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# STRICTLY DIAGONAL HOLOMORPHIC FUNCTIONS ON BANACH SPACES

In this paper we investigate the boundedness of holomorphic functionals on a Banach space with a normalized basis $\{e_n\}$ which have very special form $f(x) = f(0) + \sum_{n=1}^{\infty} c_n x_n^n$ and which we call strictly diagonal. We consider under which conditions strictly diagonal functions are entire and uniformly continuous on every ball of a fixed radius.

*Key words and phrases:* holomorphic functions on Banach space, base on Banach space.

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## INTRODUCTION AND PRELIMINARIES

Let $X$ be a separable complex Banach space with a normalized basis $\{e_n\}_{n=1}^{\infty}$. A holomorphic function $f$ on an open ball $B(0, r)$ of $X$ centered at zero (of finite or infinite radius $r$) will be called *strictly diagonal* with respect to the basis if it is of the form

$$f(x) = f(0) + \sum_{n=1}^{\infty} c_n x_n^n, \qquad x \in X, \qquad \text{where} \qquad x = \sum_{n=1}^{\infty} x_n e_n. \tag{1}$$

We can *associate* a formal power series with $f$ in such way

$$\gamma(t) = \sum_{n=0}^{\infty} c_n t^n, \qquad c_0 = f(0), \qquad t \in \mathbb{C}$$

and we will write $\gamma = \gamma_f$ and $f = f_\gamma$ if it is necessary. Note that the strictly diagonal function $f(x) = \sum_{n=1}^{\infty} x_n^n$ is the well-known example [4, p. 169] of entire function on $\ell_p$, $1 \le p < \infty$ or on $c_0$ which is not of bounded type (the radius of boundedness at zero is equal to one). On the other hand its associated series $\gamma(t)$ well defines a holomorphic function only on the open unit disk $\mathbb{D}_1 \subset \mathbb{C}$. More examples of entire holomorphic functions which are not bounded on all bounded sets can be found in [1, 2, 3].

The purpose of this paper is to examine properties of strictly diagonal holomorphic functions in terms of associated power series and construct some new interesting examples of holomorphic functions on $X$.

Le us recall that a continuous function $f \colon X \to \mathbb{C}$ is said to be *holomorphic* at a point $a \in X$ if it has power series representation

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

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in a neighborhood of $a$, where $f_n$ are continuous $n$-homogeneous polynomials. A function $f$ is *entire* if it is holomorphic at each point of $X$. The space of all entire functions on $X$ is denoted by $H(X)$.

The *radius of uniform convergence* of a function $f$ at $a$ can be calculated by formula

$$\rho_a(f) = (\limsup_{n\to\infty} \|f_n\|^{\frac{1}{n}})^{-1}$$

and coincides with the radius of boundedness. In particular, each entire function is uniformly bounded on the ball $B(a,r)$ centered at $a$ of radius $r$ if $r < \rho_a(f)$ and unbounded on $B(a,r)$ if $r > \rho_a(f)$.

For details on holomorphic functions on Banach spaces we refer the reader to [4, 5, 7].

## 1 Main results

Throughout in this section $f$ is a strictly diagonal function defined by (1).

**Theorem 1.** *Let $\delta > 0$ and*

$$\gamma(t) = \sum_{n=0}^{\infty} c_n t^n$$

*converges in the open $\delta$-disk $\mathbb{D}_\delta = \{t \in \mathbb{C} \colon |t| < \delta\}$. Then $f_\gamma \in H(X)$ and $\rho_z(f_\gamma) \geq \delta$ for every $z \in X$.*

*Proof.* For a given $x \in X$ let $n_0$ be a number such that $|x_n| \leq r < \delta$ for every $n > n_0$. Then

$$|f_\gamma(x)| \leq \Big|\sum_{k=0}^{n_0} c_k x_k\Big| + \sum_{k=n_0+1}^{\infty} |c_k||x_k| \leq \Big|\sum_{k=0}^{n_0} c_k x_k\Big| + \sum_{k=n_0+1}^{\infty} |c_k| r^k < \infty.$$

So $f_\gamma$ is well-defined at any point of $X$. Clearly $f_\gamma$ is $G$-holomorphic and

$$\rho_0(f_\gamma) = \big(\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}\big)^{-1} = \rho_0(\gamma) \geq \delta.$$

