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CONVERGENCE IN $L^p[0, 2\pi]$-METRIC OF LOGARITHMIC DERIVATIVE AND ANGULAR v-DENSITY FOR ZEROS OF ENTIRE FUNCTION OF SLOWLY GROWTH

The subclass of a zero order entire function f is pointed out for which the existence of angular v-density for zeros of entire function of zero order is equivalent to convergence in $L^p[0, 2\pi]$-metric of its logarithmic derivative.

Key words and phrases: logarithmic derivative, entire function, angular density, Fourier coefficients, slowly increasing function.

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INTRODUCTION

Let L be the class of all positive non-decreasing unbounded continuously differentiable on $[0, +\infty)$ functions v such that $rv'(r)/v(r) \to 0$ as $0 < r_0 \le r \to +\infty$. It is known (see [1, p. 15]) that the class L coincides with the class of slowly increasing functions accurate to the equivalent functions. By $H_0(v)$, $v \in L$, we denote the class of entire functions f of zero order for which $0 < \Delta = \overline{\lim\limits_{r\to+\infty}}\, n(r)/v(r) < +\infty$. Without loss of generality we assume that $f(0) = 1$.

We will say that zeros of function $f \in H_0(v)$, $v \in L$, have an angular v- density, if the limit

$$\Delta(\alpha, \beta) = \lim_{r\to+\infty} \frac{n(r, \alpha, \beta)}{v(r)}$$

exists for all α and β, that do not belong to some no more than countable set from $[0, 2\pi]$. Here $n(r, \alpha, \beta)$ is the number of zeros a_n of the function f, which lie in the sector $\{z\colon |z| \le r, \alpha \le \arg z < \beta\}$, $0 \le \alpha < \beta < 2\pi$.

We also denote by $F(z) = z\dfrac{f'(z)}{f(z)}$ the logarithmic derivative of f, by $\mathcal{E}_\eta$ the family of all measurable sets $G \subset \mathbb{R}_+$ such that $\overline{\lim\limits_{r\to+\infty}}\, \mathrm{mes}(G \cap [0, r])/r \le \eta$, $0 < \eta < 1$.

Theorem ([2]). *Let $v \in L$, $f \in H_0(v)$ and zeros of the function f have angular v-density. Then there exists a set $G \in \mathcal{E}_\eta$ such that, for arbitrary $p \in [1, +\infty)$,*

$$\left\| \frac{F(re^{i\theta}) - n(r)}{v(r)} \right\|_p \to 0, \quad r \to +\infty, \quad r \notin G.$$

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The converse statement is false. The question is under which conditions for $f \in H_0(v)$ from the convergence in $L^p[0,2\pi]$-metric of the function F the existence of angular v-density of zeros of f will follow. We note [3], that in the case of an entire function f of non integer order $\rho > 0$ the existence of angular density of its zeros is equivalent to the following

$$\left\| \frac{F(re^{i\theta})}{r^{\rho(r)}} - g(\theta) \right\|_p \to 0,\ r \to +\infty,\ r \notin G, G \in \mathcal{E}_\eta,$$

where $p \in [1,+\infty)$, $g \in L^1[0,2\pi]$, $\rho(r)$ is the proximate order of f, $\rho(r) \to \rho$, $r \to +\infty$.

In this paper we will point out the subclass of entire function f from the class $H_0(v)$, for which the existence of angular v-density of zeros of the function f will be equivalent to the convergence of the logarithmic derivative F in $L^p[0,2\pi]$-metric.

1 Main results

Let us denote by $\Gamma_m = \bigcup\limits_{j=1}^{m} \{z\colon\ \arg z = \theta_j\} = \bigcup\limits_{j=1}^{m} l_{\theta_j}$, $-\pi \le \theta_1 < \theta_2 < \ldots < \theta_m < \pi$, the finite system of rays, by $n(r,\theta_j; f) = n(r,\theta_j)$ the number of zeros of $f \in H_0(v)$ lying on the ray $l_{\theta_j} = \{z\colon\ \arg z = \theta_j\}$ and modules of which do not exceed r. Let $h_j(\theta) = (\theta - \pi - \theta_j)$, $\theta_j < \theta < \theta_j + 2\pi$, and $\widehat{h}_j(\theta)$ be its periodic continuation from $(\theta_j, \theta_j + 2\pi)$ on $\mathbb{R}$, $j = \overline{1,m}$. For $\widetilde{v} \in L$ we set

$$v(r) = \int\limits_0^r \frac{\widetilde{v}(t)}{t} dt.$$

It is easy to see that $v \in L$ and $\widetilde{v}(r) = o(v(r))$ as $r \to +\infty$.

