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NOTE ON ARENS REGULARITY OF SYMMETRIC TENSOR PRODUCTS

We investigate symmetric regularity of sums of symmetric tensor products of Banach spaces and Arens regularity of symmetric tensor products of Banach algebras. An example for the Hilbert space is obtained.

Key words and phrases: symmetric regularity, multilinear map, polynomial on Banach space, Arens regularity, tensor product.

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INTRODUCTION

Let X and Y be complex Banach spaces and $B : X \times X \to Y$ be a bilinear map. A map $\widetilde{B} : X^{**} \times X^{**} \to Y^{**}$ is said to be the Aron-Berner extension of B if it is defined by

$$\widetilde{B}(x^{**}, y^{**}) = \lim_{\alpha} B(x_\alpha, y_\beta),$$

where x_α and y_β are nets in X which are weakly-star convergent in X^{**} to x^{**} and y^{**} respectively.

The bilinear map is *regular* if

$$\lim_{\alpha,\beta} B(x_\alpha, y_\beta) = \lim_{\beta,\alpha} B(x_\alpha, y_\beta) \tag{1}$$

for all weakly-star convergent nets $(x_\alpha), (y_\beta) \subset X$ in X^{**}. X is *regular* if each bilinear form on $X \times X$ is regular. X is *symmetricaly regular* if each symmetric bilinear form on X is regular (see [3]). If A is a Banach algebra, then A is called *Arens regular* if the bilinear map associated with the algebra product $(x, y) \to xy$ is regular. In this case the Aron-Berner extension of the algebra product coincides with the Arens extension [1].

In this note we examine Arens regularity of symmetric projective tensor products of Banach algebras.

1 REGULARITY OF SUMS OF SYMMETRIC TENSOR PRODUCTS

Let us denote by $\mathcal{P}(^nX)$ the Banach space of all continuous n-homogeneous polynomials on X. A net $(x_\alpha) \subset X$ is called *n-polynomially convergent* to a functional $\varphi \in \mathcal{P}(^nX)^*$ if

$$\varphi(P) = \lim_{\alpha} P(x_\alpha)$$

for every $P \in \mathcal{P}(^nX)$. (x_α) is *polynomially convergent* if it is n-polynomially convergents for some n.

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Theorem 1. *Let (x_α) and (y_β) be polynomially convergent nets such that*

$$\lim_{\alpha,\beta} P(x_\alpha + y_\beta) \neq \lim_{\beta,\alpha} P(x_\alpha + y_\beta)$$

for a polynomial $P \in \mathcal{P}(^n X)$. Then the Banach space

$$\sum^n X := \mathbb{C} \oplus X \oplus X \otimes_{s,\pi} X \oplus \ldots \oplus \overbrace{X \otimes_{s,\pi} \ldots \otimes_{s,\pi} X}^{n}$$

is not symmetrically regular, where the symbol $\otimes_{s,\pi}$ denotes the complete symmetric projective tensor product.

Proof. Let A_P be the symmetric n-linear map associated with P. That is, $A_P(x, \ldots, x) = P(x)$. Let us define a bilinear map B_P on $\sum^n X$ by the following way: given $w, u \in \sum^n X$ can be represented by $w = w_0 + w_1 + \ldots + w_n$, $u = u_0 + u_1 + \ldots + u_n$, where

$$w_0, u_0 \in \mathbb{C}, \qquad w_k, u_k \in \otimes_{s,\pi}^k X = \overbrace{X \otimes_{s,\pi} \ldots \otimes_{s,\pi} X}^{k},$$

and $w_1 = x_1 \in X$, $u_1 = y_1 \in Y$,

$$w_k = \sum_{j=1}^{\infty} x_{kj}^{\otimes k} = \sum_{j=1}^{\infty} x_{kj} \otimes \ldots \otimes x_{kj}, \qquad u_k = \sum_{j=1}^{\infty} y_{kj}^{\otimes k} = \sum_{j=1}^{\infty} y_{kj} \otimes \ldots \otimes y_{kj}.$$

Then we set

$$\begin{aligned} B_P(w,u) = & \sum_{j_1,\ldots,j_n} A_P(u_0 x_{nj_1}, \ldots, u_0 x_{nj_n}) + \sum_{j_2,\ldots,j_n} n A_P(y_1, x_{(n-1)j_2}, \ldots, x_{(n-1)j_n}) + \ldots \\ & + \binom{n}{k} \sum_{j_1,\ldots,j_n} A_P(y_{j_1}, \ldots, y_{j_k}, x_{j_{k+1}}, \ldots, x_{j_n}) + \sum_{j_1,\ldots,j_n} A_P(w_0 y_{nj_1}, \ldots, w_0 y_{nj_n}). \end{aligned}$$

Clearly that B_P is a continuous symmetric bilinear form on $\sum^n X$ and

$$B_P(1 + x + \ldots x^{\otimes x}, 1 + y + \ldots y^{\otimes n}) = P(x + y).$$

Let ν be the 'canonical' embedding $\nu(x) = 1 + x + \ldots + x^{\otimes n}$, x_α and y_β be n- polynomially convergent nets. Then $\nu(x_\alpha)$ and $\nu(y_\alpha)$ are weakly-star convergent in $(\sum^n X)^{**}$. Hence

$$\lim_{\alpha,\beta} B_P(\nu(x_\alpha), \nu(y_\beta)) \neq \lim_{\beta,\alpha} B_P(\nu(x_\alpha), \nu(y_\beta))$$

and so B_P is not regular. Thus $\sum^n X$ is not symmetrically regular. □

For a given $x = \sum_{n=1}^{\infty} x_n e_n$ in ℓ_1 the *support* of x is the subset $\text{supp}\ \ x = \{m \in \mathbb{N} : x_m \neq 0\}$. Here $\{e_n\}$ is the standart basis of ℓ_1.