This, in particular, means that $f_\gamma$ is locally bounded at 0 and so it is holomorphic. Let $z$ be a fixed element in $X$. For any $0 < r < \delta$ let $m_0$ be a number such that $|z_n| < \frac{\delta - r}{2}$ $\forall n > m_0$. Then for every $x \in X$, $\|x\| < r$, we have

$$|f_\gamma(x+z)| \leq \Big|\sum_{k=0}^{m_0} c_k (z_k + x_k)^k\Big| + \sum_{m_0+1}^{\infty} c_k |(z_k + x_k)|^k \leq \Big|\sum_{k=0}^{m_0} c_k (z_k + x_k)^k\Big| + \Big|\gamma\Big(\frac{\delta - r}{2} + r\Big)\Big|.$$

Let us denote

$$c(z,r) := \Big|\sum_{k=0}^{m_0} c_k (z_k + x_k)^k\Big| + \Big|\gamma\Big(\frac{\delta - r}{2} + r\Big)\Big|.$$

Then for every $z \in X$ and $r < \delta$, $f_\gamma$ is bounded in $B(z,r)$ by the constant $c(z,r)$ which depends only on $z$ and $r$. That is, $\rho_z(f_\gamma) \geq \delta$. ☐

**Definition 1.1.** *A basis $\{e_n\}_{n=1}^{\infty}$ is said to be boundedly complete if for every sequence of numbers $\{b_n\}_{n=1}^{\infty}$ such that $\sup \|\sum_{n=1}^{m} b_n e_n\| < \infty$ the series $\sum_{n=1}^{\infty} b_n e_n$ converges to a vector in $X$.*

Note that the standard basis in $\ell_p$, $1 \leq p < \infty$, is boundedly complete while in $c_0$ it is not. Moreover if $\{e_n\}_{n=1}^{\infty}$ is not boundedly complete, then it contains a subsequence equivalent to the standard basis in $c_0$ (see [6]).

**Definition 1.2.** *We say that $K$ is the index of boundedness of $\{e_n\}_{n=1}^{\infty}$ if*

$$\|x\| = \|\sum_{n=1}^{\infty} x_n e_n\| = \delta > 0$$

*implies that the cardinality of set $\{x_k \colon |x_k| = \delta\}$ does not exceed $K$.*

**Theorem 2.** *Let $\{e_n\}_{n=1}^{\infty}$ be a normalized basis of a Banach space $X$ which has a finite index of boundedness $K$ and $\gamma(t) = \sum_{n=0}^{\infty} c_n t^n$ is holomorphic and bounded on the disk $\mathbb{D}_\delta$. Then $f_\gamma \in H(X)$ and for every $z \in X$, $f_\gamma$ is bounded on $B(z, \delta)$.*

*Proof.* From Theorem 1 it follows that $f_\gamma \in H(X)$. For a given $x \in X$, $\|x\| < 1$, we have

$$|f_\gamma(x)| \leq \sum_{n=0}^{K} c_n \delta^n + \sup_{|t|<\delta} |\gamma(t)|.$$

So

$$\sup_{\|x\|<1} |f_\gamma(x)| \leq \sum_{n=0}^{K} c_n \delta^n + \sup_{|t|<\delta} |\gamma(t)| < \infty$$

and $f_\gamma$ is bounded on $B(0, \delta)$. Using the same work like in Theorem 1 we can show that $f_\gamma$ is bounded on $B(z, \delta)$ for every fixed $z \in X$. $\square$

**Definition 1.3.** *Let us suppose that there are $0 < \varepsilon < 1$ and positive integer $K_\varepsilon$ such that $\|x\| = 1$ implies card $\{x_n \colon |x_n| \leq 1 - \varepsilon\} \leq K_\varepsilon < \infty$. Then we say that $K_\varepsilon$ is the index of $\varepsilon$-boundedness of the basis $\{e_n\}_{n=1}^{\infty}$.*

Clearly that if $X$ has an index of $\varepsilon$-boundedness $K_\varepsilon$ for some $\varepsilon > 0$, then $K_\varepsilon = K$.

