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BANDURA A.I.

PROPERTIES OF POSITIVE CONTINUOUS FUNCTIONS IN $\mathbb{C}^n$

The properties of classes $Q^n_{\mathbf{b}}$ and Q of positive continuous functions are investigated. We prove that some compositions of functions from Q belong to class $Q^n_{\mathbf{b}}$. A relation between functions from these classes is established.

Key words and phrases: positive function, continuous function, several complex variables.

Ivano-Frankivsk National Technical University of Oil and Gas, 15 Karpatska str., 76019, Ivano-Frankivsk, Ukraine
E-mail: andriykopanytsia@gmail.com

INTRODUCTION

Introducing entire functions of bounded L-index in direction (see [1]) we have to impose additional conditions to a continuous function $L : \mathbb{C}^n \to \mathbb{R}_+$. We suppose that $L \in Q^n_{\mathbf{b}}$ (see below (5)). It is necessary to establish criteria of boundedness of L-index in direction and to apply L-index for solutions of partial differential equations or for entire functions with "plane" zeros [3].

Such conditions describe a behavior of slice function $L(z^0 + t\mathbf{b})$, $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$. It provides that function L does not rapidly change as $|z| \to \infty$. In one-dimensional case Sheremeta M.M. [5] used a class Q of positive continuous functions $l = l(t)$, $t \in \mathbb{C}$, satisfying some additional conditions. In fact, $l(t) = \ln |t|$, $l(t) = |t|^\alpha$, $\alpha \in \mathbb{R}_+$ belong to Q.

It is interesting: what are examples of functions from $Q^n_{\mathbf{b}}$? To answer the question we consider compositions of functions from Q. Thus, it is a natural question: how to build a function $L \in Q^n_{\mathbf{b}}$ by a function $l \in Q$?

1 PRELIMINARIES AND DENOTATIONS

For $\eta > 0$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{0\}$ and a positive continuous function $L : \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda^{\mathbf{b}}_1(z, t_0, \eta) = \inf\left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z + t_0\mathbf{b})} \right\}, \tag{1}$$

$$\lambda^{\mathbf{b}}_1(z, \eta) = \inf\{\lambda^{\mathbf{b}}_1(z, t_0, \eta) : t_0 \in \mathbb{C}\}, \qquad \lambda^{\mathbf{b}}_1(\eta) = \inf\{\lambda^{\mathbf{b}}_1(z, \eta) : z \in \mathbb{C}^n\}, \tag{2}$$

and

$$\lambda^{\mathbf{b}}_2(z, t_0, \eta) = \sup\left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z + t_0\mathbf{b})} \right\}, \tag{3}$$

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$$\lambda_2^{\mathbf{b}}(z,\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta):t_0\in\mathbb{C}\},\qquad \lambda_2^{\mathbf{b}}(\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,\eta):z\in\mathbb{C}^n\}. \tag{4}$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L, which for all $\eta\geq 0$ satisfy the condition

$$0<\lambda_1^{\mathbf{b}}(\eta)\leq\lambda_2^{\mathbf{b}}(\eta)<+\infty. \tag{5}$$

For a positive continuous function $l(t)$ for $t\in\mathbb{C}$ and $t_0\in\mathbb{C}$, $\eta>0$ we denote $\lambda_1(t_0,\eta)\equiv\lambda_1^{\mathbf{b}}(0,t_0,\eta)$ and $\lambda_2(t_0,\eta)\equiv\lambda_2^{\mathbf{b}}(0,t_0,\eta)$ in the case $z=0$, $\mathbf{b}=1$, $n=1$, $L\equiv l$, and

$$\lambda_1(\eta)=\inf\{\lambda_1(t_0,\eta):t_0\in\mathbb{C}\},\qquad \lambda_2(\eta)=\sup\{\lambda_2(t_0,\eta):t_0\in\mathbb{C}\}.$$

As in [5], by Q we denote the class of positive continuous functions $l(t)$, $t\in\mathbb{C}$, which satisfy the condition: $0<\lambda_1(\eta)\leq\lambda_2(\eta)<+\infty$ for all $\eta\geq 0$. In particular, $Q=Q_{\mathbf{1}}^1$.

2 ELEMENTARY PROPERTIES OF FUNCTIONS FROM $Q_{\mathbf{b}}^n$

Investigating the properties of entire functions of bounded L-index in direction we obtained following propositions about class $Q_{\mathbf{b}}^n$.