Proposition 1. *There exists a symmetric bilinear map $B : \ell_1 \times \ell_1 \to \mathbb{C}$ and there are nets $(x_\alpha) \subset \ell_1$ and $(y_\beta) \subset \ell_1$ such that $||x_\alpha|| = ||y_\beta|| = 1$ and*

1) $\lim_{\alpha,\beta} B(x_\alpha, y_\beta) \neq \lim_{\beta,\alpha} B(x_\alpha, y_\beta)$,

2) $\text{supp}\ \ x_\alpha \cap \text{supp}\ \ y_\beta = \varnothing$ *for all* α and β.

Proof. 1) it follows from the fact that ℓ_1 is not symmetrically regular. To construct map B which satisfy both 1) and 2) conditions we will use Example 1.1 in [5]. For simplicity we consider $\ell_1(\mathbb{Z})$. Let L_+ and L_- are in $\ell_1(\mathbb{Z})^{**}$ such that L_+ is a Hahn-Banach extension of functional

$$\varphi_+(x) = \lim_{n\to+\infty} x_n,$$

$x_n \in c(\mathbb{Z})$ and L_- is a Hahn-Banach extension of

$$\varphi_-(x) = \lim_{n\to-\infty} x_n.$$

Clearly L_+ may by approximated in weak-star topology by (x_α), $x_\alpha \in \ell_1$, $||x|| = 1$, $\alpha > 0$ and L_- by y_β, $||y_\beta|| = 1$, $\beta < 0$. Also in [5] it is shown that the Arens extension of the convolution $*$ on ℓ_1 is not commutative and

$$\lim_{\alpha,\beta}(x_\alpha * y_\beta) \neq \lim_{\beta,\alpha}(x_\alpha * y_\beta).$$

So there is a linear functional f on $\ell_1(\mathbb{Z})^{**}$ such that

$$\lim_{\alpha,\beta} f(x_\alpha * y_\beta) \neq \lim_{\beta,\alpha} f(x_\alpha * y_\beta).$$

We set $B(x,y) = f(x * y)$. □

Proposition 2. *There exists a 4-homogeneous polynomial P on ℓ_2 such that*

$$\lim_{\alpha,\beta} P(x_\alpha + y_\beta) \neq \lim_{\beta,\alpha} P(x_\alpha + y_\beta)$$

for some polynomially convergent nets $(x_\alpha), (y_\beta) \subset \ell_2$.

Let $B(x,y)$ be a symmetric non-regular bilinear map on ℓ_1 and (x_α) and (y_β) as in Proposition 1. We can write

$$x_\alpha = \sum_{n=1}^{\infty} x_{\alpha,n} e_n \qquad \text{and} \qquad y_\beta = \sum_{n=1}^{\infty} y_{\beta,n} e_n,$$

where e_n is the standart basis on ℓ_2 of the form $z_\alpha = \sum_{n=1}^{\infty} \sqrt{x_{\alpha,n}} e_n$ and $r_\beta = \sum_{n=1}^{\infty} \sqrt{y_{\beta,n}} e_n$. Clearly $||z_\alpha||_{\ell_2} = ||x_\alpha||_{\ell_1} = 1$ and $||r_\beta||_{\ell_2} = ||y_\beta||_{\ell_1} = 1$. By compactness reasons nets (z_α) and (r_β) contains H_b-convergent subsets (which are polynomially convergent as well) which we will denote by the same symbols ([2]). Let us define the following polynomial on ℓ_2

$$P(x) = B\Big(\sum_{n=1}^{\infty} x_n^2 e_n, \sum_{n=1}^{\infty} x_n^2 e_n\Big),$$

where B is defined above. Since

$$\sum_{n=1}^{\infty} x_n^2 e_n \in \ell_1 \qquad \text{for every} \qquad x = \sum_{n=1}^{\infty} x_n e_n \in \ell_2,$$

P is well defined. Since nets (x_α) and (y_β) have the disjoint supports,

$$\begin{aligned} P(z_\alpha + r_\beta) &= B\Big(\sum_{n=1}^{\infty} z_{\alpha,n}^2 + \sum_{n=1}^{\infty} r_{\beta,n}^2, \sum_{n=1}^{\infty} z_{\alpha,n}^2 + \sum_{n=1}^{\infty} r_{\beta,n}^2\Big) \\ &= B(x_\alpha + y_\beta, x_\alpha + y_\beta) = B(x_\alpha, x_\alpha) + 2B(x_\alpha, y_\beta) + B(y_\beta, y_\beta). \end{aligned}$$