Theorem 1. *Let $\widetilde{v} \in L$, $f \in H_0(v)$. Suppose that zeros of the function f lie on the finite system of rays Γ_m and for each $j = \overline{1,m}$, $\Delta_j > 0$*

$$n(r,\theta_j) = \Delta_j v(r) + o(\widetilde{v}(r)), \quad r \to +\infty. \tag{1}$$

Then

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p = \left\| \frac{F(re^{i\theta}) - \Delta v(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p \to 0,\ r \to +\infty, \tag{2}$$

where $H_f(\theta) = \sum\limits_{j=1}^{m} \Delta_j \widehat{h}_j(\theta)$, $\Delta = \sum\limits_{j=1}^{m} \Delta_j$.

Theorem 2. *Let $G \in L^1[0,2\pi]$, $\widetilde{v} \in L$, $f \in H_0(v)$. Suppose that zeros of the function f lie on the finite system of rays Γ_m and*

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\tilde{v}(r)} - iG(\theta) \right\|_p \to 0,\ r \to +\infty. \tag{3}$$

Then zeros of the function f have an angular v-density, moreover $\int\limits_0^{2\pi} G(\theta) d\theta = 0$.

2 Additional results

To prove Theorems 1, 2 we will use the following results, which we formulate as lemmas.

Lemma 1 ([1]). *Let $v \in L$. Then for $k \in \mathbb{N}$*

$$r^k \int\limits_r^{+\infty} \frac{v(t)}{t^{k+1}}dt = \frac{1}{k}v(r) + o(v(r)), \quad r \to +\infty,$$

$$r^{-k} \int\limits_0^{r} \frac{v(t)}{t^{-k+1}}dt = \frac{1}{k}v(r) + o(v(r)), \quad r \to +\infty.$$

Lemma 2. *Let $v \in L$, $\varepsilon(t)$ be a function, locally integrable on $[1,+\infty)$, and $\varepsilon(t) \to 0$ as $t \to +\infty$. Then for $k \in \mathbb{N}$*

$$r^k \int\limits_r^{+\infty} \frac{\varepsilon(t)v(t)}{t^{k+1}}dt = o(v(r)), \quad r \to +\infty,$$

$$r^{-k} \int\limits_0^{r} \frac{\varepsilon(t)v(t)}{t^{-k+1}}dt = o(v(r)), \quad r \to +\infty.$$

The proof of this lemma follows from applying L'Hopital's rule.

Let $c_k(r,\Phi)$, $k \in \mathbb{Z}$, be the Fourier coefficients of function $\Phi(re^{i\theta})$ as a function of θ, that is $c_k(r,\Phi) = \frac{1}{2\pi} \int\limits_0^{2\pi} \Phi(re^{i\theta})e^{-ik\theta}, \quad r > 0.$

Lemma 3. *Let $\widetilde{v} \in L$, $f \in H_0(v)$, zeros of the function f lie on the finite system of rays Γ_m and* (1) *holds. Then there exists $r_0 > 0$ such that for $k \in \mathbb{Z} \setminus \{0\}$ the relations*

$$c_k(r,F) = -\frac{\widetilde{\Delta}_k}{k}\widetilde{v}(r) + o(\widetilde{v}(r)), \quad r \to +\infty,$$

$$|c_k(r,F)| \le \frac{2\Delta}{|k|}\widetilde{v}(r), \quad r \ge r_0, \Delta > 0, \widetilde{\Delta}_k > 0,$$

hold.

Proof. Since $n_k(r) = \sum\limits_{j=1}^{m} e^{-ik\theta_j} n(r,\theta_j)$, owing to (1) we have

$$n_k(r) = \widetilde{\Delta}_k v(r) + o(\widetilde{v}(r)), \quad r \to +\infty,$$

where $\widetilde{\Delta}_k = \sum\limits_{j=1}^{m} \Delta_j e^{-ik\theta_j}$.

