $d_0(x,y) < d_1(x,y)$ is valid.*

*Proof.* (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3) have already been proved. To show (3) $\Longrightarrow$ (1), assume that the condition of the theorem holds for some $d_0, d_1 \in Ps(X)$, and a directed set $D \subset Ps(X)$ is such that $\sup D \geqslant d_1$, hence $\sup\{d(x,y) \mid d \in D\} \geqslant d_1(x,y)$ for all $x,y \in X$. For all pairs $x,y \in X$ such that $d_0(x,y) \geqslant 0$ (and hence $d_1(x,y) > d_0(x,y)$) choose an element $d_{x,y} \in D$ such that $d_{x,y}(x,y) > d_0(x,y)$. The set of the chosen elements of $D$ is finite, $D$ is directed, hence there is $d \in D$ that succeeds all $d_{x,y}$. Obviously $d \geqslant d_0$, thus $d_0 \ll d_1$.

Proof of (3) $\Longrightarrow$ (2) is analogous. ☐

Unfortunately, for an infinite set $X$ conditions of the latter proposition are necessary but not sufficient.

**Example 2.** *Consider $X = \mathbb{N}$ with the standard metric $d(x,y) = |x-y|$ and the set of pseudometrics $D = \{d_i | i \in \mathbb{N}\}$,*

$$d_i(x,y) = \begin{cases} |x-y|, & x,y < i; \\ |x-i|, & x < i, y \geq i; \\ |i-y|, & x \geq i, y < i; \\ 0, & x,y \geq i. \end{cases}$$

*It is directed because $i \leq j$ implies $d_i \leq d_j$, and $\sup\{d_i \mid i \in \mathbb{N}\} = d$. For the metric $d' = \frac{1}{2}d$ and all points $x,y \in \mathbb{N}$ we have either $d'(x,y) = d(x,y) = 0$ or $d'(x,y) < d(x,y)$ but $d'(i,i+1) = \frac{1}{2} > d_i(i,i+1) = 0$, hence neither of $d_i$ succeeds $d'$.*

We describe a construction of a pseudometric that precedes a given one, and is obtained by "gluing" points. In what follows we denote $d(x,F) = \inf\{d(x,y) \mid y \in F\}$.

**Lemma 3.** *Let $d \in Ps(X)$ and subset $F \subset X$ be non-empty. Then the function $\acute{d}_F : X \times X \to \mathbb{R}$ that is determined with the formula*

$$\acute{d}_F(x,y) = \min\{d(x,y), d(x,F) + d(y,F)\}, \quad x, y \in X,$$

*is a pseudometric on $X$, and $\acute{d}_F \leq d$. If the set $F$ is bounded, then $d(x,y) - \acute{d}_F(x,y) \leq \operatorname{diam} F$ for all $x, y \in X$.*

*Proof.* Check the prorerties from the definition of pseudometrics for arbitrary $x, y, z \in X$:
(1) $\acute{d}_F(x,y) \geq 0$ because $d(x,y) \geq 0$ i $d(x,F) + d(y,F) \geq 0$.
(2) $\acute{d}_F(x,x) = \min\{d(x,x), d(x,F) + d(x,F)\} = 0$.
(3) $\acute{d}_F(x,y) = \min\{d(x,y), d(x,F) + d(y,F)\} = \min\{d(y,x), d(y,F) + d(x,F)\} = \acute{d}_F(y,x)$.
(4)

$$\begin{aligned}
&\acute{d}_F(x,z) + \acute{d}_F(z,y) \\
&\qquad = \min\{d(x,z), d(x,F) + d(z,F)\} + \min\{d(z,y), d(z,F) + d(y,F)\} \\
&\qquad = \min\big\{d(x,z) + d(z,y), \big(d(x,z) + d(z,F)\big) + d(y,F), \\
&\qquad\qquad \big(d(z,y) + d(z,F)\big) + d(x,F), d(x,F) + d(z,F) + d(z,F) + d(y,F)\big\} \\
&\qquad \geq \min\{d(x,y), d(x,F) + d(y,F), d(y,F) + d(x,F), d(x,F) + d(y,F) + 2d(z,F)\} \\
&\qquad = \min\{d(x,y), d(x,F) + d(y,F)\}.
\end{aligned}$$

Thus $\acute{d}_F$ is a pseudometric.