Lemma 1 ([1]). *If $L\in Q_{\mathbf{b}}^n$, then $L\in Q_{\theta\mathbf{b}}^n$ for every $\theta\in\mathbb{C}\setminus\{0\}$, and if $L\in Q_{\mathbf{b}_1}^n$ and $L\in Q_{\mathbf{b}_2}^n$ then $L\in Q_{\mathbf{b}_1+\mathbf{b}_2}^n$ for any $\mathbf{b}_1,\mathbf{b}_2\in\mathbb{C}^n$.*

For $l\in Q$ we denote

$$l_1(t,w)=(|t|+|w|+1)l(tw),\qquad l_2(t,w)=(|w|+1)l(tw),\qquad l_3(t,w)=(|t|+1)l(tw),$$

where $t,w\in\mathbb{C}$.

Lemma 2 ([2]). *If $l\in Q$, then $\forall\mathbf{b}\in\mathbb{C}^2$ $l_1\in Q_{\mathbf{b}}^2$, $l_2\in Q_{\mathbf{b}_1}^2$, $l_3\in Q_{\mathbf{b}_2}^2$, where $\mathbf{b}_1=(1,0)$, $\mathbf{b}_2=(0,1)$.*

For $l\in Q$ we denote $l_4(z)=l(|\langle z,m\rangle|)$, where $z\in\mathbb{C}^n$, $m\in\mathbb{C}^n$.

Lemma 3 ([4]). *If $l\in Q$, then $l_4\in Q_{\mathbf{b}}^n$ for every $m\in\mathbb{C}^n$ and every $\mathbf{b}\in\mathbb{C}^n$.*

For $l\in Q$ we denote $l_5(z)=l(|z|)$, $z\in\mathbb{C}^n$.

Lemma 4 ([4]). *If $l\in Q$, then $l_5\in Q_{\mathbf{b}}^n$ for every $\mathbf{b}\in\mathbb{C}^n$.*

It is easy to see that Lemmas 2, 3, 4 propose possible ways to construct a function $L\in Q_{\mathbf{b}}^n$ by a function $l\in Q$. Below we prove a generalization of Lemma 2 for $\mathbb{C}^n$ (see Theorem 1).

Let $L^*(z)$ be a positive continuous function in $\mathbb{C}^n$. The denotation $L\asymp L^*$ means that for some $\theta_1,\theta_2\in\mathbb{R}_+$, and for all $z\in\mathbb{C}^n$ the inequalities $\theta_1L(z)\leq L^*(z)\leq\theta_2L(z)$ hold.

Lemma 5. *If $L\in Q_{\mathbf{b}}^n$ and $L\asymp L^*$, then $L^*\in Q_{\mathbf{b}}^n$.*

Proof. Using the definition of $Q_{\mathbf{b}}^n$, we have

$$\begin{aligned}
&\inf_{z\in\mathbb{C}^n}\inf_{t_0\in\mathbb{C}}\inf\left\{\frac{L^*(z+t\mathbf{b})}{L^*(z+t_0\mathbf{b})}:|t-t_0|\leq\frac{\eta}{L^*(z+t_0\mathbf{b})}\right\}\\
&\quad\geq\inf_{z\in\mathbb{C}^n}\inf_{t_0\in\mathbb{C}}\inf\left\{\frac{\theta_1L(z+t\mathbf{b})}{\theta_2L(z+t_0\mathbf{b})}:|t-t_0|\leq\frac{\eta}{\theta_1L(z+t_0\mathbf{b})}\right\}\\
&\quad=\frac{\theta_1}{\theta_2}\inf_{z\in\mathbb{C}^n}\inf_{t_0\in\mathbb{C}}\inf\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:|t-t_0|\leq\frac{\eta}{\theta_1L(z+t_0\mathbf{b})}\right\}>0,
\end{aligned}$$

because $L \in Q^n_{\mathbf{b}}$. Besides,

$$\sup_{z\in\mathbb{C}^n}\sup_{t_0\in\mathbb{C}}\sup\left\{\frac{L^*(z+t\mathbf{b})}{L^*(z+t_0\mathbf{b})} : |t-t_0| \le \frac{\eta}{L^*(z+t_0\mathbf{b})}\right\}$$
$$\le \sup_{z\in\mathbb{C}^n}\sup_{t_0\in\mathbb{C}}\sup\left\{\frac{\theta_2 L(z+t\mathbf{b})}{\theta_1 L(z+t_0\mathbf{b})} : |t-t_0| \le \frac{\eta}{\theta_1 L(z+t_0\mathbf{b})}\right\}$$
$$= \frac{\theta_2}{\theta_1}\sup_{z\in\mathbb{C}^n}\sup_{t_0\in\mathbb{C}}\sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})} : |t-t_0| \le \frac{\eta}{\theta_1 L(z+t_0\mathbf{b})}\right\} < +\infty.$$

Thus $L^* \in Q^n_{\mathbf{b}}$. □

3 MAIN THEOREM

Now we prove several propositions that indicate ways of construction of functions from the class $Q^n_{\mathbf{b}}$.