So,

$$\begin{aligned} \lim_{\alpha,\beta} P(z_\alpha + r_\beta) &= \lim_{\alpha} B(x_\alpha, x_\alpha) + \lim_{\beta} B(y_\beta, y_\beta) + 2\lim_{\alpha,\beta} B(x_\alpha, y_\beta) \\ &\neq \lim_{\alpha} B(x_\alpha, x_\alpha) + \lim_{\beta} B(y_\beta, y_\beta) + 2\lim_{\beta,\alpha} B(x_\alpha, y_\beta) = \lim_{\beta,\alpha} P(z_\alpha + r_\beta). \end{aligned}$$

Corollary. *$\sum^4 \ell_2$ is not symmetrically regular. Note that in [3] it is shown that the complete projective tensor product $\ell_2 \otimes_\pi \ell_2$ is not symmetrically regular.*

2 THE CASE OF BANACH ALGEBRA

Let A be a Banach algebra. Then the complete projective tensor power $\bigotimes_\pi^n A$ is a Banach algebra too and the symmetric tensor power $\bigotimes_{s,\pi}^n A$ is a Banach subalgebra of $\bigotimes_\pi^n A$. In [4] was studied conditions of Arens regularity of $\bigotimes_\pi^n A$. Here we concentrate on $\bigotimes_{s,\pi}^n A$.

Theorem 2. *Let (x_α), (y_β) be an n-polynomial convergent nets in A such that*

$$\lim_{\alpha,\beta} P(x_\alpha \cdot y_\beta) \neq \lim_{\beta,\alpha} P(x_\alpha \cdot y_\beta) \tag{2}$$

for an arbitrary $P \in \mathcal{P}(^n A)$. Then $\bigotimes_{s,\pi}^n A$ is not regular.

Proof. If $(x_\alpha), (y_\beta) \subset A$ are n-polynomial convergent nets to $\varphi, \psi \in (\mathcal{P}(^n A))^*$ respectively, then nets $u_\alpha = x_\alpha \otimes \ldots \otimes x_\alpha$, $v_\beta = y_\beta \otimes \ldots \otimes y_\beta$ are convergent in week-star topology to $\widehat{\varphi}, \widehat{\psi} \in (\bigotimes_{s,\pi}^n A)^{**}$ respectively, i.e. for all $f \in (\bigotimes_{s,\pi}^n A)^*$

$$\widehat{\varphi}(f) = \lim_\alpha f(u_\alpha), \qquad \widehat{\psi}(f) = \lim_\beta f(v_\beta).$$

Let A_P be a symmetric n-linear map associated with P and f is the linear functional on $\bigotimes_{s,\pi}^n A$ such that $P(x) = f(x \otimes \ldots \otimes x)$.

Let us consider $P(x \cdot y)$ for arbitrary $P \in \mathcal{P}(^n A)$:

$$P(x \cdot y) = A_P(\underbrace{x \cdot y, \ldots, x \cdot y}_{n}) = f(\underbrace{x \cdot y \otimes \ldots \otimes x \cdot y}_{n}) = f(u \cdot v) = B(u, v),$$

where $u = \underbrace{x \otimes \ldots \otimes x}_{n}$, $v = \underbrace{y \otimes \ldots \otimes y}_{n}$ and

$$u \cdot v = \frac{\overbrace{\underbrace{x \cdot y \otimes \ldots \otimes x \cdot y}_{n} + \ldots + \underbrace{x \cdot y \otimes \ldots \otimes x \cdot y}_{n}}^{n}}{n} = \underbrace{x \cdot y \otimes \ldots \otimes x \cdot y}_{n}.$$

So, if

$$\lim_{\alpha,\beta} P(x_\alpha \cdot y_\beta) \neq \lim_{\beta,\alpha} P(x_\alpha \cdot y_\beta),$$

then

$$\lim_{\alpha,\beta} B(u_\alpha, v_\beta) \neq \lim_{\beta,\alpha} B(u_\alpha, v_\beta).$$

Thus B is a bilinear map on $\bigotimes_{s,\pi}^n A$ and is not regular. □

Remark. *In the case of commutative Banach algebra we can see that under conditions of Theorem 2, $\bigotimes_{s,\pi}^n A$ is not symmetrically regular.*

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У роботі досліджено симетричну регулярність сум симетричних тензорних добутків банахових просторів і регулярність за Аренсом симетричних тензорних добутків банахових алгебр. Розглянуто приклад для випадку гільбертового простору.

Ключові слова і фрази: симетрична регулярність, мультилінійне відображення, поліном на банаховому просторі, регулярність за Аренсом, тензорний добуток.

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В работе исследуется симметрическая регулярность сумм симметрических тензорных произведений банаховых пространств и регулярность по Аренсу симметрических тензорных произведений банаховых алгебр. Рассмотрен пример для случая гильбертового пространства.

Ключевые слова и фразы: симметрическая регулярность, мультилинейное отображение, полином на банаховом пространстве, регулярность по Аренсу, тензорное произведение.