Now for arbitrary $\varepsilon > 0$ choose $z, z' \in F$ such that $d(x,z) < d(x,F) + \varepsilon$, $d(y,z') < d(y,F) + \varepsilon$. Hence

$$\begin{aligned}
d(x,F) + d(y,F) &> d(x,z) + d(y,z') - 2\varepsilon \geqslant d(x,z) + d(y,z) - d(z,z') - 2\varepsilon \\
&\geqslant d(x,z) + d(y,z) - \operatorname{diam} F - 2\varepsilon \geqslant d(x,y) - \operatorname{diam} F - 2\varepsilon,
\end{aligned}$$

thus

$$\acute{d}_F(x,y) \geqslant d(x,y) - \operatorname{diam} F - 2\varepsilon,$$

then passing to the limit as $\varepsilon$ tends to 0 we obtain the required inequality. $\square$

**Theorem 1.** *For all pseudometrics $d_0, d_1$ on an infinite set $X$, $d_0 \gg d_1$ is not valid in $Ps(X)$. If $d_0 \not\equiv 0$, then $d_0 \ll d_1$ also does not hold.*

*Proof.* Let $d_0$ be way above $d_1$. Choose a sequence $x_1, x_2, \dots \in X$ of distinct points and put $\alpha_m = \max\{d_0(x_i, x_j) \mid 1 \leq i, j \leq m\} + m$ for all $m \in \mathbb{N}$. The sequence $(\alpha_m)_{m \in \mathbb{N}}$ is increasing, and the functions

$$\delta_m(a,b) = \begin{cases} 0, & a = b \text{ or } a, b \notin \{x_m, x_{m+1}, \dots\}, \\ \alpha_{\max\{i,j\}}, & a = x_i \neq b = x_j, i, j \geq m, \\ \alpha_i, & a = x_i, i \geq m, b \notin \{x_m, x_{m+1}, \dots\} \\ & \text{ or } b = x_i, i \geq m, a \notin \{x_m, x_{m+1}, \dots\}, \end{cases} \qquad a, b \in X,$$

are pseudometrics and even pseudoultrametrics. It is easy to see that $\delta_1 \geq \delta_2 \geq \dots$, $\inf\{\delta_m \mid m \in \mathbb{N}\} \equiv 0 \leqslant d_1$, but $\delta_m \nleqslant d_0$ (e.g., $\delta_m(x_m, x_{m+1}) = \alpha_{m+1} \geq d_0(x_m, x_{m+1})$). Therefore $d_0 \not\gg d_1$.

Assume now $d_0 \ll d_1$, $d_0 \not\equiv 0$. Choose a sequence $x_0, x_1, x_2, \dots \in X$ of distinct points such that $d_0(x_0, x_i) > 0$ for all $i \in \mathbb{N}$. Denote $F_i = \{x_0, x_i, x_{i+1}, x_{i+2}, \dots\}$, $i \geq 1$. Let $d'$ be the pseudometric on $X$:

$$d'(a,b) = \begin{cases} 0, & a, b \notin \{x_0, x_1, \dots\}, \\ |i-j|, & a = x_i, b = x_j, \\ i, & a = x_i, b \notin \{x_0, x_1, \dots\} \\ & \text{or } a \notin \{x_0, x_1, \dots\}, b = x_i, \end{cases} \qquad x, y \in X.$$

Show that the pseudometric $\rho = d_1 + d' \geqslant d_1$ is the least upper bound of the non-decreasing sequence of pseudometrics $\rho_i = \acute{\rho}_{F_i}$. Clearly $\rho(a, F_i \setminus \{x_0\}) \to \infty$ as $i \to \infty$ for all points $a \in X$, hence $\rho(a, F_i) \to \rho(a, x_0)$, and

$$\acute{\rho}_{F_i}(a,b) \to \min\{\rho(a,b), \rho(a,x_0) + \rho(b,x_0)\} = \rho(a,b).$$

On the other hand, none of $\rho_i$ succeeds $d_0$ because $\rho_i(x_0, x_i) = 0$ but $d_0(x_0, x_i) > 0$. Therefore $d_0$ is not way below $d_1$. □

Thus there is no non-trivial approximation in $Ps(X)$ for infinite $X$.