Theorem 1. *If $l \in Q$ and $\inf\{l(t) : t \in \mathbb{C}\} = c > 0$, then $L \in Q^n_{\mathbf{b}}$, where*

$$L(z) = \frac{1}{c}\Big(1 + \sum_{k=1}^{n}\big(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|b_j|)\big)\Big)l\Big(\prod_{j=1}^{n} z_j\Big), \quad \textit{and} \quad \prod_{j\in\varnothing}(\cdot) = 1.$$

Proof. Note that in the definition of $Q^n_{\mathbf{b}}$ it is required that inequality (5) holds for all $\eta > 0$. But in view of (1)–(4) function $\lambda^{\mathbf{b}}_1(\eta)$ is nonincreasing and $\lambda^{\mathbf{b}}_2(\eta)$ is nondecreasing. So it is sufficient to require in definition of $Q^n_{\mathbf{b}}$ that inequality (5) is true for all $\eta \ge 1$. Indeed let this inequality holds for $\eta^* > 1$. Then for all $\widetilde{\eta}$ such that $0 < \widetilde{\eta} < 1 \le \eta^* < +\infty$, the following inequalities hold $\lambda^{\mathbf{b}}_1(\widetilde{\eta}) \ge \lambda^{\mathbf{b}}_1(\eta^*) > 0$, $\lambda^{\mathbf{b}}_2(\widetilde{\eta}) \le \lambda^{\mathbf{b}}_2(\eta^*) < +\infty$. Thus inequality (5) holds for all $\eta > 0$. Below we assume that $\eta \ge 1$.

Besides, we suppose that $\inf\{l(t) : t \in \mathbb{C}\} = 1$. If this infimum does not equal 1, then we can consider the function $\widetilde{l}(t) = \dfrac{l(t)}{\inf\{l(t) : t \in \mathbb{C}\}}$, for which this equality holds.

So we consider the case $\eta \ge 1$ and $\inf\{l(t) : t \in \mathbb{C}\} = 1$. We shall prove that for all $\eta \ge 1$ the following inequalities hold

$$\inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}\Bigg\{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt|\prod_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt)\Big)$$
$$\Big/\Big(\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\Big) :$$
$$|t-t^0| \le \frac{\eta}{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)}\Bigg\} > 0 \tag{6}$$

and

$$\sup_{z\in\mathbb{C}^n}\sup_{t^0\in\mathbb{C}}\sup_{t}\Bigg\{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt|\prod_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt)\Big)$$
$$\Big/\Big(\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\Big):$$
$$|t-t^0|\le\frac{\eta}{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)}\Bigg\}<\infty. \tag{7}$$

For this end we use the fact that $l\in Q$. According to our choice $\inf\{l(t): t\in\mathbb{C}\}=1$ and

$$\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\ge 1.$$

Hence, we obtain that

$$|t-t^0|\le\frac{\eta}{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)}\le\eta. \tag{8}$$

It remains to estimate the module

$$\begin{aligned}\Big|\prod_{j=1}^{n}(z_j+b_jt)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big| &= \Big|\Big(\prod_{j=1}^{n}(z_j+b_jt)-(z_1+b_1t^0)\prod_{j=2}^{n}(z_j+b_jt)\Big)\\ &+\Big((z_1+b_1t^0)\prod_{j=2}^{n}(z_j+b_jt)-\prod_{j=1}^{2}(z_j+b_jt^0)\prod_{j=3}^{n}(z_j+b_jt)\Big)+\cdots\\ &+\Big(\prod_{j=1}^{k-1}(z_j+b_jt^0)\prod_{j=k}^{n}(z_j+b_jt)-\prod_{j=1}^{k}(z_j+b_jt^0)\prod_{j=k+1}^{n}(z_j+b_jt)\Big)+\cdots\\ &+\Big((z_j+b_nt)\prod_{j=1}^{n-1}(z_j+b_jt^0)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big)\Big|.\end{aligned} \tag{9}$$