**Example 1.** *Let $X = \bigoplus_{k=1}^{\infty} \ell_\infty^k$ (the $\ell_1$-sum). That is, for every*

$$x = \sum_{k=1}^{\infty} \sum_{j=1}^{k} x_j^k e_j^k = (x_1^1, x_1^2, x_2^2, x_1^3, x_2^3, x_3^3, \ldots), \qquad x \in X,$$

*we have*

$$\|x\| = \sum_{k=1}^{\infty} \max_{1 \leq j \leq k} |x_j^k|.$$

*Basis $\{e_j^k\}_{k=1,j=1}^{\infty,\ k}$ is boundedly complete. Indeed, let $\{b_j^k\}_{k=1,j=1}^{\infty,\ k}$ be a sequence of numbers such that $\sum_{k=1}^{m} \max_{1 \leq j \leq k} |b_j^k| < c$ for every $m$ and some $c > 0$. Then $\sum_{k=1}^{\infty} \max_{1 \leq j \leq k} |b_j^k|$ converges and so $\sum b_j^k e_j^k \in X$. On the other hand for every $K \in \mathbb{N}$ we can pick*

$$x_0 = e_1^{k+1} + \cdots + e_{k+1}^{k+1}$$

*with $\|x_0\| = 1$ and so $\{e_j^k\}_{k=1,j=1}^{\infty,\ k}$ has no finite index of boundedness.*

**Example 2.** *Let $X$ be the $\ell_1$-sum of $\ell_n$, $X = \bigoplus_{n=1}^{\infty} \ell_n^n$ and $\{e_j^k\}_{k=1,j=1}^{\infty,\ k}$ be the natural basis. This basis has the index of boundedness $K = 1$. Indeed, suppose $\|x\| = 1$ and for two different coordinates $|x_j^k| = 1$ and $|x_i^s| = 1$. We have two cases:*

*1) if $k = s$, then $\|x\| \geq (|x_j^k|^k + |x_i^k|^k)^{\frac{1}{k}} > 1$,*

*2) if $k \neq s$, then $\|x\| \geq 2$.*

*This contradicts our assumption. So, just one coordinate may have the absolute value equals one.*

*Let $0 < \varepsilon < 1$ and $K_\varepsilon$ be a fixed positive integer. Let us find $k_0 \in \mathbb{N}$ such that $(1-\varepsilon)^{k_0} < \frac{1}{K_\varepsilon+1}$. Let $m \geq 2\max(k_0, K_\varepsilon + 1)$ and*

$$x_0 = (1-\varepsilon)e_1^m + \ldots + (1-\varepsilon)e_{K_\varepsilon+1}^m,$$

*then*

$$\begin{aligned}\|x_0\|^m &= (K_\varepsilon+1)(1-\varepsilon)^m = (K_\varepsilon+1)(1-\varepsilon)^{\frac{2m}{2}} \\ &< (K_\varepsilon+1)(1-\varepsilon)^{\frac{m}{2}}(1-\varepsilon)^{\frac{m}{2}} < (K_\varepsilon+1)\frac{1}{(K_\varepsilon+1)}(1-\varepsilon)^{\frac{m}{2}},\end{aligned}$$

*that is,*

$$\|x_0\| \leq ((1-\varepsilon)^{\frac{m}{2}})^{\frac{1}{m}} = (1-\varepsilon)^{\frac{1}{2}}.$$

*It means that the index of $\varepsilon$-boundedness of the basis is greater than $K_\varepsilon$. Since $K_\varepsilon$ is arbitrary, the basis has no finite index of $\varepsilon$-boundedness.*

**Theorem 3.** *Let $\{e_n\}_{n=1}^{\infty}$ be a basis of a Banach space $X$ which has an index of $\varepsilon$-boundedness $K_\varepsilon$ for every $0 < \varepsilon < 1$ and $\gamma(t) = \sum_{n=0}^{\infty} c_n t^n$ converges in the disk $\mathbb{D}_1$. Then $f_\gamma$ is uniformly continuous on $B(z, 1)$ for every $z \in X$.*