## 2 POSET OF PSEUDOULTRAMETRICS

Consider the subset $PsU(X) \subset Ps(X)$ that consists of all pseudoultrametrics on $X$, with the restriction of the partial order. It is also a lattice, with the meets (the pairwise infima) calculated pointwise as well, but the formula for the joins (the pairwise suprema) needs to be modified. For $d_1, d_2 \in PsU(X)$ the function

$$d_*(x,y) = \inf\{\max\{\min\{d_1(t_k, t_{k+1}), d_2(t_k, t_{k+1})\} \mid 0 \leq k \leq n-1\} \mid$$
$$n \in \mathbb{N}, x = t_0, \{t_1, ..., t_{n-1}\} \subset X, t_n = y\}$$

is the infimum of $d_1$, $d_2$ in the set $PsU(X)$. The formula for the infima of arbitrary sets is modified accordingly. The pseudometrics in Example 1 are pseudoultrametrics, hence the lattice $PsU(X)$ is not distributive as well.

*Mutatis mutandis* we obtain a similar result on approximation relations in $PsU(X)$ for a finite set $X$.

**Proposition 2.** *For pseudoultrametrics $d_0$ and $d_1$ on a finite set $X$ the following statements are equivalent:*

*(1) $d_0 \ll d_1$ in $PsU(X)$;*

*(2) $d_1 \gg d_0$ in $PsU(X)$;*

*(3) for all $x, y \in X$ either $d_0(x,y) = d_1(x,y) = 0$ or $d_0(x,y) < d_1(x,y)$ is valid.*

Nonetheless, the transfer of Theorem 1 to pseudoultrametrics is not so trivial. We need to modify Lemma 3.

**Lemma 4.** *Let $d \in PsU(X)$ and subset $F \subset X$ be non-empty. Then the function $\grave{d}_F : X \times X \to \mathbb{R}$ that is determined with the formula*

$$\acute{d}_F(x,y) = \min\{d(x,y), \max\{d(x,F), d(y,F)\}\}, \quad x, y \in X,$$

*is a pseudoultrametric on $X$, and $\grave{d}_F \leq d$. If the set $F$ is bounded, then $d(x,y) \leq \max\{\grave{d}_F(x,y), \operatorname{diam} F\}$ for all $x, y \in X$.*

*Proof.* Only the triangle inequality has to be verified. For arbitrary $x,y,z \in X$:
(4)

$$\begin{aligned}
&\max\{\grave{d}_F(x,z), \grave{d}_F(z,y)\\
&\qquad = \max\big\{\min\{d(x,z), \max\{d(x,F), d(z,F)\}\}, \min\{d(z,y), \max\{d(z,F), d(y,F)\}\}\big\}\\
&\qquad = \min\big\{\max\{d(x,z), d(z,y)\}, \max\{d(x,z), d(z,F), d(y,F)\},\\
&\qquad\qquad \max\{d(z,y), d(z,F), d(x,F)\}, \max\{d(x,F), d(z,F), d(z,F), d(y,F)\}\big\}\\
&\qquad \geq \min\big\{d(x,y), \max\{d(x,F), d(y,F)\}\big\}.
\end{aligned}$$

Thus $\grave{d}_F$ is a pseudoultrametric.

Now for arbitrary $\varepsilon > 0$ choose points $z, z' \in F$ such that $d(x,z) < d(x,F) + \varepsilon$, $d(y,z') < d(y,F) + \varepsilon$. Hence

$$\begin{aligned}
\max\{d(x,F), d(y,F)\} &\geq \max\{d(x,z) - \varepsilon, d(y,z') - \varepsilon\} = \max\{d(x,z), d(y,z')\} - \varepsilon\\
&\geqslant \max\{d(x,z), d(y,z), d(z,z')\} - \varepsilon,
\end{aligned}$$