We estimate each of obtained n differences separately. In particular n-th difference can be estimated as

$$\begin{aligned}\Big|(z_j+b_nt)\prod_{j=1}^{n-1}(z_j+b_jt^0)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big| &= \prod_{j=1}^{n-1}|z_j+b_jt_0|\,|b_n|\,|t-t_0|\\ &\le\frac{\eta\prod_{j=1}^{n-1}|z_j+b_jt_0|\,|b_n|}{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\cdot\frac{1}{l\Big(\prod_{j=1}^{n}(z_j+b_jt^0)\Big)}.\end{aligned}$$

Applying the inequality (8) and using that $\eta>1$, $(n-1)$-th differences can be estimated as

$$\begin{aligned}
\Big|\prod_{j=1}^{n-2}(z_j+b_jt^0)\prod_{j=n-1}^{n}(z_j+b_jt)-\prod_{j=1}^{n-1}(z_j+b_jt^0)\,(z_j+b_jt)\Big| &= \prod_{j=1}^{n-2}|z_j+b_jt^0||b_{n-1}|\,|t-t^0|\,|z_n+b_nt| \\
&= \prod_{j=1}^{n-2}|z_j+b_jt^0|\,|b_{n-1}|\,|t-t^0|\,|z_n+b_nt^0+b_n(t-t^0)| \\
&\le \prod_{\substack{j=1\\ j\neq n-1}}^{n}|z_j+b_jt^0|\,|b_{n-1}|\,|t-t^0|+\prod_{j=1}^{n-2}|z_j+b_jt^0|\,|b_{n-1}|\,|b_n|\,|t-t^0|^2 \\
&\le \frac{\eta\prod\limits_{j=1,j\neq n-1}^{n}|z_j+b_jt_0|\,|b_{n-1}|}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)} \\
&+\frac{\eta^2\prod\limits_{j=1}^{n-2}|z_j+b_jt_0|\,|b_{n-1}|\,|b_n|}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)} \\
&\le \frac{\eta^2\Big(\prod\limits_{j=1,j\neq n-1}^{n}|z_j+b_jt_0|\,|b_{n-1}|+\prod\limits_{j=1}^{n-2}|z_j+b_jt_0|\,|b_{n-1}|\,|b_n|\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\cdot\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}.
\end{aligned}$$

For arbitrary k-th difference, $1\le k\le n$, of (9) we can obtain estimate

$$\begin{aligned}
&\Big|\prod_{j=1}^{k-1}(z_j+b_jt^0)\prod_{j=k}^{n}(z_j+b_jt)-\prod_{j=1}^{k}(z_j+b_jt^0)\prod_{j=k+1}^{n}(z_j+b_jt)\Big| \\
&\quad=\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}|z_j+b_jt|\,|b_k|\,|t-t_0| \\
&\quad=\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}|z_j+b_jt^0+b_j(t-t^0)|\,|b_k|\,|t-t_0| \\
&\quad\le\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|\,|t-t^0|)|b_k|\,|t-t_0| \\
&\quad\le\frac{\eta\,|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|\eta)}{\Big(1+\prod\limits_{j=1}^{n}(|z_j+b_jt^0|+|b_j|)-\prod\limits_{j=1}^{n}|z_j+b_jt^0|\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)} \\
&\quad\le\frac{\eta^{n-k}|b_k|\,\prod_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)}{\Big(1+\prod\limits_{j=1}^{n}(|z_j+b_jt^0|+|b_j|)-\prod_{j=1}^{n}|z_j+b_jt^0|\Big)l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}.
\end{aligned}$$

Thus, returning to (9) and considering that $\eta^j\le\eta^n$ for all j, $1\le j\le n$, we obtain the following inequality

$$\Big|\prod_{j=1}^{n}(z_j+b_jt)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big|$$

$$\leq \sum_{k=1}^{n}\frac{\eta^{n-k}|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}$$

$$\leq \eta^n\sum_{k=1}^{n}\frac{|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}$$

$$\times\frac{1}{\Big(1+\sum_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}$$

$$\leq\frac{\eta^n\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{1}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}$$

$$\leq\frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}.$$

Then for all $\eta\geq 1$

$$\inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}\Bigg\{\frac{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\frac{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}:$$

$$|t-t^0|\leq\frac{\eta}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\cdot\frac{1}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}\Bigg\}$$

$$\geq\inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}\Bigg\{\frac{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}:|t-t^0|\leq\eta\Bigg\}$$

$$\times\inf_{z\in\mathbb{C}^n}\inf_{t^0\in\mathbb{C}}\inf_{t}\Bigg\{\frac{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}:\Big|\prod_{j=1}^{n}(z_j+b_jt)-\prod_{j=1}^{n}(z_j+b_jt^0)\Big|\leq\frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt^0)\Big)}\Bigg\}. \tag{10}$$

The first factor in the obtained inequality is a fractional rational expression with the same degrees of the numerator and denominator by variable z_j, and by t, t^0, respectively. Thus the corresponding infimum is not equal to zero. Suppose that the second expression equals zero.

