ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2016, **8** (1), 16–20
doi:10.15330/cmp.8.1.16-20

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2016, Т.8, №1, С.16–20

BASIUK Y.V., TARASYUK S.I.

FOURIER COEFFICIENTS ASSOCIATED WITH THE RIEMANN ZETA-FUNCTION

We study the Riemann zeta-function $\zeta(s)$ by a Fourier series method. The summation of $\log|\zeta(s)|$ with the kernel $1/|s|^6$ on the critical line Re $s = \frac{1}{2}$ is the main result of our investigation. Also we obtain a new restatement of the Riemann Hypothesis.

Key words and phrases: Fourier coefficients, the Riemann zeta-function, Riemann Hypothesis.

Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: yuliya.basyuk.92@mail.ru (Basiuk Y.V.), svt.tarasyuk@gmail.com (Tarasyuk S.I.)

INTRODUCTION

It is known that the integral $\int\limits_{-\infty}^{\infty} \log\left|\zeta\left(\frac{1}{2}+it\right)\right| dt$, where $\zeta(s)$ is the Riemann zeta-function, diverges. M. Balazard, E.Saias, M. Yor [1] summed $\log|\zeta(s)|$ on the critical line with the kernel $1/|s|^2$. Using the fact that $f(z) = \frac{z}{1-z}\,\zeta\left(\frac{1}{1-z}\right)$, $|z| < 1$, belongs to the Hardy space $H^{\frac{1}{3}}$ and the result of Bercovici and Foias [2] on the factorization of $f(z)$, they have proved the following theorem.

Theorem ([1]).
$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^2}\,|ds| = \sum_{Re\,\rho_j>\frac{1}{2}} \log\left|\frac{\rho_j}{1-\rho_j}\right|,$$
where $\{\rho_j\}$ is the sequence of non-trivial zeroes of $\zeta(s)$.

In particular, the Riemann Hypothesis holds if and only if
$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^2}\,|ds| = 0.$$

A. Kondratyuk, P. Yatsulka [6], using the method of Fourier series, have established the following fact.

Theorem ([6]). *Let $\{\rho_j\}$ be the sequence of non-trivial zeroes of $\zeta(s)$. Then*
$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^4}\,|ds| = 1-\gamma+2\sum_{Re\,\rho_j>\frac{1}{2}} \log\left|\frac{\rho_j}{1-\rho_j}\right| + \sum_{Re\,\rho_j>\frac{1}{2}} \frac{(|\rho_j|^2 - Re\rho_j)(2Re\rho_j-1)}{|\rho_j(\rho_j-1)|^2},$$

УДК 517.53
2010 *Mathematics Subject Classification:* Primary 11M06, 30D99; Secondary 11M41.

where γ is the Euler constant. The Riemann Hypothesis holds if and only if

$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^4}\,|ds| = 1-\gamma.$$

We make the next step studying the behaviour of the Riemann zeta-function on the critical line. The summation of $\log|\zeta(s)|$ with the kernel $1/|s|^6$ on the critical line $\mathrm{Re}\ s = \frac{1}{2}$ is the main result of our research.

1 Section with results

Our result is the following.

Theorem 1. *Let $\{\rho_j\}$ be the sequence of non-trivial zeroes of $\zeta(s)$. Then*

$$\begin{aligned}\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds| = \frac{7}{2} - 4\gamma + \frac{\gamma_1-\gamma^2}{2} + 6\sum_{Re\,\rho_j>\frac{1}{2}} \log\left|\frac{\rho_j}{1-\rho_j}\right| \\ + 4\sum_{Re\,\rho_j>\frac{1}{2}} \frac{(|\rho_j|^2 - Re\rho_j)(2Re\rho_j-1)}{|\rho_j(\rho_j-1)|^2} \\ + \frac{1}{2}\sum_{Re\,\rho_j>\frac{1}{2}} \frac{Re(|\rho_j|^2-\overline{\rho}_j)^2(2Re\rho_j-1)(2|\rho_j|^2-2Re\rho_j+1)}{|\rho_j(\rho_j-1)|^4},\end{aligned} \tag{1}$$

where γ is the Euler constant,

$$\gamma_1 = -\lim_{N\to\infty}\left(\sum_{m\le N}\frac{1}{m}\log m - \frac{\log^2 N}{2}\right).$$

Also we obtain a new restatement of the Riemann Hypothesis.