thus

$$\begin{aligned}
&\max\{\operatorname{diam} F, \grave{d}_F(x,y)\}\\
&\qquad \geq \max\big\{\operatorname{diam} F, \min\{d(x,y), \max\{d(x,z), d(y,z), d(z,z')\} - \varepsilon\}\big\}\\
&\qquad = \min\big\{\max\{\operatorname{diam} F, d(x,y)\}, \max\{\operatorname{diam} F, d(x,z) - \varepsilon, d(y,z) - \varepsilon, d(z,z') - \varepsilon\}\big\}\\
&\qquad \geq \max\{\operatorname{diam} F, d(x,y)\} - \varepsilon
\end{aligned}$$

for all $\varepsilon > 0$, hence $\max\{\operatorname{diam} F, \grave{d}_F(x,y)\} \geq d(x,y)$. □

Now we are ready to prove

**Theorem 2.** *For all pseudoultrametrics $d_0, d_1$ on an infinite set $X$, $d_0 \gg d_1$ is not valid in $PsU(X)$. If $d_0 \not\equiv 0$, then $d_0 \ll d_1$ also does not hold.*

*Proof.* Recall that the pseudometrics $\delta_m$ used in the proof of Theorem 1 are pseudoultrametrics, hence the entire construction is applicable to proof of $d_0 \not\gg d_1$ in $PsU(X)$ as well.

Assume now $d_0 \ll d_1$, $d_0 \not\equiv 0$. Choose a sequence $x_0, x_1, x_2, \dots \in X$ of distinct points such that $d_0(x_0, x_i) > 0$ for all $i \in \mathbb{N}$. Put $\alpha_m = \max\{d_0(x_i, x_j) \mid 0 \leq i,j \leq m\} + m$ for all $m \geq 0$ (hence $\alpha_0 = 0$), and denote $F_i = \{x_0, x_i, x_{i+1}, x_{i+2}, \dots\}$ for all $i \in \mathbb{N}$. The formula

$$d'(a,b) = \begin{cases} 0, & a,b \notin \{x_0, x_1, \dots\} \text{ or } a = b,\\ \alpha_{\max\{i,j\}}, & a = x_i \neq b = x_j,\\ \alpha_i, & a = x_i, b \notin \{x_0, x_1, \dots\}\\ & \text{ or } a \notin \{x_0, x_1, \dots\}, b = x_i, \end{cases} \qquad x,y \in X,$$

defines a pseudoultrametric on $X$. Then the pseudoultrametric $\rho = \sup\{d_1, d'\} \geqslant d_1$ is the least upper bound of the non-decreasing sequence of pseudoultrametrics $\rho_i = \grave{\rho}_{F_i}$. Observe $\rho(a, F_i \setminus \{x_0\}) \to \infty$ as $i \to \infty$ for all points $a \in X$, hence $\rho(a, F_i) \to \rho(a, x_0)$, and

$$\grave{\rho}_{F_i}(a,b) \to \min\big\{\rho(a,b), \max\{\rho(a,x_0), \rho(b,x_0)\}\big\} = \rho(a,b).$$

Again, $\rho_i(x_0, x_i) = 0$ but $d_0(x_0, x_i) > 0$, hence $\rho_i \geq d_0$ is impossible, which contradicts to $d_0 \ll d_1$ in $PsU(X)$. □

Thus, for an infinite set $X$ the poset $PsU(X)$ is as poor in "way below" and "way above" relations as $Ps(X)$ is.

## 3 CONCLUSIONS

We have proved that the posets $Ps(X)$ and $PsU(X)$ have no nontrivial approximation relations, hence are not continuous or dually continuous. Therefore we shall restrict our attention to narrower classes of pseudometrics, namely to compact and locally compact pseudoultrametrics. This will be the topic of an upcoming publication.

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Никорович С. *Відношення апроксимації на частково впорядкованих множинах псевдометрик і псевдоультраметрик* // Карпатські матем. публ. — 2016. — Т.8, №1. — C. 150–157.

Ми доводимо, що нетривіальні відношення апроксимації знизу та апроксимації згори на частково впорядкованих множинах псевдометрик і псевдоультраметрик на фіксованій множині $X$ можливі, якщо і тільки якщо множина $X$ скінченна.

*Ключові слова і фрази:* псевдометрика, псевдоультраметрика, апроксимація знизу та згори.