Then there exists sequences (z^p), (t^0_p), for which

$$\inf_t \left\{ \frac{l\Big(\prod\limits_{j=1}^n (z_j^p + b_j t)\Big)}{l\Big(\prod\limits_{j=1}^n (z_j^p + b_j t_p^0)\Big)} : \Big|\prod_{j=1}^n (z_j^p + b_j t) - \prod_{j=1}^n (z_j^p + b_j t_p^0)\Big| \le \frac{\eta^n}{l\Big(\prod\limits_{j=1}^n (z_j + b_j t_p^0)\Big)} \right\} \underset{p\to+\infty}{\to} 0.$$

Denoting $u_p(t) = \prod\limits_{j=1}^n (z_j^p + b_j t)$, and $v_p(t_p^0) = \prod\limits_{j=1}^n (z_j^p + b_j t_p^0)$, we obtain that

$$\inf_t \left\{ \frac{l(u_p(t))}{l(v_p(t_p^0))} : |u_p(t) - v_p(t_p^0)| \le \frac{\eta}{l(v_p(t_p^0))} \right\} \underset{p\to+\infty}{\to} 0.$$

But

$$\inf_t \left\{ \frac{l(u_p(t))}{l(v_p(t_p^0))} : |u_p(t) - v_p(t_p^0)| \le \frac{\eta}{l(v_p(t_p^0))} \right\} \ge \inf_u \left\{ \frac{l(u)}{l(v_p(t_p^0))} : |u - v_p(t_0^p)| \le \frac{\eta}{l(v_p(t_p^0))} \right\},$$

and $\inf\limits_{v\in\mathbb{C}} \inf\limits_u \left\{ \frac{l(u)}{l(v)} : |u - v| \le \frac{\eta}{l(v)} \right\} = 0$, that contradicts the condition $l \in Q$. Thus, the second factor in (10) is also positive, so the inequality (6) is correct.

Using similar considerations, we can prove the similar inequality for sup. Indeed, for all $\eta \ge 1$ the following inequalities hold

$$\begin{aligned}
&\sup_{z\in\mathbb{C}^n} \sup_{t^0\in\mathbb{C}} \sup_t \left\{ \frac{\Big(1 + \sum\limits_{k=1}^n \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t| \prod\limits_{j=k+1}^n (|z_j + b_j t| + |b_j|)\Big)\Big)}{\Big(1 + \sum\limits_{k=1}^n \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t^0| \prod\limits_{j=k+1}^n (|z_j + b_j t^0| + |b_j|)\Big)\Big)} \frac{l\Big(\prod\limits_{j=1}^n (z_j + b_j t)\Big)}{l\Big(\prod\limits_{j=1}^n (z_j + b_j t^0)\Big)} : \right. \\
&\qquad \left. |t - t^0| \le \frac{\eta}{l\Big(\prod\limits_{j=1}^n (z_j + b_j t^0)\Big)} \frac{1}{\Big(1 + \sum\limits_{k=1}^n \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t^0| \prod\limits_{j=k+1}^n (|z_j + b_j t^0| + |b_j|)\Big)\Big)} \right\} \\
&\quad \le \sup_{z\in\mathbb{C}^n} \sup_{t^0\in\mathbb{C}} \sup_t \left\{ \frac{\Big(1 + \sum\limits_{k=1}^n \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t| \prod\limits_{j=k+1}^n (|z_j + b_j t| + |b_j|)\Big)\Big)}{\Big(1 + \sum\limits_{k=1}^n \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t^0| \prod\limits_{j=k+1}^n (|z_j + b_j t^0| + |b_j|)\Big)\Big)} : \right. \\
&\qquad \left. |t - t^0| \le \frac{\eta}{\Big(1 + \sum\limits_{k=1}^n \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t^0| \prod\limits_{j=k+1}^n (|z_j + b_j t^0| + |b_j|)\Big)\Big)} \right\} \\
&\quad \times \sup_{z\in\mathbb{C}^n} \sup_{t^0\in\mathbb{C}} \sup_t \left\{ \frac{l\Big(\prod\limits_{j=1}^n (z_j + b_j t)\Big)}{l\Big(\prod\limits_{j=1}^n (z_j + b_j t^0)\Big)} : \Big|\prod_{j=1}^n (z_j + b_j t) - \prod_{j=1}^n (z_j + b_j t^0)\Big| \le \frac{\eta^n}{l\Big(\prod\limits_{j=1}^n (z_j + b_j t^0)\Big)} \right\}
\end{aligned} \tag{11}$$