From formulas for calculating the coefficients $c_k(r,F)$ [2, Lemma 3] and the last identity, using Lemma 2, we obtain

$$c_k(r,F) = n_k(r) - kr^k \int\limits_r^{+\infty} \frac{n_k(t)}{t^{k+1}}dt = \widetilde{\Delta}_k v(r) + o(\widetilde{v}(r)) - k\widetilde{\Delta}_k r^k \int\limits_r^{+\infty} \frac{v(t)}{t^{k+1}}dt - kr^k \int\limits_r^{+\infty} \frac{o(\widetilde{v}(r))}{t^{k+1}}dt$$

$$= \widetilde{\Delta}_k v(r) - k\widetilde{\Delta}_k r^k \left(\frac{v(r)}{kr^k} + \frac{1}{k} \int\limits_r^{+\infty} \frac{\widetilde{v}(t)}{t^{k+1}}dt \right) + o(\widetilde{v}(r)) = -\widetilde{\Delta}_k r^k \int\limits_r^{+\infty} \frac{\widetilde{v}(t)}{t^{k+1}}dt + o(\widetilde{v}(r)),\ k \in \mathbb{N},$$

as $r \to +\infty$.

Similarly, for $k \in \mathbb{Z}$, $k < 0$,

$$c_k(r, F) = \widetilde{\Delta}_k r^k \int_0^r \frac{\widetilde{v}(t)}{t^{k+1}} dt + o(\widetilde{v}(r)), \quad r \to +\infty.$$

From this and Lemma 1 we have

$$c_k(r, F) \sim -\frac{\widetilde{\Delta}_k}{k} \widetilde{v}(r), \quad r \to +\infty,$$
$$|c_k(r, F)| \leq \frac{2\Delta}{|k|} \widetilde{v}(r), \quad r \geq r_0.$$

□

3 Proof of the main results

Proof of Theorem 1. We set

$$\begin{aligned} b_k := c_k(H_f) &= \frac{1}{2\pi} \sum_{j=1}^m \Delta_j \int_0^{2\pi} \widehat{h}_j(\theta) e^{-ik\theta} d\theta = \frac{1}{2\pi} \sum_{j=1}^m \Delta_j \int_{\theta_j}^{\theta_j+2\pi} h_j(\theta) e^{-ik\theta} d\theta \\ &= \frac{i}{k} \sum_{j=1}^m \Delta_j e^{-ik\theta_j} = \begin{cases} \dfrac{i\widetilde{\Delta}_k}{k}, & k \neq 0, \\ 0, & k = 0. \end{cases} \end{aligned} \tag{4}$$

Therefore $|b_k| \leq \dfrac{\Delta}{|k|}$, $k \neq 0$. Since, by Lemma 3, $|c_k(r, F)| \leq \dfrac{2\Delta}{|k|} \widetilde{v}(r)$, the sequence $\left(\dfrac{c_k(r, F)}{\widetilde{v}(r)} - ib_k \right)_{k \neq 0}$ belongs to the space l_q with $q > 1$, $r \geq r_0$. We have

$$c_k(r, F(z) - n(r)) = c_k(r, F) \qquad \text{for} \quad k \neq 0.$$

Thus by Hausdorff-Young theorem [4, p. 153] for $p \geq 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p \leq \left\{ \sum_{k \neq 0} \left| \frac{c_k(r, F)}{\widetilde{v}(r)} - ib_k \right|^q \right\}^{\frac{1}{q}}.$$

Since the resulting series is uniformly convergent for all $r \geq r_0$, by making the limiting transition as $r \to +\infty$ in the last inequality and owing to Lemma 3 and identity (4) we obtain

$$\left\| \frac{F(re^{i\theta}) - n(r)}{\widetilde{v}(r)} - iH_f(\theta) \right\|_p \to 0, \; r \to +\infty,$$

for $p \geq 2$. By Holder's inequality $\| \cdot \|_p \leq \| \cdot \|_2$ for $1 \leq p < 2$, that is (2) is also valid for $1 \leq p < 2$. The Theorem 1 is proved. □

Proof of Theorem 2. Let us denote by g_k the Fourier coefficients of function G, namely $g_k = c_k(G)$. Then, by (3), we obtain

$$\left|\frac{c_k(r,F)-n(r)}{\widetilde{v}(r)} - ig_k\right| = \left|\frac{c_k(r,F-n(r))}{\widetilde{v}(r)} - ig_k\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{F(re^{i\theta})-n(r)}{\widetilde{v}(r)} - iG(\theta)\right)e^{-ik\theta}d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{F(re^{i\theta})-n(r)}{\widetilde{v}(r)} - iG(\theta)\right|d\theta \leq \left\|\frac{F(re^{i\varphi})-n(r)}{\widetilde{v}(r)} - iG(\theta)\right\|_p \to 0,$$

as $r\to+\infty$. Since $c_0(r,F)=n(r)$ from the last relation we find $g_0=0$, that is

$$\int_0^{2\pi} G(\theta)d\theta = 0.$$

For $k\neq 0$ owing to $l_k(r) := c_k(r,\ln f) = \int_0^r \frac{c_k(t,F)}{t}dt$ we obtain

$$ig_k = \lim_{r\to+\infty}\frac{c_k(r,F)}{\widetilde{v}(r)} = \lim_{r\to+\infty}\frac{\int_0^r c_k(t,F)/t\,dt}{\int_0^r \widetilde{v}(t)/t\,dt} = \lim_{r\to+\infty}\frac{c_k(r,\ln f)}{v(r)}. \tag{5}$$