Theorem 2. *The Riemann Hypothesis holds if and only if*

$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds| = \frac{7}{2} - 4\gamma + \frac{\gamma_1-\gamma^2}{2}. \tag{2}$$

Proof of Theorem 1. Observe that the conformal map $z = 1 - 1/s$ transforms the domain $\left\{s : \mathrm{Re}\ s > \frac{1}{2}\right\}$ onto the unit disc $\{z : |z| < 1\}$. Consider the function

$$f(z) = (s-1)\,\zeta(s) = \frac{z}{1-z}\,\zeta\left(\frac{1}{1-z}\right).$$

We have

$$(s-1)\,\zeta(s) = 1 + \gamma(s-1) + \gamma_1(s-1)^2 + \cdots + \gamma_k(s-1)^{k+1} + \ldots, \tag{3}$$

where

$$\gamma_k = \frac{(-1)^k}{k!}\lim_{N\to\infty}\left(\sum_{m\le N}\frac{1}{m}\log^k m - \frac{\log^{k+1} N}{k+1}\right),\ k\in\mathbb{N},$$

([5, p.4]). Therefore $f(z)$ is holomorphic in the unit disk. It was showed in [3] that the function $f(z)$ belongs to the Hardy class $H^p,\ 0<p<1$. Earlier it was established in [1] and [2] that the function $f(z)$ belongs to the Hardy class $H^{\frac{1}{3}}$ and $\sigma = 0$, where σ is the singular measure from the factorization (see [4])

$$f(z) = B(z) \cdot \exp(iC) \cdot \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\sigma(\varphi)\right) \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \log|f(e^{i\varphi})|\, d\varphi\right), \tag{4}$$

where

$$B(z) = \prod_j \frac{|a_j|}{a_j} \frac{a_j - z}{1-\overline{a}_j z}$$

is the Blaschke product, $\{a_j\}$ is the sequence of zeros of $f(z)$ and $C = \operatorname{Im} f(0)$ is a real constant.

Consider the Fourier coefficient of $\log|f(re^{i\theta})|$:

$$c_k(r,f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} \log|f(re^{i\theta})| d\theta,\ r \le 1.$$

Note that $c_{-k}(r,f) = \overline{c_k(r,f)}$.

It follows from (3) that $f(0) = 1$, and (4) yields

$$c_0(1,f) = -\log|B(0)| = \sum_j \log\frac{1}{|a_j|}$$

and

$$\log|f(re^{i\theta})| = \log|B(re^{i\theta})| + \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}} \log|f(e^{i\varphi})|\, d\varphi. \tag{5}$$

In some neighborhood of the origin, the function $F(z) = \log f(z)$, $\log f(0) = 0$, is holomorphic. Let $F(z) = A_1 z + A_2 z^2 + \dots$ be its Maclaurin expansion. According to (3)

$$A_1 = \gamma;\ A_2 = \frac{\gamma_1 - \gamma^2}{2}.$$

On the other hand,

$$\log|f(re^{i\varphi})| = \operatorname{Re}\log f(re^{i\varphi}) = \frac{F+\overline{F}}{2} = \frac{\gamma r(e^{i\varphi}+e^{-i\varphi})}{2} + \frac{(\gamma_1-\gamma^2)r^2(e^{2i\varphi}+e^{-2i\varphi})}{4} + \dots,$$

where r is sufficiently small.