$$\leq \sup_{z\in\mathbb{C}^n}\sup_{t^0\in\mathbb{C}}\sup_{t}\left\{\frac{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+|!b_jt|\prod\limits_{j=k+1}^{n}(|z_j+b_jt|+|b_j|)\Big)\Big)}{\Big(1+\sum\limits_{k=1}^{n}\Big(|b_k|\prod\limits_{j=1}^{k-1}|z_j+b_jt^0|\prod\limits_{j=k+1}^{n}(|z_j+b_jt^0|+|b_j|)\Big)\Big)}:|t-t^0|\leq\eta\right\}.$$

As above for infimum in the first brackets we obtain a fractional rational expression with the same degrees of the numerator and denominator by z_j, and by t, t^0 respectively. Hence corresponding supremum does not equal infinity. Suppose that the second expression is equal to infinity. Then there exist (z^p), (t^0_p) with property

$$\sup_{t}\left\{\frac{l\Big(\prod\limits_{j=1}^{n}(z_j^p+b_jt)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j^p+b_jt_p^0)\Big)}:\Big|\prod_{j=1}^{n}(z_j^p+b_jt)-\prod_{j=1}^{n}(z_j^p+b_jt_p^0)\Big|\leq\frac{\eta^n}{l\Big(\prod\limits_{j=1}^{n}(z_j+b_jt_p^0)\Big)}\right\}\underset{p\to+\infty}{\to}\infty.$$

Denoting $u_p(t)=\prod\limits_{j=1}^{n}(z_j^p+b_jt)$, and $v_p(t_p^0)=\prod\limits_{j=1}^{n}(z_j^p+b_jt_p^0)$, we obtain

$$\sup_{t}\left\{\frac{l(u_p(t))}{l(v_p(t_p^0))}:|u_p(t)-v_p(t_p^0)|\leq\frac{\eta}{l(v_p(t_p^0))}\right\}\underset{p\to+\infty}{\to}\infty.$$

But

$$\sup_{t}\left\{\frac{l(u_p(t))}{l(v_p(t_p^0))}:|u_p(t)-v_p(t_p^0)|\leq\frac{\eta}{l(v_p(t_p^0))}\right\}\leq\sup_{u}\left\{\frac{l(u)}{l(v_p(t_p^0))}:|u-v_p(t_0^p)|\leq\frac{\eta}{l(v_p(t_p^0))}\right\},$$

and $\sup_{v\in\mathbb{C}}\sup_u\left\{\frac{l(u)}{l(v)}:|u-v|\leq\frac{\eta}{l(v)}\right\}=\infty$, that contradicts the condition $l\in Q$. Thus, the second factor in (11) is also positive, so the inequality (7) is valid. Hence, we deduce that the function

$$\frac{1}{c}\Big(1+\sum_{k=1}^{n}(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|b_j|))\Big)l\Big(\prod_{j=1}^{n}z_j\Big)$$

belongs to the class $Q^n_{\mathbf{b}}$. □

4 Remarks to Main Theorem

Remark 1. *The condition* $\inf\{l(t):t\in\mathbb{C}\}=c>0$ *is not essential. In fact, every function* $l\in Q$, *which satisfies the equality* $\inf\{l(t):t\in\mathbb{C}\}=0$, *can be replaced by the function* $l(t)+1$, *which also belongs to the class* Q.

Proof. Indeed, for the positive continuous function $l(t)$ the inequality holds

$$\frac{l(t)}{l(t_0)}\leq\frac{l(t)+1}{l(t_0)+1}<\frac{l(t)}{l(t_0)}+1, \tag{12}$$

where the right part is true for all $t,t_0\in\mathbb{C}$, and the left part is true for all $t,t_0\in\mathbb{C}$ such that $l(t)\leq l(t_0)$. The right inequality is equivalent to the following

$$l(t_0)(l(t)+1)<(l(t)+l(t_0))(l(t_0)+1)\quad\text{or}\quad l(t_0)l(t)+l(t_0)<l(t)l(t_0)+l^2(t_0)+l(t)+l(t_0),$$

i. e. $0 < l^2(t_0) + l(t)$. But this inequality holds for the function $l(t)$ for all $t, t_0 \in \mathbb{C}$.