By identities (see, for instance, [5, Lemma 1])

$$l_k(r) = -r^k\int_r^{+\infty}\frac{n_k(t)}{t^{k+1}}dt = -r^k\sum_{j=1}^m e^{-ik\theta_j}\int_r^{+\infty}\frac{n(r,\theta_j)}{t^{k+1}}dt,\ k=\overline{1,m},$$

we have the linear system of equations with respect to the quantities $n(r,\theta_j)$, $j=\overline{1,m}$,

$$\begin{cases}\sum_{j=1}^m e^{-i\theta_j}n(r,\theta_j) = rl_1'(r) - l_1(r),\\ \sum_{j=1}^m e^{-i2\theta_j}n(r,\theta_j) = rl_2'(r) - 2l_2(r),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \sum_{j=1}^m e^{-im\theta_j}n(r,\theta_j) = rl_m'(r) - ml_m(r).\end{cases}$$

Since

$$\begin{vmatrix} e^{-i\theta_1} & e^{-i\theta_2} & \dots & e^{-i\theta_m}\\ e^{-i2\theta_1} & e^{-i2\theta_2} & \dots & e^{-i2\theta_m}\\ . & . & . & .\\ e^{-im\theta_1} & e^{-im\theta_2} & \dots & e^{-im\theta_m}\end{vmatrix} \neq 0,$$

we have

$$n(r,\theta_j) = \sum_{k=1}^m b_{kj}(kl_k(r) - rl_k'(r)) = \sum_{k=1}^m b_{kj}(kl_k(r) - c_k(r,F)),$$

where $b_{kj}\in\mathbb{C}$. Taking into consideration (5) and the last identities we obtain for $j=\overline{1,m}$

$$n(r,\theta_j) = (1+o(1))i\sum_{k=1}^m b_{kj}(kg_kv(r) - g_k\widetilde{v}(r)) = i\sum_{k=1}^m b_{kj}kg_kv(r) + o(v(r))$$
$$= \Delta_j v(r) + o(v(r)),\quad r\to+\infty.$$

Hence, zeros of the function f have an angular v-density. □

Remark. *By the conditions of Theorem 2 it is easy to verify that* $G(\theta) = H_f(\theta)$ *for almost all* $\theta \in [0, 2\pi]$.

References

[1] Seneta E. Regularly varying functions. Nauka, Moscow, 1985.

[2] Zabolotskyj M.V., Mostova M.R. *Logarithmic derivative and the angular density of zeros for a zero-order entire function*. Ukrainian Mat. Zh. 2014, **66** (4), 530–540. doi:10.1007/s11253-014-0950-7 (translation of Ukrain. Mat. Zh. 2014, **66** (4), 473–481. (in Ukrainian))

[3] Vasylkiv Ya.V. *Asymptotic behavior of the logarithmic derivatives and the logarithms of meromorphic functions of completely regular growth in* $L^p[0, 2\pi]$*-metrics. II*. Mat. Stud. 1999, **12** (2), 135–144. (in Ukrainian)

[4] Zigmund A. Trigonometric series. Mir, Moscow, 1965. (in Russian)

[5] Bodnar O.V., Zabolotskyj M.V. *The criteria for regularity of growth of the logarithm module and argument of an entire function*. Ukrainian Mat. Zh. 2010, **62** (7), 1028–1039. doi:10.1007/s11253-010-0411-x (translation of Ukrain. Mat. Zh. 2010, **62** (7), 885–893. (in Ukrainian))

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Виділено підклас цілих функцій f нульового порядку, для яких поняття існування кутової v-щільності нулів f та збіжність в $L^p[0, 2\pi]$-метриці її логарифмічної похідної є рівносильними.

Ключові слова і фрази: логарифмічна похідна, ціла функція, кутова щільність, коефіцієнти Фур'є, повільно зростаюча функція.