The relation (5) implies, for small r,

$$\frac{\gamma_1-\gamma^2}{4} r^2 = c_{-2}(r,B) + r^2\, c_{-2}(1,f).$$

In [7], the expression for the Fourier coefficient of the Blaschke product was obtained

$$c_{-2}(r,B) = \frac{r^2}{4}\sum_{j=1}^{\infty} \frac{1}{\overline{a}_j^2}\left(|a_j|^4 - 1\right)$$

for $r < |a_1|$. Thus,

$$c_{-2}(1,f) = \frac{\gamma_1-\gamma^2}{4} - \frac{1}{4}\sum_{j=1}^{\infty}\frac{1}{\overline{a}_j^2}\left(|a_j|^4-1\right). \tag{6}$$

Note that

$$c_{-2}(1,f) = \frac{1}{4} + \frac{1}{2\pi}\int_0^{2\pi} e^{2i\theta} \log\left|\zeta\left(\frac{1}{1-e^{i\theta}}\right)\right| d\theta. \tag{7}$$

Return to the variable s. Taking (6) and (7) into account, we obtain

$$\begin{aligned}&\frac{1}{4} + \frac{1}{2\pi}\int\limits_{\operatorname{Re} s=\frac{1}{2}} \left(1-\frac{1}{s}\right)^2 \frac{\log|\zeta(s)|}{|s|^2}\,|ds|\\ &= \frac{\gamma_1-\gamma^2}{4} + \frac{1}{4}\sum_{\operatorname{Re}\rho_j>\frac{1}{2}} \frac{\overline{\rho}_j^{\,2}(2\operatorname{Re}\rho_j-1)(2|\rho_j|^2-2\operatorname{Re}\rho_j+1)}{(\overline{\rho}_j-1)^2\,|\rho_j|^4}.\end{aligned} \tag{8}$$

Taking the real parts of both sides (8), we get

$$\begin{aligned}&\frac{1}{4} + \frac{1}{2\pi}\int_{\operatorname{Re} s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^2}\,|ds| - \frac{1}{2\pi}\int_{\operatorname{Re} s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^4}\,|ds| + \frac{1}{2\pi}\int_{\operatorname{Re} s=\frac{1}{2}} \operatorname{Re}\left(\overline{s}^2\right)\frac{\log|\zeta(s)|}{|s|^6}\,|ds|\\ &= \frac{\gamma_1-\gamma^2}{4} + \frac{1}{4}\sum_{\operatorname{Re}\rho_j>\frac{1}{2}} \frac{\operatorname{Re}(|\rho_j|^2-\overline{\rho}_j)^2(2\operatorname{Re}\rho_j-1)(2|\rho_j|^2-2\operatorname{Re}\rho_j+1)}{|\rho_j(\rho_j-1)|^4}.\end{aligned}$$

Note that

$$\begin{aligned}\int\limits_{\operatorname{Re} s=\frac{1}{2}} \operatorname{Re}\left(\overline{s}^2\right)\frac{\log|\zeta(s)|}{|s|^6}\,|ds| &= 2\int_0^\infty \left(\frac{1}{4}-t^2\right)\frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{\left(\frac{1}{4}+t^2\right)^3}dt\\ &= -2\int_0^\infty \frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{\left(\frac{1}{4}+t^2\right)^2}dt + \int_0^\infty \frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{\left(\frac{1}{4}+t^2\right)^3}dt\\ &= -\int_{\operatorname{Re} s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^4}\,|ds| + \frac{1}{2}\int_{\operatorname{Re} s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds|.\end{aligned}$$

Using the results from [1] and [6], we obtain (1). The proof is completed. ☐

Proof of Theorem 2. If the Riemann Hypothesis is true, then the series at the right hand side of (1) are absent, and we have (2)

$$\frac{1}{2\pi}\int_{\operatorname{Re} s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds| = \frac{7}{2} - 4\gamma + \frac{\gamma_1-\gamma^2}{2}.$$

Now assume that relation (2) holds. If the Riemann Hypothesis is not true, then in (1)

$$6\sum_{\operatorname{Re}\rho_j>\frac{1}{2}} \log\left|\frac{\rho_j}{1-\rho_j}\right| + 4\sum_{\operatorname{Re}\rho_j>\frac{1}{2}} \frac{(|\rho_j|^2-\operatorname{Re}\rho_j)(2\operatorname{Re}\rho_j-1)}{|\rho_j(\rho_j-1)|^2} > 0.$$