*Proof.* Let us prove the statement for the case $B(0,1)$. The general case follows from there like in Theorem 1. Note that $\gamma(t)$ is uniformly continuous on the closed disk $\overline{\mathbb{D}}_\rho$ for every $0 < \rho < 1$. For a given $0 < \varepsilon < 1$ let $\omega > 0$ be such that

$$|\gamma(t_1) - \gamma(t_2)| < \varepsilon \tag{2}$$

if only $|t_1 - t_2| < \delta$ for $t_1, t_2 \in \overline{\mathbb{D}}_{1-\varepsilon/2}$. Let $x, y \in X$, $\|x\| \leq 1$, $\|y\| \leq 1$,

$$x = \sum_{n=1}^{\infty} x_n e_n, \quad y = \sum_{n=1}^{\infty} y_n e_n.$$

Then there is a number $m \leq K_\varepsilon + 1$ such that for

$$\widetilde{x} = \sum_{n=m}^{\infty} x_n e_n \quad \text{and} \quad \widetilde{y} = \sum_{n=m}^{\infty} y_n e_n$$

$\|\widetilde{x}\| < 1 - \varepsilon$ and $\|\widetilde{y}\| < 1 - \varepsilon$. Clearly that $f_\gamma$ is uniformly continuous on $B(0,1)$ if and only if

$$f_\gamma^c := \sum_{n=m}^{\infty} c_n x_n^n$$

is uniformly continuous on $B(0,1)$. If $\|\widetilde{x} - \widetilde{y}\| < \delta$, then $\|x_k - y_k\| < \delta$ for $k \geq m$. Let $r = \sup_{k \geq m} \|x_k - y_k\|$. Then from (2) we obtain

$$\|f_\gamma(x) - f_\gamma(y)\| = |\sum_{n=m}^{\infty} c_n(x_n^n - y_n^n)| \leq \sum_{n=m}^{\infty} |c_n(x_n^n - y_n^n)| \leq \sum_{n=m}^{\infty} c_n r^n < \varepsilon.$$

□

**Example 3.** *Let $\gamma(t) = \sum_{n=1}^{\infty} t^n$, then the entire function $f_\gamma$ is uniformly continuous on a unit ball centered at any point in $\ell_p$, $1 \le p < \infty$. But it is not bounded in the unit ball in $c_0$. Indeed let $x^n = e_1 + e_2 + \ldots + e_n \in c_0$, then $f(x^n) = n \to \infty$. By the same way it is possible to show that if $\gamma(t)$ is unbounded in $\mathbb{D}_1 \subset \mathbb{C}$, then $f_\gamma(x)$ is unbounded in the unit ball of $c_0$.*

**Proposition 1.1.** *$f_\gamma$ is bounded on $B(z,r) \subset c_0$ for every $z \in c_0$ if and only if $\gamma(t)$ converges absolutely on $\overline{\mathbb{D}}_r$.*

*Proof.* If $\gamma(t)$ converges absolutely on $\overline{\mathbb{D}}_r$, then it is easy that $f_\gamma$ is bounded on $B(z,r) \subset c_0$ for every $z \in c_0$. To prove the converse statement without loss of the generality we assume that $r = 1$. If $\gamma(t) = \sum_{n=1}^{\infty} c_n t^n$ does not converges absolutely on $\overline{\mathbb{D}}_1$, then there are numbers $b_n$, $|b_n| = 1$, such that $\sum_{n=1}^{\infty} c_n b_n \to \infty$ as $n \to \infty$.

Let $x_n = \sqrt[n]{b_n}$ and $x^n = \sum_{n=1}^{m} x_n e_n$. Clearly $\|x^m\|_{c_0} = 1$ and $f_\gamma(x^m) = \sum_{n=1}^{m} c_n b_n = m \to \infty$ so $f_\gamma(x)$ is unbounded on $B(0,1)$. □

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Досліджено обмеженість голоморфних функцій на банахових просторах з базисом $\{e_n\}$, які мають дуже спеціальний вигляд $f(x) = f(0) + \sum_{n=1}^{\infty} c_n x_n^n$ і які ми називаємо строго діагональними. Розглянуто при яких умовах строго діагональні функції будуть цілими і рівномірно обмеженими на всіх кулях фіксованого радіуса.

*Ключові слова і фрази:* голоморфні функції на банахових просторах, базиси в банахових просторах.