From the left part we similarly obtain $l(t)l(t_0) + l(t) \le l(t_0)(l(t)+1)$. Hence $l(t) \le l(t_0)$.

Evaluating the supremum for the right part of inequality (12) and the infimum for the left side and using that $l(t) \in Q$, we obtain

$$\begin{aligned} 0 < \inf\left\{\frac{l(t)}{l(t_0)} : |t-t_0| \le \frac{\eta}{l(t_0)}, t\in\mathbb{C}\right\} &\le \inf\left\{\frac{l(t)}{l(t_0)} : |t-t_0| \le \frac{\eta}{l(t_0)+1}, t\in\mathbb{C}\right\} \\ &\le \inf\left\{\frac{l(t)+1}{l(t_0)+1} : |t-t_0| \le \frac{\eta}{l(t_0)+1}, t\in\mathbb{C}\right\} \\ &\le \sup\left\{\frac{l(t)+1}{l(t_0)+1} : |t-t_0| \le \frac{\eta}{l(t_0)+1}, t\in\mathbb{C}\right\} \\ &\le \sup\left\{\frac{l(t)}{l(t_0)}+1 : |t-t_0| \le \frac{\eta}{l(t_0)+1}, t\in\mathbb{C}\right\} \\ &\le \sup\left\{\frac{l(t)}{l(t_0)}+1 : |t-t_0| \le \frac{\eta}{l(t_0)}, t\in\mathbb{C}\right\} < \infty. \end{aligned}$$

These inequalities imply $l(t)+1 \in Q$. □

Remark 2. *In fact, analysis of the proof of Theorem 1 indicates that we can somehow decrease function L. For each* $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$, *such that* $\prod_{j=1}^n |b_j| \ne 0$, $l \in Q$ *and* $\inf\{l(t) : t\in\mathbb{C}\} = c > 0$, *we have* $L \in Q^n_{\mathbf{b}}$, *where*

$$L(z) = \frac{1}{c}\Big(\sum_{k=1}^n \big(|b_k| \prod_{j=1}^{k-1}|z_j| \prod_{j=k+1}^n (|z_j|+|b_j|)\big)\Big).$$

The appearance of term 1 in the proof of Theorem 1 is necessary for lower estimate of the function $\Big(\sum\limits_{k=1}^n \big(|b_k| \prod\limits_{j=1}^{k-1}|z_j| \prod\limits_{j=k+1}^n (|z_j|+|b_j|)\big)\Big)^j$, where $j = 1, 2, \ldots, n$. We can take the direction $\widetilde{\mathbf{b}} = \mathbf{b} \Big/ \prod\limits_{j=1}^n |b_j|$ instead of $\mathbf{b}$ under the previous condition $\prod\limits_{j=1}^n |b_j| \ne 0$, because by Lemma 1 the function L belongs to the class $Q^n_{\theta\mathbf{b}}$, with $\theta = \frac{1}{\prod_{j=1}^n |b_j|}$.

Then all considerations of previous theorem should be repeated, omitting the term 1 in the appropriate places. Alternatively we can take a larger function.

Remark 3. *If* $l^* \in Q$, $l \in Q$, $\inf\{l(t) : t\in\mathbb{C}\} = c > 0$, *and for all* $z\in\mathbb{C}^n$ *the following inequalities hold*

$$l^*\Big(\prod_{j=1}^n z_j\Big) \ge c_1\Big(1 + \sum_{k=1}^n \Big(|b_k|\prod_{j=1}^{k-1}|z_j| \prod_{j=k+1}^n (|z_j|+|b_j|)\Big)\Big)$$

and

$$l^*\Big(\prod_{j=1}^n z_j\Big) \le c_2\Big(\prod_{j=1}^n (|z_j|+|b_j|) - \prod_{j=1}^n |z_j|\Big),$$

then $L \in Q^n_{\mathbf{b}}$, *where* $L(z) = \frac{1}{c}\, l^*\Big(\prod_{j=1}^n z_j\Big) l\Big(\prod_{j=1}^n z_j\Big)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, $c_1 > 0$, $c_2 > 0$.