Examine carefully the series

$$\sum_{\operatorname{Re}\rho_j>\frac{1}{2}} \frac{\operatorname{Re}(|\rho_j|^2-\overline{\rho}_j)^2(2\operatorname{Re}\rho_j-1)(2|\rho_j|^2-2\operatorname{Re}\rho_j+1)}{|\rho_j(\rho_j-1)|^4}.$$

We are interested in when all terms of this series are positive. The following conditions appear

$$\mathrm{Re}(|\rho_j|^2 - \overline{\rho}_j)^2 > 0.$$

If $0 < \mathrm{Re}\,\rho_j < 1$ and $|\ \mathrm{Im}\,\rho_j\ | > \frac{1}{2} + \frac{1}{\sqrt{2}}$, then $\mathrm{Re}(|\rho_j|^2 - \overline{\rho}_j)^2 > 0$.

It is known (see [8]) that the first $10^{22} + 1$ non-trivial zeros of the Riemann zeta-function lie on the critical line. In particular, $\mathrm{Im}\,\rho_1 = 14,1347\dots$.

These facts imply $\mathrm{Re}(|\rho_j|^2 - \overline{\rho}_j)^2 > 0$ for all non-trivial zeros ρ_j that lie inside the critical strip $0 < \mathrm{Re}\,s < 1$.

Hence, if the Riemann Hypothesis is not true, then

$$\frac{1}{2\pi} \int_{\mathrm{Re}\,s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds| > \frac{7}{2} - 4\gamma + \frac{\gamma_1 - \gamma^2}{2}.$$

This is a contradiction with (2) which finishes the proof. □

REFERENCES

[1] Balazard M., Saias E., Yor M. *Notes sur la fonction ζ de Riemann, 2*. Adv. Math. 1999, **143** (2), 284–287. doi:10.1006/aima.1998.1797 (in French)

[2] Bercovici H., Foias C. *A real variable restatement of Riemann's hypothesis*. Israel J. Math. 1984, **48** (1), 57–68. doi:10.1007/BF02760524

[3] Eroğlu K.I., Ostrovskii I.V. *On an application of the Hardy classes to the Riemann zeta-function*. Turkish J. Math. 2001, **25**, 545–551.

[4] Hoffman K. Banach spaces of analytic functions. Dover, New York, 1988.

[5] Ivic A. The Riemann zeta-function. Wiley, New York, 1985.

[6] Kondratyuk A., Yatsulka P. Summation of the Riemann zeta-function logarithm on the critical line. Proc. of the fourth intern. conf. on analytic number theory and spatial fessellations "Voronoy's impact on modern science", Kyiv, Ukraine, September 22–28, 2008, 59–63.

[7] McLane G.R., Rubel L.A. *On the growth of Blaschke products*. Canad. J. Math. 1969, **21**, 595–601. doi: 10.4153/CJM-1969-067-3

[8] Odlyzko A. *The* 10^{22}*-nd zero of the Riemann zeta function*. Contemp. Math. 2001, **290**, 139–144.

Received 17.11.2015

Revised 17.02.2016

Басюк Ю.В., Тарасюк С.І. *Коефіцієнти Фур'є, асоційовані з дзета-функцією Рімана* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 16–20.

Ми вивчаємо дзета-функцію Рімана $\zeta(s)$, використовуючи метод коефіцієнтів Фур'є. Підсумовування $\log|\zeta(s)|$ з ядром $1/|s|^6$ на критичній прямій $\mathrm{Re}\ s = \frac{1}{2}$ є головним результатом нашого дослідження. Також отримали твердження, рівносильне гіпотезі Рімана.

Ключові слова і фрази: коефіцієнти Фур'є, дзета-функція Рімана, гіпотеза Рімана.