Proof. Without loss of generality, we may suppose $\inf\{l(t) : t \in \mathbb{C}\} = 1$ as in Theorem 1. Then we can repeat the considerations of this theorem, taking everywhere the function $l^*\Big(\prod\limits_{j=1}^{n} z_j\Big)$ instead of

$$1 + \sum_{k=1}^{n} \Big(|b_k| \prod_{j=1}^{k-1} |z_j| \prod_{j=k+1}^{n} (|z_j| + |b_j|)\Big).$$

Therefore we obtain

$$\Big|\prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0)\Big| \le \eta^n \frac{\sum\limits_{k=1}^{n} \Big(|b_k| \prod\limits_{j=1}^{k-1} |z_j + b_j t^0| \prod\limits_{j=k+1}^{n} (|z_j + b_j t^0| + |b_j|)\Big)}{\min\{1, c_1^n\} l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)}$$
$$\le \frac{\eta^n}{\min\{c_1, c_1^{n+1}\} l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)}.$$

Denoting $\widetilde{c} = \min\{c_1, c_1^{n+1}\}$, for all $\eta \ge 1$ we obtain the following inequality

$$\inf_{z\in\mathbb{C}^n} \inf_{t^0\in\mathbb{C}} \inf_{t} \left\{ \frac{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : |t - t^0| \le \frac{\eta}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\}$$
$$\ge \inf_{z\in\mathbb{C}^n} \inf_{t^0\in\mathbb{C}} \inf_{t} \left\{ \frac{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : \Big|\prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0)\Big| \le \frac{\eta^n}{\widetilde{c}\, l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\}$$
$$\times \inf_{z\in\mathbb{C}^n} \inf_{t^0\in\mathbb{C}} \inf_{t} \left\{ \frac{l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : \Big|\prod_{j=1}^{n}(z_j + b_j t) - \prod_{j=1}^{n}(z_j + b_j t^0)\Big| \le \frac{\eta^n}{\widetilde{c}\, l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\}. \tag{13}$$

Since $l(t) \in Q$, by similar considerations as in Theorem 1 it can be showed that the product in (13) is greater than zero. It is obviously that we can prove

$$\sup_{z\in\mathbb{C}^n} \sup_{t^0\in\mathbb{C}} \sup_{t} \left\{ \frac{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t)\Big)}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} : |t - t^0| \le \frac{\eta}{l^*\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big) l\Big(\prod\limits_{j=1}^{n}(z_j + b_j t^0)\Big)} \right\} < \infty. \tag{14}$$

In view of (13), (14) we obtain that the function $l^*\Big(\prod_{j=1}^{n} z_j\Big) l\Big(\prod_{j=1}^{n} z_j\Big)$ belongs to the class $Q^n_{\mathbf{b}}$. □

Remark 4. *We can take the following functions*

$$\prod_{j=1}^{n}(|z_j|+|b_j|)-\prod_{j=1}^{n}|z_j| \qquad \text{or} \qquad \sum_{k=1}^{n}\Big(|b_k|\prod_{\substack{j=1\\ j\neq k}}^{n}(|z_j|+|b_j|)\Big)$$

instead of the expression $\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|b_j|)\Big)$ *in Theorem 1.*

It follows from Lemma 5 and notion

$$1+\prod_{j=1}^{n}(|z_j|+|b_j|)-\prod_{j=1}^{n}|z_j| \asymp 1+\sum_{k=1}^{n}\Big(|b_k|\prod_{\substack{j=1\\ j\neq k}}^{n}(|z_j|+|b_j|)\Big) \asymp 1+\sum_{k=1}^{n}\Big(|b_k|\prod_{j=1}^{k-1}|z_j|\prod_{j=k+1}^{n}(|z_j|+|\mathbf{b}_j|)\Big).$$

Proposition 1. *If $L\in Q^n_{\mathbf{b}}$, then for every $z^0\in\mathbb{C}^n$ we have $l_{z^0}\in Q$ $(l_{z^0}(t)\equiv L(z^0+t\mathbf{b}))$.*

Proof. We remark that (1)–(5) imply for every $z^0\in\mathbb{C}^n$, $t\in\mathbb{C}$

$$\forall\eta>0\ \ 0<\lambda_1^{\mathbf{b}}(z,\eta)\le\lambda_1^{\mathbf{b}}(z,t_0,\eta)\le 1\le\lambda_2^{\mathbf{b}}(z,t_0,\eta)\le\lambda_2^{\mathbf{b}}(z,\eta)<+\infty.$$

These inequalities imply that $l_{z^0}\in Q$. □

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Досліджено властивості класів $Q^n_{\mathbf{b}}$ та Q додатних непервних функцій. Доведено, що деякі композиції функцій із класу Q належать класу $Q^n_{\mathbf{b}}$. Встановлено зв'язок між функціями цих класів.

Ключові слова і фрази: додатна функція, неперервна функція, декілька комплексних змінних.